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Natural classes and torsion free classes in categories of modules

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Abstract. The relation between natural classes and torsion free classes of modules is studied. The mapping ϕ: R-nat $\rightarrow \mathcal{P}$ between corresponding lattices is defined and some properties of ϕ are shown, in particular, the compatibility of ϕ with operations of unions in lattices.

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1 Preliminaries

The abstract class of modules $\mathcal{K} \subseteq R$-Mod, (i.e. the class closed under isomorphisms) is called *natural* (or *saturated*) if it is closed with respect to submodules, direct sums and essential extensions (or injective envelopes). This type of classes of modules was studied from diverse points of view in a series of works, for example in [1–4]. The purpose of this note is to elucidate the relation between the natural classes and torsions ($\equiv$ hereditary radicals) of R-Mod, in special, torsion free classes of R-modules. It is well known that every torsion r of R-Mod determines two classes of modules:

$$\mathcal{T}_r = \{{}_R M \mid r(M) = M\}, \quad \mathcal{F}_r = \{{}_R M \mid r(M) = 0\}.$$

The class of the form $\mathcal{T}_r$, where r is a torsion, is called *torsion class* and is characterized as a class closed under submodules, direct sums, homomorphic images and extensions. Dually, the class of the form $\mathcal{F}_r$, where r is a torsion, is called *torsion free class* and can be described as a class closed under submodules, direct products and essential extensions (or injective envelopes). We note that every torsion free class is closed also under extensions. These and other facts on torsions can be found in the books [5–8].

In such a way all results on torsions (and on radicals) can be expounded by classes of modules, using the classes of the form $\mathcal{T}_r$ and $\mathcal{F}_r$. The relation between these two types of classes can be expressed by the following *operators of Hom-orthogonality*:

$$\mathcal{K} \subseteq R\text{-Mod}, \qquad \mathcal{K}^{\uparrow} = \{{}_R X \mid Hom_R(X,Y) = 0 \quad \forall\, Y \in \mathcal{K}\},$$
$$\mathcal{K}^{\downarrow} = \{{}_R Y \mid Hom_R(X,Y) = 0 \quad \forall\, X \in \mathcal{K}\}.$$

For every torsion r of R-Mod the following relations are true:

$$\mathcal{T}_r = \mathcal{F}_r^{\uparrow}, \quad \mathcal{F}_r = \mathcal{T}_r^{\downarrow}.$$

In the following statement we give an account of elementary properties of the operators of Hom-orthogonality [6, 8].

Lemma 1.1. (1) *The operators* ($\uparrow$) *and* ($\downarrow$) *are anti-monotone, i.e. they converte the inclusions of classes: if* $\mathcal{K}_1 \subseteq \mathcal{K}_2$*, then*

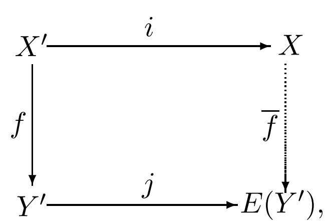
$$\mathcal{K}_1^{\uparrow} \supseteq \mathcal{K}_2^{\uparrow}, \quad \mathcal{K}_1^{\downarrow} \supseteq \mathcal{K}_2^{\downarrow}.$$

(2) *For every* $\mathcal{K} \subseteq R$*-Mod the class* $\mathcal{K}^{\uparrow}$ *is a* **radical class***, i.e. it is closed under homomorphic images, direct sums and extensions.*
(3) *For every* $\mathcal{K} \subseteq R$*-Mod the class* $\mathcal{K}^{\downarrow}$ *is a* **semisimple class***, i.e. it is closed under submodules, direct products and extensions.*
(4) *For every* $\mathcal{K} \subseteq R$*-Mod the class* $\mathcal{K}^{\downarrow\uparrow}$ *is the smallest radical class containing* $\mathcal{K}$*.*
(5) *For every* $\mathcal{K} \subseteq R$*-Mod the class* $\mathcal{K}^{\uparrow\downarrow}$ *is the smallest semisimple class containing* $\mathcal{K}$*.*

The abstract class $\mathcal{K} \subseteq R$-Mod is called *hereditary class* if it is closed under submodules, and $\mathcal{K}$ is called *stable class* if it is closed under essential extensions (if $\mathcal{K}$ is hereditary, then the last condition is equivalent to the closeness under injective envelopes). It is known that if $(\mathcal{T}, \mathcal{F})$ is a torsion theory in the sense of S.E. Dickson (i.e. $\mathcal{T} = \mathcal{F}^{\uparrow}$ and $\mathcal{F} = \mathcal{T}^{\downarrow}$), then the class $\mathcal{T}$ is hereditary if and only if $\mathcal{F}$ is stable. This statement is a corollary of the following facts [6, 8].

Lemma 1.2. (1) *If* $\mathcal{K}$ *is a hereditary and stable class, then* $\mathcal{K}^{\uparrow}$ *is hereditary.*
(2) *If* $\mathcal{K}$ *is a hereditary class, then* $\mathcal{K}^{\downarrow}$ *is a stable class.*

Proof. 1). Let $X \in \mathcal{K}^{\uparrow}$ and $X' \subseteq X$. If $X' \notin \mathcal{K}^{\uparrow}$, then there exists $0 \neq f : X' \to Y$, $Y \in \mathcal{K}$ and denoting $Y' = Im\, f \neq 0$, we have $Y' \in \mathcal{K}$. Now we consider the diagram:



where $E(Y')$ is the injective envelope of $Y' \in \mathcal{K}$ and i, j are inclusions. Then $E(Y') \in \mathcal{K}$ and there exists $\overline{f} : X \to E(Y')$ which extends f. From $f \neq 0$ it follows $\overline{f} \neq 0$, a contradiction with $X \in \mathcal{K}^{\uparrow}$.

2) Let $Y \in \mathcal{K}^{\downarrow}$ and $Y \subseteq^{*} Z$ (where $\subseteq^{*}$ is the essential inclusion). If $Hom_R(X, Z) \neq 0$ for some $X \in \mathcal{K}$, then there exists $0 \neq f : X \to Z$ with $0 \neq Im\, f \subseteq Z$. From $Y \subseteq^{*} Z$ it follows $Y \cap Im\, f \neq 0$. Denoting $X' = f^{-1}(Y \cap Im\, f)$, we have $X' \in \mathcal{K}$ and the restriction of f to X' is a non-zero homomorphism

$0 \neq f' : X' \to Y \cap Im\, f = Y'$, where $Y' \in \mathcal{K}^{\downarrow}$ (since $\mathcal{K}^{\downarrow}$ is hereditary), so $Hom_R(X', Y') \neq 0$, a contradiction. Therefore $Hom_R(X, Z) = 0$ for every $X \in \mathcal{K}$, i.e. $Z \in \mathcal{K}^{\downarrow}$. □

Corollary 1.3. (1) *If* $\mathcal{K}$ *is a natural class, then* $\mathcal{K}^{\uparrow}$ *is a radical and hereditary class, i.e. a torsion class.*
(2) *If* $\mathcal{K}$ *is a natural class, then the class* $\mathcal{K}^{\uparrow\downarrow}$ *is semisimple and stable, i.e. a torsion free class.*

Further we will use the following notations:
R-tors – the set (lattice) of all torsions of R-Mod;
R-nat – the set (lattice) of all natural clases of R-Mod;
$\boldsymbol{\mathcal{R}}$ – the set of all torsion classes of R-Mod;
$\boldsymbol{\mathcal{P}}$ – the set of all torsion free classes of R-Mod.

It is known that R-nat can be transformed in a lattice and this lattice is boolean [1, 3, etc]. Similarly, the sets $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{P}}$ are transformed in a natural way in lattices, where the order relation is the inclusion and the lattice operations "$\wedge$" and "$\vee$" are defined as follows:

$$\text{in } \boldsymbol{\mathcal{R}}: \quad \bigwedge_{\alpha \in \mathfrak{A}} \mathcal{R}_{\alpha} = \bigcap_{\alpha \in \mathfrak{A}} \mathcal{R}_{\alpha}, \quad \bigvee_{\alpha \in \mathfrak{A}} \mathcal{R}_{\alpha} = \cap \{\mathcal{R} \in \boldsymbol{\mathcal{R}} \,|\, \mathcal{R} \supseteq \mathcal{R}_{\alpha} \;\; \forall \alpha \in \mathfrak{A}\};$$

$$\text{in } \boldsymbol{\mathcal{P}}: \quad \bigwedge_{\alpha \in \mathfrak{A}} \mathcal{P}_{\alpha} = \bigcap_{\alpha \in \mathfrak{A}} \mathcal{P}_{\alpha}, \quad \bigvee_{\alpha \in \mathfrak{A}} \mathcal{P}_{\alpha} = \cap \{\mathcal{P} \in \boldsymbol{\mathcal{P}} \,|\, \mathcal{P} \supseteq \mathcal{P}_{\alpha} \;\; \forall \alpha \in \mathfrak{A}\}.$$

Since there exists a monotone bijection between torsions and torsion free classes, we have a lattice isomorphism $\boldsymbol{\mathcal{R}} \cong R$-tors. The anti-monotone bijection between $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{P}}$ is established by the operators of Hom-orthogonality ($\uparrow$) and ($\downarrow$), and these operators are compatible with lattice operations in the following sense.

Proposition 1.4. *For every sets* $\{\mathcal{R}_{\alpha} \,|\, \alpha \in \mathfrak{A}\} \subseteq \boldsymbol{\mathcal{R}}$ *and* $\{\mathcal{P}_{\alpha} \,|\, \alpha \in \mathfrak{A}\} \subseteq \boldsymbol{\mathcal{P}}$ *the following relations are true:*

$$\text{a)} \quad \Big(\bigwedge_{\alpha \in \mathfrak{A}} \mathcal{P}_{\alpha}\Big)^{\uparrow} = \bigvee_{\alpha \in \mathfrak{A}} (\mathcal{P}_{\alpha}^{\uparrow}); \qquad \text{b)} \quad \Big(\bigwedge_{\alpha \in \mathfrak{A}} \mathcal{R}_{\alpha}\Big)^{\downarrow} = \bigvee_{\alpha \in \mathfrak{A}} (\mathcal{R}_{\alpha}^{\downarrow});$$

$$\text{c)} \quad \Big(\bigvee_{\alpha \in \mathfrak{A}} \mathcal{P}_{\alpha}\Big)^{\uparrow} = \bigwedge_{\alpha \in \mathfrak{A}} (\mathcal{P}_{\alpha}^{\uparrow}); \qquad \text{d)} \quad \Big(\bigvee_{\alpha \in \mathfrak{A}} \mathcal{R}_{\alpha}\Big)^{\downarrow} = \bigwedge_{\alpha \in \mathfrak{A}} (\mathcal{R}_{\alpha}^{\downarrow}).$$

Therefore, the lattice $\boldsymbol{\mathcal{R}}$ is anti-isomorphic to the lattice $\boldsymbol{\mathcal{P}}$.

Remark. Some results on the lattice of natural classes are contained in [9] and [10]. In particular, the lattice R-nat is described as a skeleton (boolean part) of the frame of closed classes of R-Mod.

2 Natural classes and torsion free classes

From the definitions of §1 it is clear that every torsion free class (i.e. of the form $\mathcal{F}_r$, where r is a torsion) is natural, so we have the inclusion $i : \mathcal{P} \to R$-nat. Now we define an inverse mapping $\phi : R\text{-nat} \to \mathcal{P}$, considering that for every $\mathcal{K} \in R$-nat the class $\phi(\mathcal{K})$ is the smallest torsion free class containing $\mathcal{K}$ (i.e. the intersection of all torsion free classes of R-Mod which contain $\mathcal{K}$).

From the Lemma 1.1 and Corollary 1.3 it follows

Proposition 2.1. *For every natural class $\mathcal{K}$ the following relation is true:*

$$\phi(\mathcal{K}) = \mathcal{K}^{\uparrow\downarrow}.$$

The next specification of this relation follows from the fact that every class $\mathcal{K} \in R$-nat is hereditary (to compare with Prop. 2.5, chapter VI of [5]).

Proposition 2.2. *For every natural class $\mathcal{K}$ of R-Mod we have*:

$$\phi(\mathcal{K}) = \{{}_RY \,|\, \forall\ 0 \neq Y' \subseteq Y,\ \exists \text{epi}\ 0 \neq f : Y' \to Y'',\ Y'' \in \mathcal{K}\},$$

i.e. $\phi(\mathcal{K})$ consists of all modules Y such that for every non-zero submodule $Y' \subseteq Y$ there exists a non-zero epimorphism $f : Y' \to Y''$ with $Y'' \in \mathcal{K}$.

Proof. Denote by $\overline{\mathcal{K}}$ the class of right part of this relation.
$\phi(\mathcal{K}) \subseteq \overline{\mathcal{K}}$: From definitions we have:

$$\begin{aligned}
\mathcal{K}^{\uparrow\downarrow} &= \{{}_RY \,|\, Hom_R(X,Y) = 0\ \forall\ X \in \mathcal{K}^{\uparrow}\} = \\
&= \{{}_RY \,|\, Hom_R(X,Z) = 0\ \forall\ Z \in \mathcal{K} \Rightarrow Hom_R(X,Y) = 0\} = \\
&= \{{}_RY \,|\, Hom_R(X,Y) \neq 0 \Rightarrow \exists\, Z \in \mathcal{K},\ Hom_R(X,Z) \neq 0\}.
\end{aligned}$$

Let $Y \in \phi(\mathcal{K}) = \mathcal{K}^{\uparrow\downarrow}$ and $0 \neq Y' \subseteq Y$. Then $Hom_R(Y',Y) \neq 0$, therefore there exists $Z \in \mathcal{K}$ such that $Hom_R(Y',Z) \neq 0$. For $0 \neq f : Y' \to Z$ and $Y'' = Im\, f$, we obtain a non-zero epimorphism $\overline{f} : Y' \to Y'' \subseteq Z$, where $Y'' \in \mathcal{K}$ (since $\mathcal{K}$ is hereditary), therefore $Y \in \overline{\mathcal{K}}$.
$\phi(\mathcal{K}) \supseteq \overline{\mathcal{K}}$: Let $Y \in \overline{\mathcal{K}}$ and we will prove that $Hom_R(X,Y) = 0$ for every $X \in \mathcal{K}^{\uparrow}$. Suppose the contrary: there exists an $X \in \mathcal{K}^{\uparrow}$ such that $Hom_R(X,Y) \neq 0$. Then we have $0 \neq f : X \to Y$ and denote $0 \neq Y' = Im\, f \subseteq Y$. Since $Y \in \overline{\mathcal{K}}$, there exists a non-zero epimorphism $0 \neq g : Y' \to Y'',\ Y'' \in \mathcal{K}$. Therefore we have a non-zero epimorphism $0 \neq gf : X \to Y' \to Y'',\ Y'' \in \mathcal{K}$, in contradiction with $X \in \mathcal{K}^{\uparrow}$. □

Now we will show another description of the class $\phi(\mathcal{K})$ for $\mathcal{K} \in R$-nat, using the closeness properties. Comparing the respective definitions, it is clear that for the natural class $\mathcal{K}$ to be torsion free class it is necessary in addition to be closed under direct products. In continuation we will prove that to obtain the class $\phi(\mathcal{K})$ for $\mathcal{K} \in R$-nat it is sufficient to close the class $\mathcal{K}$ with respect to submodules and direct products. For that we consider the class of all modules of R-Mod *cogenerated* by the natural class $\mathcal{K}$:

$$Cog\,(\mathcal{K}) = \{{}_RM \mid \exists \text{ mono } 0 \to M \to \prod_{\alpha\in\mathfrak{A}} M_\alpha,\ M_\alpha \in \mathcal{K}\},$$

i.e. $Cog\,(\mathcal{K})$ is the smallest class of R-Mod, which contains $\mathcal{K}$ and is closed under submodules and direct products.

Proposition 2.3. *For every class $\mathcal{K} \in R$-nat the following relation is true*:

$$\phi(\mathcal{K}) = Cog\,(\mathcal{K}).$$

Proof. Firstly we verify that the class $Cog\,(\mathcal{K})$ is torsion free. From definition it follows that the class $Cog\,(\mathcal{K})$ is closed under submodules and direct products, so it remains to prove that $Cog\,(\mathcal{K})$ is a stable class.

Let $M \in Cog\,(\mathcal{K})$ and $E(M)$ be the injective envelope of M. Then there exists a monomorphism $0 \to M \xrightarrow{\phi} \prod_{\alpha\in\mathfrak{A}} M_\alpha,\ M_\alpha \in \mathcal{K}$. Since $\prod_{\alpha\in\mathfrak{A}} E(M_\alpha)$ is an injective module, the inclusion $\prod_{\alpha\in\mathfrak{A}} M_\alpha \subseteq \prod_{\alpha\in\mathfrak{A}} E(M_\alpha)$ can be extended to a monomorphism $\psi : E\Big(\prod_{\alpha\in\mathfrak{A}} M_\alpha\Big) \to \prod_{\alpha\in\mathfrak{A}} E(M_\alpha)$. Now we consider the diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & M & \xrightarrow{\quad\varphi\quad} & \prod_{\alpha\in\mathfrak{A}} M_\alpha & & \\
& {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle i} & & \\
& E(M) & \xrightarrow{\quad\overline{\varphi}\quad} & E(\prod_{\alpha\in\mathfrak{A}} M_\alpha) & \xrightarrow{\quad\psi\quad} & \prod_{\alpha\in\mathfrak{A}} E(M_\alpha),
\end{array}$$

where i, j are inclusions. By injectivity of $E\Big(\prod_{\alpha\in\mathfrak{A}} M_\alpha\Big)$ the monomorphism $i\varphi$ can be extended to a $\overline{\varphi} : E(M) \to E\Big(\prod_{\alpha\in\mathfrak{A}} M_\alpha\Big)$ and, since $M \subseteq^* E(M)$, $\overline{\varphi}$ is a monomorphism. So we obtain a monomorphism $\psi\overline{\varphi} : E(M) \to \prod_{\alpha\in\mathfrak{A}} E(M_\alpha)$, where $E(M_\alpha) \in \mathcal{K}$ for every $\alpha \in \mathfrak{A}$, since $\mathcal{K}$ is a stable class. Therefore $E(M) \in Cog\,(\mathcal{K})$ and so the class $Cog\,(\mathcal{K})$ is stable.

Taking into account that $\mathcal{K} \subseteq Cog\,(\mathcal{K})$, from the preceding result it follows the inclusion $\mathcal{K}^{\uparrow\downarrow} \subseteq Cog\,(\mathcal{K})$, since $\mathcal{K}^{\uparrow\downarrow}$ is the smallest torsion free class containing $\mathcal{K}$ (Corollary 1.3).

It remains to prove that $Cog\,(\mathcal{K}) \subseteq \mathcal{K}^{\uparrow\downarrow}$. Let $M \in Cog\,(\mathcal{K})$, i.e. we have a monomorphism $0 \to M \xrightarrow{\varphi} \prod_{\alpha\in\mathfrak{A}} M_\alpha$, $M_\alpha \in \mathcal{K}$ for every $\alpha \in \mathfrak{A}$. We will verify that $Hom_R(X, M) = 0$ for every $X \in \mathcal{K}^{\uparrow}$.

Suppose the contrary: there exists $X \in \mathcal{K}^{\uparrow}$ such that $Hom_R(X, M) \neq 0$. Then we have $0 \neq f : X \to M$ and since φ is mono, there exists $\beta \in \mathfrak{A}$ such that $p_\beta \varphi f \neq 0$:

$$X \xrightarrow{f} M \xrightarrow{\varphi} \prod_{\alpha\in\mathfrak{A}} M_\alpha \xrightarrow{p_\beta} M_\beta,$$

where p_β is the canonical projection. Therefore, $Hom_R(X, M_\beta) \neq 0$, where $M_\beta \in \mathcal{K}$ and $X \in \mathcal{K}^{\uparrow}$, a contradiction. $\square$

The studied mapping $\phi : R$-nat $\to \boldsymbol{\mathcal{P}}$ can be extended to a mapping $\psi : R$-nat $\to \boldsymbol{\mathcal{R}}$, where $\boldsymbol{\mathcal{R}}$ is the lattice of torsion classes of R-Mod, taking by definition:

$$\psi(\mathcal{K}) = [\phi(\mathcal{K})]^{\uparrow}$$

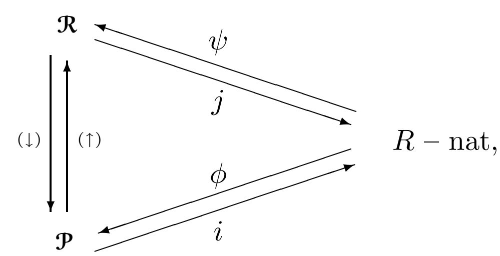
(since $\phi(\mathcal{K})$ is stable, $[\phi(\mathcal{K})]^{\uparrow}$ is a torsion class by Corollary 1.3). By Prop. 2.1 $\phi(\mathcal{K}) = \mathcal{K}^{\uparrow\downarrow}$, so we have:

$$\psi(\mathcal{K}) = (\mathcal{K}^{\uparrow\downarrow})^{\uparrow} = \mathcal{K}^{\uparrow}.$$

Moreover, we can define the mapping $j : \boldsymbol{\mathcal{R}} \to R$-nat by the rule:

$$j\,(\mathcal{R}) = \mathcal{R}^{\downarrow},\ \mathcal{R} \in \boldsymbol{\mathcal{R}}.$$

So we obtain the diagram:



where i is the inclusion. By definitions it is clear that ϕ is a monotone mapping, while ψ and j are anti-monotone. The following relations (commutativity of the diagram) are obvious:

$$j \cdot \psi = i \cdot \phi, \quad i = j \cdot (\uparrow), \quad \psi \cdot i = (\uparrow), \quad \psi \cdot j = (\downarrow).$$

As we have seen above, the operators $(\uparrow)$ and $(\downarrow)$ are compatible with lattice operations of $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{P}}$ (Prop. 1.4), i.e. these mappings convert the lattice operations. Now we will study the similar question for the mappings ϕ and ψ. We begin with the following remark.

Lemma 2.4. *The mapping ϕ preserves the lattice operations if and only if the mapping ψ converts these operations.*

Proof. Let, for example, ϕ preserves the unions:

$$\phi\Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big) = \bigvee_{\alpha\in\mathfrak{A}} \phi(\mathcal{K}_\alpha).$$

Then applying Prop. 1.4 we obtain:

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)^{\uparrow\downarrow} = \bigvee_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow\downarrow}) \Leftrightarrow \Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)^{\uparrow} = \Big(\big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\big)^{\uparrow\downarrow}\Big)^{\uparrow} =$$

$$= \Big(\bigvee_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow\downarrow})\Big)^{\uparrow} = \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow\downarrow\uparrow}) = \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow}),$$

i.e. ϕ preserves the unions if and only if ψ transforms the unions of classes in intersections. □

From this statement it follows that it is sufficient to prove the respective relations only for one of the mappings ϕ or ψ. Now we will show that the mapping ψ converts the unions of the lattice R-nat in the intersections of the lattice $\boldsymbol{\mathfrak{R}}$. For that we would remind that the unions of classes of R-nat can be characterized as follows:

$$\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha = \{{}_RM \mid \exists \bigoplus_{\alpha\in\mathfrak{A}} M_\alpha \subseteq^* M, \quad M_\alpha \in \mathcal{K}_\alpha\} \text{ (see [1, Theor. 2.15])},$$

where $\mathcal{K}_\alpha \in R$-nat for every $\alpha \in \mathfrak{A}$ and $\subseteq^*$ is the essential inclusion.

Theorem 2.5. *For every set of natural classes* $\{\mathcal{K}_\alpha \mid \alpha \in \mathfrak{A}\}$ *the following relation is true*:

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)^{\uparrow} = \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow}),$$

i.e. ψ *converts the unions of* R*-nat in the intersections of* $\boldsymbol{\mathfrak{R}}$.

Proof. ($\subseteq$). From $\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha \supseteq \mathcal{K}_\alpha$ for every $\alpha \in \mathfrak{A}$ it follows $\Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)^{\uparrow} \subseteq \mathcal{K}_\alpha^{\uparrow}$ for every $\alpha \in \mathfrak{A}$, therefore $\Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)^{\uparrow} \subseteq \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow})$.

($\supseteq$). Let $X \in \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow})$. We must prove that $Hom_R(X, M) = 0$ for every $M \in \bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha$. Suppose the contrary: there exists $M \in \bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha$ such that $Hom_R(X, M) \neq 0$. From the description of the class $\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha$ indicated above, it follows that there exists a direct sum $\bigoplus_{\alpha\in\mathfrak{a}} M_\alpha \subseteq^* M$ with $M_\alpha \in \mathcal{K}_\alpha$ for every $\alpha \in \mathfrak{A}$. Then we have a non-zero homomorphism $0 \neq f : X \to M$, $0 \neq Im\, f \subseteq M$ and the essential inclusion $\bigoplus_{\alpha\in\mathfrak{A}} M_\alpha \subseteq^* M$ implies $\big(\bigoplus_{\alpha\in\mathfrak{A}} M_\alpha\big) \cap Im\, f \neq 0$. Therefore, there exists an element $0 \neq m_{\alpha_1} + \cdots + m_{\alpha_k} \in Im\, f$ with $m_{\alpha_i} \in M_{\alpha_i}$. Then $0 \neq Rm_{\alpha_1} + \cdots + Rm_{\alpha_k} \subseteq Im\, f$ and it is obvious that there exists $\alpha_i \in \mathfrak{A}$ such that $0 \neq Rm_{\alpha_i} \subseteq Im\, f$. So we obtain a non-zero homomorphism from $f^{-1}(Rm_{\alpha_i})$ in M_{α_i}:

$$X \supseteq f^{-1}(Rm_{\alpha_i}) \xrightarrow{f} \bigoplus_{\alpha\in\mathfrak{A}} M_\alpha \xrightarrow{p_{\alpha_i}} M_{\alpha_i},\ M_{\alpha_i} \in \mathcal{K}_{\alpha_i}.$$

On the other hand, since $X \in \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow})$, we obtain $X \in \mathcal{K}_{\alpha_i}^{\uparrow}$, where $\mathcal{K}_\alpha^{\uparrow}$ is a hereditary class, so $f^{-1}(Rm_{\alpha_i}) \in \mathcal{K}_{\alpha_i}^{\uparrow}$. This means that $f^{-1}(Rm_{\alpha_i})$ has no non-zero homomorphism in the modules of $\mathcal{K}_{\alpha_i}$, a contradiction. □

From Theorem 2.5 and Lemma 2.4 immediatly follows

Corollary 2.6. *The mapping ϕ preserves the unions, i.e.*

$$\Big(\bigvee_{\alpha\in\mathfrak{K}} \mathcal{K}_\alpha\Big)^{\uparrow\downarrow} = \bigvee_{\alpha\in\mathfrak{K}} (\mathcal{K}_\alpha^{\uparrow\downarrow})$$

for every set $\{\mathcal{K}_\alpha \,|\, \alpha \in \mathfrak{A}\}$ of natural classes.

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