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Finite difference schemes for problems of mixture of two component elastic materials

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Abstract. In this paper we consider the numerical approximation of the solution of the 2D unsteady equations of mixture on a rectangular domain using the operator-splitting schemes for solving unsteady elasticity problems. Its major peculiarity is that transition to the next time level is performed by solving separate elliptic problems for each component of the displacement vector. The previous results make it possible to design efficient numerical algorithms for two component mixture elasticity equations.

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1 Introduction

The continuum theory of mixtures has been a subject of study in recent years. The linearized theory of elasticity for the indicated medium was given by T.R. Steel. [1] The two-dimensional problems for the isotropic mixture are considered by T.R. Steel [2] and M.O. Basheleishvili [3]. Some three-dimensional basic problems for indicated medium are considered by D.G. Natroshvili, A.J. Jagmaidze and M.J. Svanadze [4].

In this work, we develop our study using the finite difference methodology for spaces discretization. For dynamic problems of continuum mechanics the unsteady system of elastic mixture equations is used. These equations constitute a hyperbolic system of equations of second order. Stability analysis of the proposed schemes is made in framework of the general theory of stability for operator-difference schemes [5]. Discretization in space is performed in such a way that all basic properties of the differential operator are preserved in the corresponding grid Hilbert spaces. Finally, an additive scheme (of predictor-corrector type) is constructed using a triangular splitting for the discrete matrix operator.

2 Differential problem

For simplicity let us treat the transient problem of elasticity of mixture where there is no dependence on the longitudinal coordinate. Let us then consider the stressed state of an elastic isotropic body of mixture with rectangular section Ω. In the two-dimensional case the basic equations of the theory of the elastic mixture have the form [6–8]:

$$\begin{aligned} &\rho_{11}\frac{\partial^2 u}{\partial t^2} - \rho_{12}\frac{\partial^2 v}{\partial t^2} + \alpha\left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\right) - \\ &- \left(a_1\,\Delta u + b_1\, grad\, div\, u + c\,\Delta v + d\, grad\, div\, v\right) = f_1\left(x,t\right), \\ &\rho_{22}\frac{\partial^2 v}{\partial t^2} - \rho_{12}\frac{\partial^2 u}{\partial t^2} - \alpha\left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\right) - \\ &- \left(c\,\Delta u + d\, grad\, div\, u + a_2\,\Delta v + b_2\, grad\, div\, v\right) = f_2\left(x,t\right), \end{aligned} \tag{1}$$

where $u = (u_1, u_2)$, $v = (v_1, v_2)$ are partial displacements, a_1, b_1, c, d, a_2, b_2 are the known constants characterizing the physical properties of the mixture, Δ is the two-dimensional Laplacian, f is the vector of volumetric forces, *grad* and *div* are the operators on the field theory, ρ_1 and ρ_2 are the partial densities (positive constants), $\alpha \geq 0$,

$$a_j = \mu_j - \lambda_5, \quad b_j = \mu_j + \lambda_j + \lambda_5 + \frac{(-1)^j\,\rho_{3-j}\alpha_2}{\rho_1+\rho_2},$$

$$\rho_{jj} = \rho_j + \rho_{12}, \quad j = 1,2, \quad c = \mu_3 + \lambda_5,$$

$$d = \mu_3 + \lambda_3 + \lambda_5 - \frac{\rho_1\alpha_2}{\rho_1+\rho_2} = \mu_3 + \lambda_4 - \lambda_5 + \frac{\rho_2\alpha_2}{\rho_1+\rho_2}, \quad \alpha_2 = \lambda_3 - \lambda_4,$$

μ_1, μ_2, μ_3, λ_1, λ_2, ..., λ_5 are elastic constants of the mixture [1, 6, 10].

In the sequel it will be assumed that the following conditions are fulfilled [1, 6, 10]:

$$\begin{aligned} &\mu_1 > 0, \quad \mu_1\mu_2 > \mu_3^2, \quad \lambda_1 - \frac{\rho_2\alpha_2}{\rho_1+\rho_2} + \frac{2}{3}\mu_1 > 0, \\ &\lambda_5 \leq 0, \quad \rho_{11} > 0, \quad \rho_{11}\rho_{22} > \rho_{12}^2, \\ &\left(\lambda_1 - \frac{\rho_2\alpha_2}{\rho_1+\rho_2} + \frac{2}{3}\mu_1\right)\left(\lambda_2 + \frac{\rho_1\alpha_2}{\rho_1+\rho_2} + \frac{2}{3}\mu_2\right) > \left(\lambda_3 - \frac{\rho_1\alpha_2}{\rho_1+\rho_2} + \frac{2}{3}\mu_3\right)^2. \end{aligned} \tag{2}$$

The system of equations (1) is supplemented with the corresponding boundary and initial conditions. Namely, assume that the boundary $\partial\Omega$ is fixed, i.e. there is no displacement

$$u\,(x,t) = 0, \quad v(x,t) = 0, \quad x \in \partial\Omega. \tag{3}$$

The initial state is specified by

$$u\,(x,t) = u^0\,(x)\,, \quad v\,(x,t) = v^0\,(x)\,, \quad x \in \Omega, \tag{4}$$

$$\frac{\partial u}{\partial t}\,(x,0) = u^1(x), \quad \frac{\partial v}{\partial t}\,(x,0) = v^1(x), \quad x \in \Omega. \tag{5}$$

To formulate the operator for (1)-(5), we first introduce appropriate functional spaces and operators. Let us consider the standard Hilbert space $L_2(\Omega)$ the set of square-integrable scalar valued functions defined on Ω, with the scalar product and the corresponding norm

$$(u,v)=\int_{\Omega} u(x)\,v(x)\,dx, \quad \|u\|=(u,u)^{\frac{1}{2}},$$

and the Hilbert space H $=(L_2(\Omega))^4$ with the inner product for 4D vector valued functions u and v, given by

$$(u,v)=\sum_{i=1}^{4}(u_i,v_i)$$

$\overset{0}{W}{}_2^1(\Omega)$ denotes the usual Sobolev space of functions vanishing at the boundary $\partial\Omega$, with the inner product and norm defined by

$$(u,v)_{\overset{0}{W}{}_2^1(\Omega)}=\sum_{\alpha=1}^{2}\int_{\Omega}\frac{\partial u}{\partial x_\alpha}\frac{\partial v}{\partial x_\alpha}dx, \quad \|u\|_{\overset{0}{W}{}_2^1(\Omega)}=(u,u)^{\frac{1}{2}}_{\overset{0}{W}{}_2^1(\Omega)},$$

and let $V=\left(\overset{0}{W}{}_2^1(\Omega)\right)^4$.

On the H we consider the unbounded operator written in operator matrix form as

$$A\,v=\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14}\\ A_{21} & A_{22} & A_{23} & A_{24}\\ A_{31} & A_{32} & A_{33} & A_{34}\\ A_{41} & A_{42} & A_{43} & A_{44}\end{pmatrix} v. \tag{6}$$

Where (see (1))

$$A_{11}=-(a_1+b_1)\frac{\partial^2}{\partial x_1^2}-b_1\frac{\partial^2}{\partial x_2^2}, \quad A_{12}=A_{21}=-b_1\frac{\partial^2}{\partial x_1\partial x_2},$$

$$A_{22}=-b_1\frac{\partial^2}{\partial x_1^2}-(a_1+b_1)\frac{\partial^2}{\partial x_2^2}, \quad A_{13}=A_{31}=-(c+d)\frac{\partial^2}{\partial x_1^2}-d\frac{\partial^2}{\partial x_2^2},$$

$$A_{14}=A_{23}=A_{32}=A_{4'1}=-d\frac{\partial^2}{\partial x_1\partial x_2}, \quad A_{24}=A_{42}=-d\frac{\partial^2}{\partial x_1^2}-(c+d)\frac{\partial^2}{\partial x_2^2},$$

$$A_{33}=-(a_2+b_2)\frac{\partial^2}{\partial x_1^2}-b_2\frac{\partial^2}{\partial x_2^2}, \quad A_{34}=A_{43}=-b_2\frac{\partial^2}{\partial x_1\partial x_2},$$

$$A_{44}=-b_2\frac{\partial^2}{\partial x_1^2}-(a_2+b_2)\frac{\partial^2}{\partial x_2^2}$$

The operator A has the domain $D(A)=\{v\in V\,|Av\in H\}$ dense in H.

We have $(Av,v)\geq 0$. In this situation we will write $A\geq 0$ in H.

Besides, it is known that A is maximal monotone, and

$$(Av,u)=(v,Au)\,,$$

i.e., A is selfadjoint in H.

Finally, the following energetic equivalence holds

$$-b\left(\widetilde{\Delta}\,v,v\right)\leq (Av,v)\leq -\left(a+b\right)\left(\widetilde{\Delta}\,v,v\right), \tag{7}$$

where

$$\widetilde{\Delta}=\begin{pmatrix}\Delta & 0 & 0 & 0\\ 0 & \Delta & 0 & 0\\ 0 & 0 & \Delta & 0\\ 0 & 0 & 0 & \Delta\end{pmatrix}$$

and $b=\min\left\{b_1,d,b_2\right\}$, $(a+b)=\max\left\{(a_1+b_1)\,,(c+d)\,,(a_2+b_2)\right\}.$

Problem (1)-(5) can be written in differential operator form as the abstract initial value problem

$$\rho\frac{d^2v}{dt^2}+\alpha\frac{dv}{dt}+Av=f, \tag{8}$$

$$v\left(0\right)=v^0,\quad \frac{dv}{dt}\left(0\right)=v^1, \tag{9}$$

with the unique solution if $v^0\in D\left(A\right)$ and $v^1\in H$.

The operator A is selfadjoint and positive on space H and, moreover, is energetically equivalent to the analog for Laplace operator. The construction of discrete analogs for A will be oriented to the fulfillment of the same important properties.

3 Space discretization

In considering difference schemes for the solution of problem (1)-(5), we begin with making space approximation. We consider the problem on the rectangle

$$\Omega=\left\{x\mid x=(x_1,x_2)\,,\quad 0<x_\alpha<l_a,\ \alpha=1,2\right\}$$

discretized by a uniform rectangular grid mesh steps h_α, $\alpha=1,2$. Let ω be the set of internal nodes of the grid

$$\omega=\left\{x\mid x=(x_1,x_2)\,,\,x_\alpha=i_\alpha h_\alpha,\ i_\alpha=1,\ 2...,\ N_\alpha-1,\ N_\alpha h_\alpha=l_\alpha,\ \alpha=1,2\right\},$$

and the $\partial\omega$ the set of boundary nodes. The finite difference solution of problem (1)-(4) will be denoted by $v_h(x,t)$, $x\in\omega\cup\partial\omega$, $0<t\leq T$. Using the standard index-free notation of the theory of difference schemes [8], for the right and left difference derivatives we write

$$w_x=\frac{w\left(x+h\right)-w\left(x\right)}{h},\quad w_{\overline{x}}=\frac{w\left(x\right)-w\left(x-h\right)}{h},$$

and the second difference derivative is given by the expression

$$w_{\overline{x}x} = \frac{1}{h}\left(w_x - w_{\overline{x}}\right) = \frac{w\left(x+h\right) - 2w\left(x\right) + w\left(x-h\right)}{h^2}.$$

For grid functions equal to zero on $\partial\omega$ we define the Hillbert space $L_2\left(\omega\right)$ where the inner product and norm are as follows

$$(y, w) = \sum_{x\in\omega} y\left(x\right) w\left(x\right) h_1 h_2, \quad \|y\| = (y, y)^{\frac{1}{2}}.$$

For the vector grid functions $u\left(x\right)$, $v\left(x\right)$ equal to zero on $\partial\omega$ we introduce $\tilde{H} = \left(L_2\left(\omega\right)\right)^4$ with the inner product and norm given by

$$(u, v) = (u_1, v_1) + (u_2, v_2), \quad \|u\| = (u, u)^{\frac{1}{2}}.$$

Also, given a self-adjoint and positive definite operator C, $\tilde{H}_C$ denotes the space $\tilde{H}$ provided by the scalar product $(u, v)_C = (Cu, v)$ and norm $\|u\|_C = (Cu, u)^{\frac{1}{2}}$.

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} & \tilde{A}_{34} \\ \tilde{A}_{41} & \tilde{A}_{42} & \tilde{A}_{43} & \tilde{A}_{44} \end{pmatrix}, \tag{10}$$

$$\tilde{A}_{11} y = -a_1 y_{\overline{x_1}x_1} - b_1 \Delta_h y, \quad \tilde{A}_{12} y = \tilde{A}_{21} y = -\frac{b_1}{2}\left(y_{\overline{x_1}x_2} + y_{x_1\overline{x_2}}\right),$$

$$\tilde{A}_{22} y = -b_1 \Delta_h y - a_1 y_{\overline{x_2}x_2},$$

$$\tilde{A}_{13} y = \tilde{A}_{31} y = -c y_{\overline{x_1}x_1} - d\Delta_h y,$$

$$\tilde{A}_{14} y = \tilde{A}_{23} y = \tilde{A}_{32} y = \tilde{A}_{41} y = -\frac{d}{2}\left(y_{\overline{x_1}x_2} + y_{x_1\overline{x_2}}\right),$$

$$\tilde{A}_{24} y = \tilde{A}_{42} y = -d\Delta_h y - c y_{\overline{x_2}x_2},$$

$$\tilde{A}_{33} y = -a_2 y_{\overline{x_1}x_1} - b_2 \Delta_h y, \quad \tilde{A}_{34} y = \tilde{A}_{43} = -\frac{b_1}{2}\left(y_{\overline{x_1}x_2} + y_{x_1\overline{x_2}}\right),$$

$$\tilde{A}_{44} y = -b_1 \Delta_h y - a_1 y_{\overline{x_2}x_2}.$$

Here, we use the standard 5-point approximation of the Laplace operator

$$\Delta_h y = y_{\overline{x_1}x_1} + y_{x_2\overline{x_2}}.$$

For the grid functions $u(x)$ and $v(x)$ from $\tilde{H}$ we have

$$\left(\tilde{A}v, u\right) = \left(v, \tilde{A}u\right),$$

i.e., the operator $\tilde{A}$ is selfadjoint.

Besides, we have

$$-b\left(\tilde{\Delta}_h v, v\right) \leq \left(\tilde{A}v, v\right) \leq -\left(a+b\right)\left(\tilde{\Delta}_h v, v\right), \qquad (11)$$

where

$$\tilde{\Delta}_h = \begin{pmatrix} \Delta_h & 0 & 0 & 0 \\ 0 & \Delta_h & 0 & 0 \\ 0 & 0 & \Delta_h & 0 \\ 0 & 0 & 0 & \Delta_h \end{pmatrix}.$$

The relation (11) is a discrete analog of (7) given for the differential operator $\tilde{A}$. We approximate the differential operator A by the difference operator $\tilde{A}$, a self-adjoint and positive definite operator.

After approximation in space and denoting by $u\left(x,t\right)$, $x \in \omega \cup \partial\omega$, $0 < t \leq T$, the semi-discrete solution at time t, we have the initial value problem

$$\rho\frac{d^2u}{dt^2} + \alpha\frac{du}{dt} + \tilde{A}u = f\left(x,t\right), \quad x \in \omega, \quad 0 < t \leq T, \qquad (12)$$

$$u\left(0\right) = v_o\left(x\right), \quad \frac{du}{dt}\left(x,0\right) = v_1(x), \quad x \in \omega. \qquad (13)$$

4 Approximation in time

For simplicity, we consider a uniform grid in $[0,T]$, with step $\tau > 0$. Let $u_n\left(x\right) = u\left(x,t_n\right)$, $t_n = n\tau$, $n = 0,1,\ldots,N$, $N\tau = T$. The simplest second-order scheme for problem (12),(13) is

$$\rho\frac{u_{n+1} - 2u_n + u_{n-1}}{\tau^2} + \alpha\frac{u_{n+1} - u_{n-1}}{2\tau} + \tilde{A}u_n = f_n, \quad n = 1,2,\ldots,N, \qquad (14)$$

with prescribed $u_0,\ u_1$.

Let us highlight the class of additive schemes called alternating triangular methods. The schemes of this type for evolutionary equations of the first order have been proposed and investigated by A.A. Samarskii in [11]. Here we consider the possibilities of using this approach to construct additive schemes for system of second-order equations.

The alternating triangular method is constructed on the basis of the operator splitting:

$$\tilde{A} = \tilde{A}^{(1)} + \tilde{A}^{(2)}, \quad \left(\tilde{A}^{(1)}\right)^* = \tilde{A}^{(2)}, \qquad (15)$$

where, taking into account (10), we define

$$\tilde{A}^{(1)} = \begin{pmatrix} \frac{1}{2}\tilde{A}_{11} & 0 & 0 & 0 \\ \tilde{A}_{21} & \frac{1}{2}\tilde{A}_{22} & 0 & 0 \\ \tilde{A}_{31} & \tilde{A}_{32} & \frac{1}{2}\tilde{A}_{33} & 0 \\ \tilde{A}_{41} & \tilde{A}_{42} & \tilde{A}_{43} & \frac{1}{2}\tilde{A}_{44} \end{pmatrix},$$

$$\tilde{A}^{(1)} = \begin{pmatrix} \frac{1}{2}\tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & \frac{1}{2}\tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ 0 & 0 & \frac{1}{2}\tilde{A}_{33} & \tilde{A}_{34} \\ 0 & 0 & 0 & \frac{1}{2}\tilde{A}_{44} \end{pmatrix}. \quad (16)$$

Let us consider a simple predictor-corrector scheme for the numerical solution of problem (12), (13). At the predictor stage we calculate $\tilde{u}_{n+1}$ from

$$\rho\frac{u_{n+1} - 2u_n + u_{n-1}}{\tau^2} + \alpha\frac{u_{n+1} - u_{n-1}}{2\tau} + \tilde{A}^{(1)}\frac{\tilde{u}_{n+1} - u_{n-1}}{2} + \tilde{A}^{(2)}u_n = f_n. \quad (17)$$

After that, at the corrector stage, we improve the solution for the next time level:

$$\rho\frac{u_{n+1} - 2u_n + u_{n-1}}{\tau^2} + \alpha\frac{u_{n+1} - u_{n-1}}{2\tau} + \\ +\tilde{A}^{(1)}\frac{\tilde{u}_{n+1} - u_{n-1}}{2} + \tilde{A}^{(2)}\frac{u_{n+1} - u_{n-1}}{2} = f_n. \quad (18)$$

Schemes (17), (18) can be written as follows

$$\left(\rho E + \frac{\tau^2}{2}\tilde{A}^{(1)}\right)\frac{1}{\rho}\left(\rho E + \frac{\tau^2}{2}\tilde{A}^{(2)}\right)\frac{u_{n+1} - 2u_n + u_{n-1}}{\tau^2} + \\ +\alpha\frac{u_{n+1} - u_{n-1}}{2\tau} + \tilde{A}u_n = f_n, \quad n = 1, 2, \dots, N,$$

where E denotes the single operator.

The generalization of this scheme is the factorized scheme

$$D\frac{u_{n+1} - 2u_n + u_{n-1}}{\tau^2} + \alpha\frac{u_{n+1} - u_{n-1}}{2\tau} + \tilde{A}u_n = f_n, \quad n = 1, 2, \dots, N, \quad (19)$$

$$D = \left(\rho E + \sigma\tau^2\tilde{A}^{(1)}\right)\frac{1}{\rho}\left(\rho E + \sigma\tau^2\tilde{A}^{(2)}\right). \quad (20)$$

This schemes is second order in time, since $D = \rho E + \mathrm{O}\left(\tau^2\right)$. To advance to a next time-level, its implementation requires to solve four grid elliptic problems, one for each component of the solution.

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