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On commutative Moufang loops with some restrictions for subgroups of its multiplication groups

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Abstract. Let $\mathfrak{M}$ be the multiplication group of a commutative Moufang loop Q. In this paper it is proved that if all infinite abelian subgroups of $\mathfrak{M}$ are normal in $\mathfrak{M}$, then Q is associative. If all infinite nonabelian subgroups of $\mathfrak{M}$ are normal in $\mathfrak{M}$, then all nonassociative subloops of Q are normal in Q, all nonabelian subgroups of $\mathfrak{M}$ are normal in $\mathfrak{M}$ and the commutator subgroup $\mathfrak{M}'$ is a finite 3-group.

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While considering different classes of algebras (groups, rings, loops) it is of crucial importance to analyze the existence of their subalgebras with certain predefined features. To this end, it is advisable to consider the construction of algebras under condition that they don't have any subalgebras with predefined features. Before providing the findings that we'll need further, we will remind some notions from group's theory and commutative Moufang loops (abbreviated CMLs), found in [1] and [2] respectively.

A finite nonabelian group is called a *Miller-Moreno's group* if all its proper subgroups are abelian. An *IP-group* (respect. $\overline{IH}$*-group*) is an infinite group if all its proper infinite abelian (respect. nonabelian) groups are normal within. But if all its nonabelian subgroups are normal, then such a group is called a *metagamiltonian group.*

In [1] the construction of IH-groups is described, elements of infinite order and periodic IH-group, which does not satisfy the minimum condition for abelian subgroups. It also describes the solvable $\overline{IH}$-groups with finite or infinite commutator subgroup and the (solvable) metagamiltonian or the non-metagamiltonian $\overline{IH}$-groups are characterized.

A *commutative Moufang loop* (abbreviated CMLs) is characterized by the identity $x^2 \cdot yz = xy \cdot xz$.

The *multiplicative group* $\mathfrak{M}(Q)$ of the CML Q is the group generated by all the *translations* $L(x)$, where $L(x)y = xy$. The subgroup $I(Q)$ of the group $\mathfrak{M}(Q)$, generated by all the *inner mappings* $L(x,y) = L^{-1}(xy)L(x)L(y)$ is called the *inner mapping group* of the CLM Q. The inner mappings are automorphisms in the CML. A subloop H of the CML Q is called *normal* in Q if $x \cdot yH = xy\cdot$ for all $x, y \in Q$. The subloop H is normal in Q if $\mathfrak{I}(Q)H = H$.

The *associator* (a, b, c) of the elements a, b, c of the CML Q is defined by the equality $ab \cdot c = (a \cdot bc)(a, b, c)$. The *associator subloop* Q' of CML Q is generated by all associators (x, y, z), $x, y, z \in Q$. The *centre* $Z(Q)$ of the CML Q is a normal subloop $Z(Q) = \{x \in Q | (x, y, z) = 1 \forall y, z \in Q\}$.

Let Q be an arbitrary CML and let H be a subset of the set Q. Let $\mathbf{M}(H)$ denote a subgroup of the multiplicative group $\mathfrak{M}(Q)$ of the CML Q, generated by the set $\{L(x) | \forall x \in H\}$. Takes place

Lemma 1 [3]. *Let the commutative Moufang loop Q with the multiplicative group $\mathfrak{M}$, $Z(\mathfrak{M})$, which is the centre of the group $\mathfrak{M}$ and the centre $Z(Q)$ decompose into a direct product $Q = D \times H$, moreover, $D \subseteq Z(Q)$. Then $\mathfrak{M} = \boldsymbol{M}(D) \times \boldsymbol{M}(H)$, besides, $\boldsymbol{M}(D) \subseteq Z(\mathfrak{M}), \boldsymbol{M}(D) \cong D$.*

Further, in papers [3–5] the CML is characterized with the help of various systems of subloops of CML, and also various systems of multiplication groups of CML. In particular, is proved

Lemma 2. *The following statements are equivalent for an arbitrary non-associative CML Q with the multiplication group $\mathfrak{M}$:*

1) Q satisfies the minimum condition for subloops;

2) $\mathfrak{M}$ satisfies the minimum condition for subgroups;

3) Q is a direct product of a finite number of quasicyclic groups, lying in the centre of Q, and a finite loop;

4) $\mathfrak{M}$ is a direct product of a finite number of quasicyclic groups, lying in the centre of $\mathfrak{M}$, and a finite group;

5) Q satisfies the minimum condition for non-normal subloops;

6) $\mathfrak{M}$ satisfies the minimum condition for non-normal subgroups;

7) $\mathfrak{M}$ satisfies the minimum condition for nonabelian subgroups;

8) $\mathfrak{M}$ satisfies the minimum condition for abelian subgroups.

The following are a natural reducing of statements 5) and 6):

i) all infinite associative subloops of Q are normal in Q;

ii) all infinite nonassociative subloops of Q are normal in Q;

iii) $\mathfrak{M}$ is an IH-group;

iv) $\mathfrak{M}$ is an $\overline{IH}$-group;

The structure of CML with conditions i), ii) is examined in [5]. It is proved that the CML with condition i) is associative, the CML with condition ii) has a finite associator subloop and in such a CML any nonassociative (finite or infinite) subloops are normal. In this paper it is proved that the CML with condition iii) is associative. It is proved also that the CML with condition iv) satisfies the condition ii), its multiplication group $\mathfrak{M}$ is metagamiltonian and it commutators subgroup $\mathfrak{M}'$ is a finite 3-group.

Lemma 3. *If the element a of an infinite order or of order three of the multiplication group $\mathfrak{M}$ of arbitrary CML generates a normal subgroup, then it belongs to the centre $Z(\mathfrak{M})$ of the group $\mathfrak{M}$.*

Proof. We denote $(x,y) = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$. Then $x^y = x(x,y)$. If the element $1 \neq a \in \mathfrak{M}$ generates a normal subgroup, then $a^b = a^k$ for a certain natural number k and for arbitrary fixed element $b \in \mathfrak{M}$. We have $a(a,b) = a^k$, $(a,b) = a^{k-1}$. If $k = 1$, then $(a,b) = 1$. Hence $a \in Z(\mathfrak{M})$. Let us now suppose that $k > 1$. Let $a^3 = 1$. Then $k = 2$ and $a = (a,b)$. The multiplication group $\mathfrak{M}$ is locally nilpotent [3]. Then $a = (a,b) = ((a,b),b) = (((a,b),b),b) = \ldots = 1$. We have obtained a contradiction, as $a \neq 1$. By [2, Theorem 11.4] the commutator subgroup of the group $\mathfrak{M}$ is locally finite 3-group. If a has an infinite order, then for a certain natural number n $(a^{k-1})^{3^n} = (a,b)^{3^n} = 1$, $a = 1$. We have obtained a contradiction again. Therefore the case of $k > 1$ is impossible. This completes the proof of Lemma 3.

Theorem 1. *If the multiplication group $\mathfrak{M}$ of CML Q is a IH-group, then $\mathfrak{M}$ is abelian and, consequently, the CML Q is associative.*

Proof. We suppose that the group $\mathfrak{M}$ is nonabelian. In this cases $\mathfrak{M}$ must be periodic. We suppose the contrary, that the group $\mathfrak{M}$, then the CML Q, is not periodic as well. Let a be an element of infinite order in Q. By [2] the element a^3 belongs to the centre $Z(Q)$ of CML Q. Then it is easy to show, considering the definition of group $\mathfrak{M}$, that the element $\alpha = L(a^3)$ belongs to the centre $Z(\mathfrak{M})$ of group $\mathfrak{M}$. Hence, the group $A =< \alpha >$ is an infinite abelian normal subgroup of group $\mathfrak{M}$. Let β be an arbitrary periodic element of group $\mathfrak{M}$ and let $B =< \beta >$. As $A \subseteq Z(\mathfrak{M})$, then it is easy to show that the product AB will be an infinite abelian subgroup of the group $\mathfrak{M}$. By supposition the subgroup AB is normal in $\mathfrak{M}$, hence if φ is an inner automorphism of group $\mathfrak{M}$, then $AB = \varphi(AB) = \varphi(A) \cdot \varphi(B) = A \cdot \varphi(B)$. Consequently, $AB = A \cdot \varphi(B)$. Let β_1 be an arbitrary element in B. Then there exists such elements $\alpha_1 \in A$, $\beta_2 \in B$ that $\varphi(\beta_1) = \alpha_1\beta_2$ or $\varphi(\beta_1)\beta_2^{-1} = \alpha_1$. As β_1 is a periodic element then $\varphi(\beta_1)$ also is a periodic element and α_1 as an element of infinite cyclic group is not periodic. Further, as $\varphi(\beta_1), \beta_2 \in \mathfrak{M}$, then let $\varphi(\beta_1) = L(u_1)\ldots L(u_k)$, $\beta_2^{-1} = L(v_1)\ldots L(v_n)$, where $u_i, v_j \in Q$. We denote by H the subloop of CML Q generated by the set $\{a, u_1, \ldots, u_k, v_1, \ldots, v_n\}$. The CML H is finitely generated, then by Bruck-Slaby's Theorem [2, Theorem 10.1] it is centrally nilpotent. Again by [2, Theorem 11.5] its multiplication group $\mathfrak{M}(H)$ is nilpotent. Further, we denote by $\varphi(\overline{\beta}_1), \overline{\beta}_2^{-1}, \overline{\alpha}_1$ the restriction on H of mappings $\varphi(\beta_1), \beta_2^{-1}, \alpha_1$ of set Q. It is obvious that $\varphi(\overline{\beta}_1), \overline{\beta}_2^{-1}, \overline{\alpha}_1 \in \mathfrak{M}(H)$, that $\varphi(\overline{\beta}_1), \overline{\beta}_2^{-1}$ are periodic elements and $\overline{\alpha}_1$ is an element of infinite order. The periodic elements form a subgroup in nilpotent groups, hence the product $\varphi(\overline{\beta}_1) \cdot \overline{\beta}_2^{-1}$ is a periodic element. From the equality $\varphi(\beta_1)\beta_2^{-1} = \alpha_1$ follows the equality $\varphi(\overline{\beta}_1) \cdot \overline{\beta}_2^{-1} = \overline{\alpha}_1$ and further $\overline{\alpha}_1 = 1$, $\varphi(\overline{\beta_1}) = \overline{\beta}_2$, $\varphi(B) = B$. We get that any element in $\mathfrak{M}$ generates a normal subgroup. Hence any subgroup from $\mathfrak{M}$ is normal in $\mathfrak{M}$. Then $\mathfrak{M}$ is a hamiltonian group.

Indeed, arbitrary hamiltonian groups are described by the next theorem. A *hamiltonian group* can be decomposed into a direct product of the group of quaternions and abelian groups, whose each element's order is not higher than 2. Conversely, a group that has such a decomposition is hamiltonian. A *group of quaternions* is the group generated by the generators a, b and that satisfies the identical relations $\alpha^4 = 1, \alpha^2 = \beta^2, \beta^{-1}\alpha\beta = \alpha^{-1}$. In [2, Theorem 11.4] it is proved that the quotient group $\mathfrak{M}/Z(\mathfrak{M})$ is a locally finite 3-group. Then from $\alpha^4 = 1$ it follows $\alpha \in Z(\mathfrak{M})$, from $\beta^{-1}\alpha\beta = \alpha^{-1}$ it follows $\alpha^2 = 1$, from $\alpha^2 = \beta^2$ it follows $\beta^2 = 1$ and, finally, from $\beta^2 = 1$ it follows $\beta \in Z(\mathfrak{M})$. We get that the *hamiltonian subgroup of multiplication group of CML is abelian.*

It follows from the aforementioned that the multiplication group $\mathfrak{M}$ of CML Q is abelian. But this contradicts our supposion about the nonabelian group $\mathfrak{M}$. Consequently, the group $\mathfrak{M}$ is periodic.

From Lemmas 1.4 and 3.1 of [3] it follows that the periodic multiplication group $\mathfrak{M}$ of CML Q decomposes into a direct product of its maximal p-subgroups $\mathfrak{M}_p$, in addition $\mathfrak{M}_p$ belongs to the centre $Z(\mathfrak{M})$ for $p \neq 3$. We denote $\mathfrak{M} = \mathfrak{N} \times \mathfrak{M}_3$, where $\mathfrak{N} = \prod_{p\neq 3} \mathfrak{M}_p$. We suppose that $\mathfrak{N}$ is an infinite group and let α be an arbitrary element in $\mathfrak{M}_3$. If $\mathfrak{A} =< \alpha >$ then by supposition the infinite abelian group $\mathfrak{N} \times \mathfrak{A}$ is normal in $\mathfrak{M}$. Let φ be an inner automorphism of $\mathfrak{M}$. Then $\mathfrak{N} \times \mathfrak{A} = \varphi(\mathfrak{N} \times \mathfrak{A}) = \varphi\mathfrak{N} \times \varphi\mathfrak{A} = \mathfrak{N} \times \varphi\mathfrak{A}$, i.e. $\mathfrak{N} \times \mathfrak{A} = \mathfrak{N} \times \varphi\mathfrak{A}$. Then for a certain $\alpha_1 \in \mathfrak{A}$ there exist such elements $\alpha_2 \in \mathfrak{A}$, $\beta \in \mathfrak{N}$ that $\beta\alpha_2 = \varphi\alpha_1$. Further, $\beta^3\alpha_2^3 = \varphi\alpha_1^3$, $\beta^{3^k}\alpha_2^{3^k} = \varphi\alpha_1^{3^k}$ and for a certain integer n $\alpha_2^{3^n} = \varphi\alpha_1^{3^n} = 1$. Hence $\beta^{3^k} = 1$. But the order of β doesn't divide 3. Hence $\beta = 1$ and we get $\varphi\alpha_1 = \alpha_2$, $\varphi A = A$. From here it follows that the subgroup $\mathfrak{M}_3$ is hamiltonian. Hence, as in the above case, $\mathfrak{M}_3$, then also $\mathfrak{M}$, are abelian groups. Consequently, in the decomposition $\mathfrak{M} = \mathfrak{N} \times \mathfrak{M}_3$ we should consider the case when the subgroup $\mathfrak{N}$ is finite. Further, without breaking the generality, we will consider that $\mathfrak{M}$ is a 3-group.

We suppose that the CML $\mathfrak{M}$ doesn't satisfy the minimum condition for subgroups. Then by 8) of Lemma 2 $\mathfrak{M}$ contains an abelian subgroup that doesn't satisfy the minimum condition for subgroups. Then it contains an infinite elementary abelian group $\mathfrak{H}$. Let $\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2 \times \ldots \times \mathfrak{H}_n \times \ldots$ be a decomposition of $\mathfrak{H}$ into a direct product of cyclic groups of order 3. For any element $\alpha \in \mathfrak{H}$ there will be such an infinite subgroup $\mathfrak{H}(\alpha) \subseteq \mathfrak{H}$ that $< \alpha > \cap \mathfrak{H}(x) = 1$. Let $\mathfrak{H}(\alpha) = \mathfrak{H}^1(\alpha) \times \mathfrak{H}^2(\alpha)$ be a certain decomposition of group $\mathfrak{H}(\alpha)$ in direct product of two infinite factors. As the cyclic group $< \alpha >$ is, obvious, the intersection of two infinite associative subgroups $< \alpha > \mathfrak{H}^1(\alpha)$ and $< \alpha > \mathfrak{H}^2(\alpha)$, then it is normal in $\mathfrak{M}$. Given the arbitrary element $\alpha \in \mathfrak{H}$ it follows that all factors $\mathfrak{H}_n$ are normal in $\mathfrak{M}$. Every factor $\mathfrak{H}_n$ is a cyclic group of order 3 and by Lemma 3 belongs to the centre $Z(\mathfrak{M})$. Hence $\mathfrak{H} \subseteq Z(\mathfrak{M})$. Let now β be an arbitrary element from $\mathfrak{M}$, let $\mathfrak{H}(\beta)$ be an infinite subgroup of $\mathfrak{H}$ such that $< \beta > \cap \mathfrak{H}(\beta) = 1$, and let $\mathfrak{H}(\beta) = \mathfrak{H}^1(\beta) \times \mathfrak{H}^2(\beta)$ be a certain decomposition of group $\mathfrak{H}(\beta)$ into a direct product of two infinite factors. As $\mathfrak{H}(\beta) \subseteq Z(\mathfrak{M})$, then $\mathfrak{H}^1(\beta), \mathfrak{H}^2(\beta)$ are normal in $\mathfrak{M}$ and the products $\beta\mathfrak{H}^1(\beta), \beta\mathfrak{H}^2(\beta)$ are infinite abelian subgroups. Then $\beta\mathfrak{H}^1(\beta), \beta\mathfrak{H}^2(\beta)$ are normal subgroups, hence also $< \beta >=< \beta > \mathfrak{H}^1(y) \cap \beta\mathfrak{H}^2(\beta)$ is also normal subgroup. We get that any element in $\mathfrak{M}$ generates a normal subgroup in $\mathfrak{M}$. Consequently, $\mathfrak{M}$ is a hamiltonian group and, as proved above, it is abelian. This contradicts our assumption about nonabelian group $\mathfrak{M}$. Hence $\mathfrak{M}$ satisfies the minimum condition for subgroups.

From minimum condition for subgroups for $\mathfrak{M}$ it follows by Lemma 2 that $\mathfrak{M} = \mathfrak{B} \times \mathfrak{C}$, where $\mathfrak{C} \subseteq Z(\mathfrak{M})$ and $\mathfrak{B}$ is a finite group. If γ is an arbitrary element in $\mathfrak{M}$ then $< \gamma > \mathfrak{C}$ is an infinite abelian subgroup. Further, from the normality of $< \gamma > \mathfrak{C}$ in $\mathfrak{M}$ follows the normality of $< \gamma >$ in $\mathfrak{M}$. Hence $\mathfrak{M}$ is a hamiltonian group. According to the above proofs $\mathfrak{M}$ is an abelian group. This completes the proof of Theorem 1.

Let us now consider a CML with certain restriction on nonabelian subgroups of it multiplication group. We suppose that the multiplication group $\mathfrak{M}$ of the CML Q doesn't have proper infinite nonabelian subgroups. Then by Lemma 2 $\mathfrak{M}$ satisfies the minimum condition for subgroups and $\mathfrak{M} = \mathfrak{K} \times \mathfrak{G}$, where $\mathfrak{K}$ is a direct product of a finite number of quasicyclic groups, lying in the centre $Z(\mathfrak{M})$ of the group $\mathfrak{M}$, $\mathfrak{G}$ is a finite group. But as $\mathfrak{M}$ doesn't have proper infinite nonabelian subgroups then $\mathfrak{K}$ is a quasicyclic group, $\mathfrak{G}$ is a Miller-Moreno group.

By Lemma 2 the CML Q satisfies the minimum condition for sublooops and $Q = D \times H$, where D is a direct product of a finite number of quasicyclic groups, lying in the centre $Z(Q)$ of the CML Q, H is a finite loop. Further, by Lemma 1 $\mathfrak{M} = M(D) \times M(H)$ and $M(D) \cong D$. Consequently, $M(H) \cong \mathfrak{G}$. Further, if for certain $a, b, c \in H$ $ab \cdot c \neq a \cdot bc$ then $L(c)L(a)b \neq L(a)L(c)b$, $L(c)L(a) \neq L(a)L(c)$. Hence if the CML H contains proper nonassiciative subloops, then the group $M(H)$ contains proper nonabelian subgroups. A CML is diassociative [2]. Then from the relation $M(H) \cong \mathfrak{G}$ it follows that the CML H is generated by three elements. Consequently, we proved.

Proposition 1. *A multiplication group $\mathfrak{M}$ of infinite nonassociative CML Q does not contain proper infinite nonabelian subgroups if and only if $Q = D \times H$, where D is a quasicyclic group, H is a nonassociative 3-generate loop or $\mathfrak{M} = D \times \mathfrak{G}$, where $\mathfrak{G}$ is a Miller-Moreno group.*

Theorem 2. *If the multiplication group $\mathfrak{M}$ of the CML Q is an $\overline{IH}$-group, then:*

1) $\mathfrak{M}$ is a metagamiltonian group;

2) all nonassociative sibloops of CML Q are normal in it;

3) if $\mathfrak{M}$ is non-periodic, then the commutator subgroup $\mathfrak{M}'$ of group $\mathfrak{M}$ is a finite abelian 3-group;

4) if $\mathfrak{M}$ is periodic, then the commutator subgroup $\mathfrak{M}'$ of group $\mathfrak{M}$ is solvable of a class not greater tha three finite 3-group;

Proof. By [3] the multiplication group $\mathfrak{M}$ of an arbitrary CML is locally nilpotent. Then by [1, Theorem 1.18] $\mathfrak{M}$ posed a centrally system with cyclic factors of simple orders. In this cases, if $\mathfrak{M}$ is an $\overline{IH}$-group, then by [1, Proposition 6.5] $\mathfrak{M}$ is solvable. Corollary 6.11 from [1] stipulated that a non-metagamiltonian solvable $\overline{IH}$-group satisfies the minimum condition for subgroups. Then from Lemma 2 it follows that the multiplication $\overline{IH}$-group $\mathfrak{M}$ is metagamiltonian. Consequently, the item 1) is proved.

Now let H be an arbitrary nonassociative subloop of the CML Q and let its multiplication group $\mathfrak{M}$ be an $\overline{IH}$-group. Then the subgroup $\mathfrak{N}$, generated by mappings $L(a)$, $a \in Q$, is nonabelian and by item 1) is normal in $\mathfrak{M}$. The set $\mathfrak{N}a$, $a \in Q$, partitions Q and

$$\mathfrak{N}a \cdot \mathfrak{N}b = L(\mathfrak{N}b)\mathfrak{N}a = \mathfrak{N}L(\mathfrak{N}b)a = \mathfrak{N}(\mathfrak{N}b \cdot a) =$$

$$= \mathfrak{N}(L(a)\mathfrak{N}b) = \mathfrak{N}(\mathfrak{N}L(a)b) = \mathfrak{N}L(a)b = \mathfrak{N}(ab).$$

If $\mathfrak{N}(ba) = \mathfrak{N}(ca)$, then

$$\mathfrak{N}b \cdot a = L(a)\mathfrak{N}b = \mathfrak{N}L(a)b = \mathfrak{N}(ba) = \mathfrak{N}(ca) = \mathfrak{N}L(a)c = L(a)\mathfrak{N}c = \mathfrak{N}c \cdot a,$$

so $\mathfrak{N}b = \mathfrak{N}c$. Hence the mapping φ defined by $\varphi a = \mathfrak{N}a$ is a homomorphism of the CML Q upon a loop φQ and the kernel $\mathfrak{N}1 = H$, of φ, is a normal subloop of Q. Consequently, any nonassociative subloop H of Q is normal in Q, i.e. the item 2) is proved.

By Theorem 6.3 from [1] the commutator subgroup $\mathfrak{M}'$ of non-periodic $\overline{IH}$-group is a finite abelian p-group. But the commutator subgroup of multiplication group of arbitrary CML is a 3-group [2, Theorem 11.4]. Hence the commutator subgroup $\mathfrak{M}'$ is a finite abelian 3-group, i.e. the item 3) is proved.

By Theorem 6.7 from [1] all solvable $\overline{IH}$-groups with infinite commutator subgroup satisfy the minimum condition for subgroups. But from 4) of Lemma 2 it follows that in these cases the commutator subgroup $\mathfrak{M}'$ of $\mathfrak{M}$ is finite. Hence the commutator subgroup $\mathfrak{M}'$ of $\overline{IH}$-group $\mathfrak{M}$ is finite and by [2, Theorem 11.4] is a finite 3-group.

Now let us suppose that the second commutator subgroup $\mathfrak{M}^{(2)}$ of the group $\mathfrak{M}$ is nonabelian. Then any subgroup that contains $\mathfrak{M}^{(2)}$ is nonabelian, and by item 2), it is normal in $\mathfrak{M}$. Obviously, the group $\mathfrak{M}/\mathfrak{M}^{(2)}$ is hamiltonian and as shown during the proof of Theorem 1, it is an abelian group. Therefore, $\mathfrak{M}' \subseteq \mathfrak{M}^{(2)}$, i.e. $\mathfrak{M}' = \mathfrak{M}^{(2)}$. But the commutator subgroup $\mathfrak{M}'$ is a finite 3-group, hence it is nilpotent. Therefore $\mathfrak{M}' \neq \mathfrak{M}^{(2)}$. Contradiction. Consequently, $\mathfrak{M}^{(2)}$ is an abelian subgroup, and the group $\mathfrak{M}$ is solvable of class not greater than three. This completes the proof of Theorem 2.

As by Theorem 2 the commutator subgroup $\mathfrak{M}'$ of the multiplication $\overline{IH}$-group is finite, then from [6, 7] it follows that $\mathfrak{M}$ is a group with finite classes of conjugate elements and the number of elements in each class doesn't exceed the number $|\mathfrak{M}'|$. Further, in [3] it is proved that the quotient group $\mathfrak{M}/Z(\mathfrak{M})$ of an arbitrary multiplication group $\mathfrak{M}$ by it centre $Z(\mathfrak{M})$ is a locally finite 3-group. Thus from [6, 8] it follows that if $\mathfrak{M}$ is an $\overline{IH}$-group, then any element in $\mathfrak{M}/Z(\mathfrak{M})$ is contained in a normal finite 3-group.

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