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On an algebraic method in the study of integral equations with shift

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Abstract. The work is centred on the sdudy of algebra $\mathfrak{A}$ generated by singular integral operators with shifts with continuous coefficients. We determine the set of maximal ideals of quotient algebra $\hat{\mathfrak{A}}$, $\hat{\mathfrak{A}} = \mathfrak{A}/\mathfrak{T}$, with respect to the ideal of compact operators. Prove that the bicompact of maximal ideals of $\hat{\mathfrak{A}}$ is isomorphic to the topological product $(\Gamma \times j) \times (\Gamma \times k)$, where $j = \pm 1$ and $k = \pm 1$. Necessary and sufficient condition are established for operators of $\mathfrak{A}$ to be noetherian and to admit equivalent regularization in space $L_p(\Gamma, \rho)$, regularizators for noetherian operators are constructed. The study is done in the space $L_p(\Gamma, \rho)$ with weight $\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta^k}$ and is based on the theory of Ghelfand [1] concerning Banach algebras.

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I. We remind that an operator $A \in L(\mathfrak{B})$ admits a regularization if there exists an operator $M \in L(\mathfrak{B})$ so that $AM = I + T_1, MA = I + T_2$, where T_1 and T_2 are compact operators in the space $\mathfrak{B}$. The class of operators which admit a regularization is of special interest due to the fact that operators of this class have the fallowing properties (E.Noether theorems):

1) Ecuations $Ax = 0$ and $A^*\varphi = 0$ have a finite number of linear independent solutions.

2) Ecuation $Ax = y$ is solvable if and ouly if the right hand part is orthogonal to all solutions of equation $A^*\varphi = 0$.

Operators with properties 1) and 2) are called *noetherian* and are essential generalizations of the class of operators of the form $I + T, T$ compact, for which theorems similar to that of Fredholm are true.

Let $A \in L(\mathfrak{B})$ be a noetherian operator. If it is known the operator M, the regularizator for A, then the problem of solvability of the equation

$$Ax = y \tag{1}$$

can be reduced to the solvability of the equation

$$MAx = My, \tag{2}$$

where the operator $MA - I$ is compact. To the last equation different methods to solve equations of the form $(I + T)x = y$ can be applied, where T is a compact operator. Of special interest is the case when the ecuations $Ax = y$ and $MAx = My$ are equivalent[1] for every y. This is true if and only if $\ker M = \{0\}$.

We say that an operator A admits an equivalent regularization if it possesses a regularizator M for which equations (1) and (2) are equivalent for every $y \in \mathfrak{B}$. In this case the operator M is called *equivalent regularizator* for the operator A. From what we said above it results that operator M is an equivalent regularizator for A if it is a regularizator for A and is left invertible.

It is well known [2] that a singular integral operator[2] $A = aI + bS + T$ admits a regularization if and only if $a^2(t) - b^2(t) \neq 0$ for all $t \in T$. For example, as a regularizator one can take the operator $R = \frac{a}{a^2 - b^2}I - \frac{b}{a^2 - b^2}S$. Under these conditions the operator A^*, obviously, also admits a regularization and thus for A and A^* E.Noether theorems hold.

The main result of this work is given by (see [3])

Theorem 1. *The operator*

$$A = aI + bS + (cI + dS)V + T, a, b, c, d \in C_\alpha(\Gamma) \tag{3}$$

admits a regularization in $L_p(\Gamma, \rho)$ if and only if

$$(a(t) + b(t))^2 - (c(t) + d(t))^2 \neq 0, (a(t) - b(t))^2 - (c(t) - d(t))^2 \neq 0 \tag{4}$$

for every $t \in T$. Under conditions (4) the operator

$$R = \frac{\alpha}{\alpha^2 - \delta^2}P + \frac{\beta}{\beta^2 - \gamma^2}Q - (\frac{\delta}{\alpha^2 - \delta^2}P + \frac{\gamma}{\beta^2 - \gamma^2}Q)V, \tag{5}$$

where $\alpha = a + b, \beta = a - b, \delta = c + d, \gamma = c - d, P = \frac{1}{2}(I + S), Q = \frac{1}{2}(I - S)$, is a regularizator for A.

II. Denote by $C_\omega(\Gamma)(\subset C(\Gamma)$ the set of functions $a(t)$ continuous on Γ and satifying the condition $a(\omega(t)) = a(t)$. Evidently, this set forms a commutative algebra with identity and the norm $||a||_{C_\omega(\Gamma)} = ||a||_{C(\Gamma)}$. It is also obvious that every function of the form $a(t) = b(t) \cdot b(\omega(t))$, where $b \in C(\Gamma)$, is contained in $C_\omega(\Gamma)$. The converse of this statement is also true: every function $a \in C_\omega(\Gamma)$ may be represented in the form $a(t) = b(t) \cdot b(\omega(t))$ where $b \in C(\Gamma)$. We can join these remarks in the assertion that the algebra $C_\omega(\Gamma)$ is characterized by the relation

$$C_\alpha(\Gamma) = \{b(t) \cdot b(\omega(t)) | b \in C(\Gamma)\}.$$

Representation of functions from $C_\omega(\Gamma)$ in the form $a(t) = b(t) \cdot b(\omega(t))$ is unique up to some constant factors c_1 and $c_2, c_1 \cdot c_2 = 1$. Later on we shall assume that $c_1 = c_2 = 1$. Thus, for example, if Γ is the unit circle and $\omega(t) = -t$, then the

[1] Equations $Ax = y$ and $MAx = My$ are called equivalent if they have the same set of solutions.
[2] By T with indices we denote compact operators.

functions $a_1(t) = -t^2, a_2(t) = t^2$ belong to $C_\omega(\Gamma)$ and they can be represented as $a_1(t) = t \cdot (-t)$ and, respectively $a_2(t) = it \cdot (-it)$.

Let Γ be a closed Liapunov type contour, S a singular integral operator with Cauchy kernel and V an operator of shifting, $(V\varphi)(t) = \varphi(\alpha(t))$, where the function $\omega : \Gamma \to \Gamma$ satisfies conditions:

a) $\omega(\omega(t)) \equiv \omega(t), (\omega(t) \neq t)$;

b) there exists derivative $\omega^{'}(t) \neq 0$;

c) $\omega^{'}(t) \in H_\mu(\Gamma)$.

Denote by $\mathfrak{A}$ the algebra generated by S, V and the set of alle operators of multiplication by functions $a(t), a \in C_\omega(\Gamma)$. $\mathfrak{A}$ is a subalgebra of algebra $L(L_p(\Gamma, \rho))$ formed by the set of linear and bounded operators acting in the space $L_p(\Gamma, \rho)$.

Theorem 2. $\mathfrak{A}$ *is a closed algebra.*

In the proof of this theorem we use properties of S and V, caracterization of algebra $C_\omega(\Gamma)$ and the following result [2].

Lemma 1. *If the operator* $(M\varphi) = a(t)\varphi(t)$ *of multiplication by function* $a(t)$ *continuuous on* Γ, *can be represented in the form* $M = B + T$, *where* B *is invertible and* T *is an operator compact in* $L_p(\Gamma, \rho)$, *then* $a(t)$ *is not vanished on* Γ.

Remark. The norm in algebra $\mathfrak{A}$, defined as operator norm, is topologically equivalent to the norm

$$||A||_1 = \max|a(t)| + \max|b(t)| + \max|c(t)| + \max|d(t)| + ||T||.$$

The set $\mathfrak{T} = \mathfrak{T}(L(L_p(\Gamma, \rho)))$ of compact operators in the space $L_p(\Gamma, \rho)$ is included in $\mathfrak{A}$ and form a twosided closed ideal. Consider the quotient algebra $\hat{\mathfrak{A}} = \mathfrak{A}/\mathfrak{T}$, which is also a Banach algebra. Four continuous functions $a(t), b(t), c(t)$ and $d(t)$ define uniquely a coset $\hat{\mathfrak{A}}$ and; conversely, every element belonging to some coset of $\hat{\mathfrak{A}}$ is of the form $aI + bS + (cI + dS)V + T$, where T is a compact operator. Really, if the elements $aI + bS + (cI + dS)V + T$ and $a_1I + b_1 + (c_1I + d_1S)V + T_1$ are in some coset, then their difference $(a - a_1)I + (b - b_1)S + ((c - c_1)I + (d - d_1)S)V + T - T_1$ must be a compact operator. Under these condutions from Theorem 2 one can deduce that the operators $(a - a_1)I, (b - b_1)I, (c - c_1)I, (d - d_1)I$ are compact, but from Lemma 1 this is possible if and only if $a(t) \equiv a_1(t), b(t) \equiv b_1(t), c(t) \equiv c_1(t), d(t) \equiv d_1(t)$.

Let us return to algebra $\hat{\mathfrak{A}}$. The element of $\hat{\mathfrak{A}}$ determined by the functions $a(t), b(t), c(t)$ and $d(t)$ is denoted by $\{aI + bS + (cI + dS)V\}$. From properties of operators S and V [4–6] and by direct calculations we get

Theorem 3. *The algebra* $\hat{\mathfrak{A}}$ *is commutative and, besides, the equality*

$$\{aI + bS + (cI + dS)V\} \cdot \{a_1I + b_1S + (c_1I + d_1S)V\} =$$

$$= \{aa_1 + bb_1 + cc_1 + dd_1)I + (ab_1 + a_1b + cd_1 + c_1d)S+ \tag{6}$$

$$+((ac_1 + a_1c + bd_1 + b_1d)I + (ad_1 + a_1d + bc_1 + b_1c)S)V\}$$

is true.

The norm in $\hat{\mathfrak{A}}$ is defined by the equality

$$|\{aI + bS + (cI + dS)V\}| = \inf_{T\in\mathfrak{T}} ||aI + bS + (cI + dS)V||$$

and it is topologically equivalent to the norm

$$|\{aI + bS + (cI + dS)V\}|_1 = \max|a(t)| + \max|b(t)| + \max|c(t)| + \max|d(t)|.$$

III. Further, elements of algebra $\hat{\mathfrak{A}}$ will be expressed in the form

$$\{aP + bQ + (cP + dQ)V\}, a, b, c, d \in C_\omega(\Gamma), \tag{7}$$

where $P = \dfrac{1}{2}(I + S)$ and $Q = \dfrac{1}{2}(I - S)$.

We shall describe all maximal ideals of $\hat{\mathfrak{A}}$. This result will enable us to establish necessary and sufficient condition under which elements of $\hat{\mathfrak{A}}$ are invertible. Using this result we shall also construct regularizators for noetherian operators.

Theorem 4. *The set of elements $\{ap + bQ + (cP + dQ)V\} \in \hat{\mathfrak{A}}$ does form a maximal ideal of $\hat{\mathfrak{A}}$ if the function $a(t) + c(t)$ is vanished at the same point $t_0 \in \Gamma$. The set of elements $\{aP + bQ + (cP + dQ)V\} \in \hat{\mathfrak{A}}$ for which one of the functions $a(t) - c(t), b(t) + d(t)$, or $b(t) - d(t)$ is vanished at the same point (every function at its own point) also form a maximal ideal. There are no other maximal ideals.*

By virtue of I.Ghelfand [1] results, according to which an element of some Banach algebra is invertible if and only if it does not belong to ony maximal ideal, we obtain the following

Theorem 5. *An element $\{aP + bQ + (cP + dQ)V\} \in \hat{\mathfrak{A}}$ is invertible in $\hat{\mathfrak{A}}$ if and only if the functions $a(t) \pm c(t)$ and $b(t) \pm d(t)$ are not vanished on contour Γ.*

We shall establish some other properties of algebra $\hat{\mathfrak{A}}$. Observe that the intersection of all maximal ideals of $\hat{\mathfrak{A}}$ coincides to the null ideal. In fact, by Theorem 4, if $\{aP+bQ+(cP+dQ)V\} \in \cap M_1$, then $a(t)+c(t) \equiv 0$, $a(t)-c(t) \equiv 0$, $b(t)+d(t) \equiv 0$ and $b(t) - d(t) \equiv 0$, that is $\{aP + bQ + (cP + dQ)V\} = \{0\}$. Consequently,

1°. Algebra has no radical.

2°. $\hat{\mathfrak{A}}$ is an involution algebra.

Define involution by

$$\{aP + bQ + (cP + dQ)V\}' = \{\bar{a}P + \bar{b}Q + (\bar{c}P + \bar{d}Q)V\}.$$

All properties of involution are evident. We shall show only that for every element $\{aP + bQ + (cP + dQ)V\} \in \hat{\mathfrak{A}}$ there exists in $\hat{\mathfrak{A}}$ the element

$$[I + \{(aP + bQ + (cP + dQ)V) \cdot (\bar{a}P + \bar{b}Q + (\bar{c}P + \bar{d}Q)V\}]^{-1}.$$

Compute

$$[I + \{ap + bQ + (cP + dQ)V) \cdot (\bar{a}P + \bar{b}Q + (\bar{c}P + \bar{d}Q)V\}] =$$

$$= \{(1 + |a|^2 + |c|^2)P + (1 + |b|^2 + |d|^2)Q + ((a\bar{c} + \bar{a}c)P + +(b\bar{d} + \bar{b}d)Q)V\},$$

$$I + |b(t)|^2 + |d(t)|^2 \pm (b(t)\bar{d}(t) + \bar{b}(t)d(t) = 1 + |b(t) \pm d(t)|^2 > 0.$$

Hence, there exists

$$[I + \{aP + bQ + (cP + dQ)V) \cdot (\bar{a}P + \bar{b}Q + (\bar{c}P + \bar{d}Q)V\}]^{-1} =$$

$$= \left\{ \begin{array}{l} \dfrac{1 + |a|^2 + |c|^2}{(1 + |a - c|^2)(1 + |a + c|^2)} P + \dfrac{1 + |b|^2 + |d|^2}{(1 + |b - d|^2)(1 + |b + d|^2)} Q - \\ - \dfrac{a\bar{c} + \bar{a}c}{(1 + |a - c|^2)(1 + |a + c|^2)} P + \dfrac{b\bar{d} + \bar{b}d}{(1 + |b - d|^2)(1 + |b + d|^2)} Q)V \end{array} \right\}$$

and this element belougs to $\hat{\mathfrak{A}}$. Property 2^o is proved.

Denote by $\mathfrak{M}$ the bicompact of maximal ideals of $\hat{\mathfrak{A}}$.

3°. $\mathfrak{M}$ is isomorphic to the topological product $(\Gamma \times j) \times (\Gamma \times k) : \mathfrak{M} = (\Gamma \times j) \times (\Gamma \times k)$, where $j = \pm 1$ and $k = \pm 1$.

It is known [1] that every commutative Banach algebra without radical is isomorphically mapped into an algebra of functions defined on bicompact of maximal ideals. It is easy to observe that in our case to the element $A = \{aP + bQ + (cP + dQ)V\} \in \hat{\mathfrak{A}}$ the function $A(M) = (a(t) + jc(t))(b(t) + kf(t))$ corresponds.

4°. Algebra $\hat{\mathfrak{A}}$ is a symmetric algebra without radical.

In commutative and symmetric algebra $\mathfrak{R}$ every element is invertible or is a generalized zero divisor (see [1]), that is, there exists a sequence $(y_n), y_n \in \mathfrak{R}, |y_n| = 1$ and $\lim\limits_{n \to \infty} ||y_n x|| = 0$. Thus, every element $A = \{aP + bQ + (cP + dQ)V\}$, for which one of the functions $a(t) + c(t), a(t) - c(t), b(t) + d(t)$ or $b(t) - d(t)$ is vanished on Γ, is a generalized zero divisor.

Obviously, $L(L_p(\Gamma, \rho)) \setminus \mathfrak{T}$ is a (noncommutative) Banach algebra including $\hat{\mathfrak{A}}$.

5°. An element $A \in \mathfrak{A}$ is invertible in $L(L_p(\Gamma, \rho)) \setminus \mathfrak{T}$ if and only if it is invertible in $\hat{\mathfrak{A}}$.

In fact, let A be invertible in $L(L_p(\Gamma, \rho)) \setminus \mathfrak{T}$ and suppose it is not invertible in $\hat{\mathfrak{A}}, A^{-1} \notin \hat{\mathfrak{A}}$. Then, by virtue of 4^o, A is a generalized zero divisor. But this is impossible, since in this case the invertible operator A should be a generalized zero divisor in $L(L_p(\Gamma, \rho)) \setminus \mathfrak{T}$.

IV. Let us approach the problem of regularization of singular integral operators with shift ω, $A = aI + bS = (cI + dS)V + T$. It is easy to observe that the operator A admits a regularization in algebra $L(L_p(\Gamma, \rho))$ if and only if the element $\{aI + bS + (cI + dS)V\} \in \hat{\mathfrak{A}}$ is invertibile in $L(L_p(\Gamma, \rho)) \setminus \mathfrak{T}$. In order to apply assertions of Theorem 5 and property 5^o we use the operators $P = \frac{1}{2}(I + S)$, $Q = \frac{1}{2}(I - S)$, $I = P + Q$ and $S = P - Q$. Then the operators A is transcribed as, $A = \alpha P + \beta Q +$

$(\delta P+\gamma Q)V+T$, where $\alpha = a+b, \beta = a-b, \delta = c+d, \gamma = c-d$. From Theorem 5 and property 5^o it results that $\{\alpha P + \beta Q + (\delta P + \gamma Q)V\}$ is invertible in $L(L_p(\Gamma, \rho)) \setminus \mathfrak{T}$ if and only if the functions $\alpha^2(t) - \delta^2(t)$ and $\beta^2(t) - \gamma^2(t)$ do not vanish on Γ. In other words, a singular integral operator A with shift, $A = ai + bS + (cI + dS)V + T$, admits a regularization in $L(L_p(\Gamma, \rho))$ if only if

$$\alpha^2(t) - \delta^2(t) = (a(t) + b(t))^2 - (c(t) + d(t))^2 \neq 0,$$

$$\beta^2(t) - \gamma^2(t) = (c(t) + d(t))^2 - (c(t) + d(t))^2 \neq 0.$$

Thus, condition (4) of Theorem 1 are satisfied. With the help of judgements used in the proof of Theorem 5 it is supplementary obtained that $AR = I + T_1$ and $RA = I + T_2$, where R is defined by relation (5) and T_1, T_2 are compact operators.

Theorem 6. *The operator $A = \alpha P + \beta Q + (\delta P + \gamma Q)V + T$ admits an equivalent regularization if and only if the following conditions*

$$\alpha^2(t) - \delta^2(t) \neq 0, \beta^2(t) - \gamma^2(t) \neq 0, ind\frac{\alpha^2(t) - \delta^2(t)}{\beta^2(t) - \gamma^2(t)} \leq 0$$

are verified. Under these conditions

$$IndA = -\frac{1}{2} ind\frac{\alpha^2(t) - \delta^2(t)}{\beta^2(t) - \gamma^2(t)}.$$

For $IndA < 0$ all solutions to equation $Ax = y$ are obtained from the relation $x = Rz$, where z runs all solutions to equation $RAz = y$ and R is defined by (5).

Cases when the function of shifting, ω, changes the orientation of contour Γ and systems of singular integral equation with shift will be approached, possibly, in other works of the author.

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