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# Characteristic functions of Markovian random evolutions in $\mathbb{R}^m$

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**Abstract.** The recurrent and integral relations for characteristic functions of Markovian random evolution in $\mathbb{R}^m$, $m \geq 2$, are presented.

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The particular models of random evolutions in various Euclidean spaces of lower dimensions were studied in [1-5]. In this note we announce the recent results on the characteristic functions for the most general $m$-dimensional random evolution.

The subject of our interests is the following stochastic motion. A particle starts its motion from the origin $x_1 = \cdots = x_m = 0$ of the space $\mathbb{R}^m$, $m \geq 2$ at time $t = 0$. The particle is endowed with constant, finite speed $c$. The initial direction is a random $m$-dimensional vector with uniform distribution on the unit $m$-sphere

$$S_1^m = \left\{ \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : \; x_1^2 + \cdots + x_m^2 = 1 \right\}.$$

The particle changes direction at random instants which form a homogeneous Poisson process of rate $\lambda > 0$. At these moments it instantaneously takes on the new direction with uniform distribution on $S_1^m$, independently of its previous motion.

Let $\mathbf{X}(t) = (X_1(t), \dots, X_m(t))$ be the position of the particle at an arbitrary time $t > 0$. At first, we concentrate our attention on the conditional distributions

$$Pr\{\mathbf{X}(t) \in d\mathbf{x} \mid N(t) = n\} =$$

$$= Pr\{X_1(t) \in dx_1, \dots, X_m(t) \in dx_m \mid N(t) = n\}, \quad n \geq 1$$

where $N(t)$ is the number of Poisson events that have occurred in the interval $(0, t)$ and $d\mathbf{x} = dx_1 \dots dx_m$ is the infinitesimal volume in the space $\mathbb{R}^m$.

Consider the conditional characteristic functions:

$$H_n(t) = E\left\{ e^{i(\boldsymbol{\alpha}, \mathbf{X}(t))} \mid N(t) = n \right\}, \qquad n \geq 1, \tag{1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ is the real $m$-dimensional vector of inversion parameters and $(\boldsymbol{\alpha}, \mathbf{X}(t))$ denotes the scalar (inner) product of the vectors $\boldsymbol{\alpha}$ and $\mathbf{X}(t)$.

Computing the expectation in (1) we obtain

$$H_n(t) = \frac{n!}{t^n} \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \dots \int_{\tau_{n-1}}^t d\tau_n \times$$
$$\times \left\{ \prod_{j=1}^{n+1} \left[ 2^{(m-2)/2} \Gamma\left(\frac{m}{2}\right) \frac{J_{(m-2)/2}(c(\tau_j - \tau_{j-1})\|\boldsymbol{\alpha}\|)}{(c(\tau_j - \tau_{j-1})\|\boldsymbol{\alpha}\|)^{(m-2)/2}} \right] \right\}. \qquad (2)$$

For the particular cases $m = 2$ (planar motion) and $m = 4$ (four-dimensional motion) the conditional characteristic functions (2) were explicitly computed in [2] (see formula (18) therein), and in [1] (see formula (15) therein), respectively.

We introduce the function

$$\varphi(t) = 2^{(m-2)/2} \, \Gamma\left(\frac{m}{2}\right) \, \frac{J_{(m-2)/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{(m-2)/2}}, \qquad m \geq 2. \qquad (3)$$

Then (2) can rewritten in the following form

$$H_n(t) = \frac{n!}{t^n} \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \dots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \varphi(\tau_j - \tau_{j-1}) \right\}, \qquad n \geq 1. \qquad (4)$$

Denote the integral factor in (4) as follows

$$\mathcal{I}_n(t) = \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \dots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \varphi(\tau_j - \tau_{j-1}) \right\}, \qquad n \geq 1. \qquad (5)$$

The following theorem states that, for different $n \geq 1$, the functions (5) are connected with each other by a convolution-type recurrent relation.

**Theorem 1.** *For any $n \geq 1$ the following recurrent relation holds*

$$\mathcal{I}_n(t) = \int_0^t \varphi(t-\tau) \, \mathcal{I}_{n-1}(\tau) \, d\tau = \int_0^t \varphi(\tau) \, \mathcal{I}_{n-1}(t-\tau) \, d\tau, \qquad n \geq 1, \qquad (6)$$

*where, by definition, $\mathcal{I}_0(x) = \varphi(x)$.*

Note that formula (6) can be rewritten in the following convolution form

$$\mathcal{I}_n(t) = \varphi(t) * \mathcal{I}_{n-1}(t), \qquad n \geq 1. \qquad (7)$$

**Corollary 1.1.** *For any $n \geq 1$ the following relation holds*

$$\mathcal{I}_n(t) = [\varphi(t)]^{*(n+1)}, \qquad n \geq 1, \qquad (8)$$

*where the symbol $*(n+1)$ means the $(n+1)$-multiple convolution.*

Application of the Laplace transform

$$\mathcal{L}\left[f(t)\right](s) = \int_0^\infty e^{-st} f(t) \, dt, \qquad \text{Re } s > 0,$$

to the equality (8) leads to the following important result.

**Corollary 1.2.** *For any $n \geq 1$ the Laplace transform of functions (5) has the form*

$$\mathcal{L}\left[\mathcal{I}_n(t)\right](s) = \left(\mathcal{L}\left[\varphi(t)\right](s)\right)^{n+1}, \qquad n \geq 1. \tag{9}$$

These results show that the function $\varphi(t)$ given by (3) plays a key role in our analysis. The reason is that $\varphi(t)$ is exactly the characteristic function (Fourier transform) of the uniform distribution on the surface of the $m$-sphere $S_{ct}^m$ of the radius $ct$.

From both the Theorem 1 and its corollaries we see that the conditional characteristic functions $H_n(t)$ and their Laplace transforms, in fact, are expressed in terms of function $\varphi(t)$. Formula (9) shows that the possibility of obtaining the explicit form of the conditional characteristic functions (4) entirely depends on whether the exact Laplace transform of the function $\varphi(t)$ and its inverse Laplace transform can be explicitly computed.

Our next result presents a general formula for the conditional characteristic functions $H_n(t)$ in terms of inverse Laplace transform.

**Theorem 2.** *For any $n \geq 1$ and any $t > 0$ the conditional characteristic functions (4) are given by*

$$H_n(t) = \frac{n!}{t^n}\,\mathcal{L}^{-1}\left[\left(\frac{1}{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}}\,F\left(\frac{1}{2},\frac{m-2}{2};\frac{m}{2};\frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2+(c\|\boldsymbol{\alpha}\|)^2}\right)\right)^{n+1}\right](t), \tag{10}$$

*where $\mathcal{L}^{-1}$ means the inverse Laplace transform and*

$$F(\xi,\eta;\zeta;z) = {}_2F_1(\xi,\eta;\zeta;z) = \sum_{k=0}^{\infty}\frac{(\xi)_k(\eta)_k}{(\zeta)_k}\frac{z^k}{k!}$$

*is the standard hypergeometric function.*

In view of (4), the characteristic function of $\mathbf{X}(t),\ t \geq 0$, is given by the uniformly converging series

$$H(t) = e^{-\lambda t}\sum_{n=0}^{\infty}\lambda^n\,\mathcal{I}_n(t). \tag{11}$$

The following theorem presents the integral equation for the function $H(t)$.

**Theorem 3.** *The characteristic function $H(t),\ t \geq 0$, satisfies the following convolution-type Volterra integral equation of second kind with the kernel $e^{-\lambda t}\varphi(t)$:*

$$H(t) = e^{-\lambda t}\varphi(t) + \lambda\int_0^t e^{-\lambda(t-\tau)}\varphi(t-\tau)H(\tau)\,d\tau, \qquad t \geq 0. \tag{12}$$

The integral equation (12) can be rewritten in the following convolution form

$$H(t) = e^{-\lambda t}\varphi(t) + \lambda\left[\left(e^{-\lambda t}\varphi(t)\right) * H(t)\right], \qquad t \geq 0. \tag{13}$$

From this we immediately obtain the general formula for the Laplace transform of the characteristic function $H(t)$:

$$\mathcal{L}\left[H(t)\right](s) = \frac{\mathcal{L}\left[\varphi(t)\right](s+\lambda)}{1-\lambda\,\mathcal{L}\left[\varphi(t)\right](s+\lambda)}\ , \qquad \text{Re}\, s>0. \tag{14}$$

The explicit form of (14) is

$$\mathcal{L}\left[H(t)\right](s) = \frac{F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right)}{\sqrt{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2} - \lambda\, F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right)}. \tag{15}$$

From (11) and (8) it follows that the solution of equation (13) has the form

$$H(t) = e^{-\lambda t}\sum_{n=0}^{\infty} \lambda^n\ \left[\varphi(t)\right]^{*(n+1)}\ . \tag{16}$$

One should emphasize that, although formula (16) gives a general form of the characteristic function $H(t)$, the multiple convolutions of the function $\varphi(t)$ with itself can scarcely be explicitly evaluated for arbitrary dimension.

From (12) we can see that

$$H(t)\big|_{t=0} = 1, \qquad \left.\frac{\partial H(t)}{\partial t}\right|_{t=0} = 0,$$

and, therefore, the transition density $f(\mathbf{x},t),\ \mathbf{x}\in\mathbb{R}^m,\ t\ge 0$, of the process $\mathbf{X}(t)$ satisfies the following initial conditions

$$f(\mathbf{x},t)\big|_{t=0} = \delta(\mathbf{x}), \qquad \left.\frac{\partial f(\mathbf{x},t)}{\partial t}\right|_{t=0} = 0,$$

where $\delta(\mathbf{x})$ is the $m$-dimensional Dirac delta-function.

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