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Discontinuous term of the distribution for Markovian random evolution in $\mathbf{R}^3$

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Abstract. We consider the random motion at constant finite speed in the space R^3 subject to the control of a homogeneous Poisson process and with uniform choice of directions on the unit 3-sphere. We obtain the explicit forms of the conditional characteristic function and conditional distribution when one change of direction occurs. We show that this conditional distribution represents a discontinuous term of the transition function of the motion.

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In this note we obtain the discontinuous term of the distribution for the three-dimensional random motion at arbitrary finite speed (so-called, random evolution). This is motivated by the previous works on planar random motions by Stadje (1987), Masoliver *et al.* (1993), Kolesnik and Orsingher (2005) where the explicit form of the transition function of the process was obtained by substantially different methods. It was shown that the transition density of the motion is discontinuous on the boundary of the diffusion area, and the discontinuous term of the distribution is the Green's function to the two-dimensional wave equation. Amazingly, this discontinuous term is determined by the conditional distribution, corresponding to the case when only one change of direction occurs (see Kolesnik and Orsingher (2005, Remark 1)). The three-dimensional motion with unit speed was examined by Stadje (1989) where the transition density was derived by means of recurrent arguments. This transition density consists of two terms (see Stadje (1989, formulae (1.3) and (4.21)). The first one is a continuous function and has the form of a fairly complicated integral, which can scarcely be exactly evaluated. The second term is a logarithmic function which is discontinuous on the boundary of the diffusion area.

Here we derive this discontinuous term for a motion at arbitrary finite speed by means of characteristic functions and show that, similarly to the planar case, it is determined by the conditional distribution corresponding to the case when only one change of direction occurs. Such a behaviour of the distribution near the border of diffusion area is a very interesting feature of the two and three-dimensional motions. However, the nature of this phenomenon is not entirely clear.

Let's consider a particle starting its motion from the origin $x_1 = x_2 = x_3 = 0$ of the space R^3 at time $t = 0$. The particle is endowed with constant, finite speed c.

The initial direction is a three-dimensional random vector with uniform distribution on the unit 3-sphere

$$S_1 = \left\{(x_1, x_2, x_3) \in R^3 : \ x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

The particle changes direction at random instants which form a homogeneous Poisson process of rate $\lambda > 0$. At these moments it instantaneously takes on the new direction with uniform distribution on S_1, independently of its previous motion.

Let $X(t) = (X_1(t), X_2(t), X_3(t))$ be the position of the particle at an arbitrary time $t > 0$. Consider the conditional distributions

$$Pr\{X(t) \in dx \mid N(t) = n\} =$$

$$= Pr\{X_1(t) \in dx_1, X_2(t) \in dx_2, X_3(t) \in dx_3 \mid N(t) = n\}, \quad n \geq 1,$$

where $N(t)$ is the number of Poisson events that have occurred in the interval $(0, t)$ and $dx = dx_1 dx_2 dx_3$ is the infinitesimal volume of the space R^3.

At any time $t > 0$ the particle, with probability 1, is located in the three-dimensional ball of radius ct

$$B_{ct} = \left\{(x_1, x_2, x_3) \in R^3 : \ x_1^2 + x_2^2 + x_3^2 \leq c^2 t^2\right\}.$$

The distribution $Pr\left\{X(t) \in dx\right\}, \ x \in B_{ct}, \ t \geq 0$, consists of two components. The singular component corresponds to the case when no Poisson event occurs in the interval $(0, t)$ and is concentrated on the sphere

$$S_{ct} = \partial B_{ct} = \left\{(x_1, x_2, x_3) \in R^3 : \ x_1^2 + x_2^2 + x_3^2 = c^2 t^2\right\}.$$

In this case the particle is located on the sphere S_{ct} and the probability of this event is

$$Pr\left\{X(t) \in S_{ct}\right\} = e^{-\lambda t}.$$

If one or more than one Poisson events occur, the particle is located strictly inside the ball B_{ct}, and the probability of this event is

$$Pr\left\{X(t) \in Int\ B_{ct}\right\} = 1 - e^{-\lambda t}.$$

The part of the distribution $Pr\left\{X(t) \in dx\right\}$ corresponding to this case is concentrated in the interior

$$Int\ B_{ct} = \left\{(x_1, x_2, x_3) \in R^3 : \ x_1^2 + x_2^2 + x_3^2 < c^2 t^2\right\},$$

and forms its absolutely continuous component.

Therefore there exists the density $p(x, t) = p(x_1, x_2, x_3, x_4, t), \ \ x \in Int\ B_{ct}, \ t > 0$, of the absolutely continuous component of the distribution $Pr\left\{X(t) \in dx\right\}$.

If $N(t) = n$, the displacement of the particle $X(t)$ at any time $t > 0$ is determined by the coordinates

$$X_k(t) = c \sum_{j=1}^{n+1} (s_j - s_{j-1}) x_j^k, \qquad k = 1, 2, 3, \tag{1}$$

where x_j^k are the components of the independent random vectors $x_j = (x_j^1, x_j^2, x_j^3)$, $j = 1, ..., n+1$, uniformly distributed on the unit sphere S_1; the $s_j,\ j = 1, ..., n$, represent the instants at which Poisson events occur, and $s_0 = 0,\ s_{n+1} = t$.

The conditional characteristic function can be written as follows:

$$H_n(\alpha, t) = E\left\{ e^{i(\alpha, X(t))} \mid N(t) = n \right\}, \quad n \geq 1, \tag{2}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in R^3$ is the real vector of inversion parameters and $(\alpha, X(t))$ denotes the scalar (inner) product of the vectors α and $X(t)$.

By substituting (1) into (2) we have

$$H_n(\alpha, t) = E\left\{ \exp\left(ic \sum_{k=1}^{3} \alpha_k \sum_{j=1}^{n+1} (s_j - s_{j-1}) x_j^k \right) \right\} =$$

$$= E\left\{ \exp\left(ic \sum_{j=1}^{n+1} (s_j - s_{j-1})(\alpha, x_j) \right) \right\},$$

where (α, x_j) is the scalar (inner) product of the vectors α and x_j. Computing the expectation in this last equality we obtain

$$H_n(\alpha, t) = \frac{n!}{t^n} \int_0^t ds_1 \int_{s_1}^t ds_2 \dots \int_{s_{n-1}}^t ds_n \left\{ \prod_{j=1}^{n+1} \left[\frac{1}{mes\ S_1} \int_{S_1} e^{ic(s_j - s_{j-1})(\alpha, x_j)}\, dx_j \right] \right\}.$$

The surface integral over the unit sphere S_1 in the last equality can easily be computed by passing to three-dimensional polar coordinates, and it is

$$\int_{S_1} e^{ic(s_j - s_{j-1})(\alpha, x_j)}\, dx_j = 4\pi\, \frac{\sin(c(s_j - s_{j-1})\|\alpha\|)}{c(s_j - s_{j-1})\|\alpha\|}, \tag{3}$$

where $\|\alpha\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$. Taking into account that $mes\ S_1 = 4\pi$, we obtain

$$H_n(\alpha, t) = \frac{n!}{t^n} \int_0^t ds_1 \int_{s_1}^t ds_2 \dots \int_{s_{n-1}}^t ds_n \left\{ \prod_{j=1}^{n+1} \frac{\sin(c(s_j - s_{j-1})\|\alpha\|)}{c(s_j - s_{j-1})\|\alpha\|} \right\}. \tag{4}$$

This expression can scarcely be explicitly computed for arbitrary $n \geq 1$. However, for the important particular case $n = 1$ this expression can be evaluated, and its inverse Fourier transform leading to the conditional distribution, corresponding to the case when only one change of direction occurs, can be explicitly given.

Theorem. *For any $t > 0$ the conditional distribution corresponding to the only change of direction has the form*

$$Pr\{X(t) \in dx \mid N(t) = 1\} = \frac{1}{4\pi (ct)^2 \|x\|}\ \ln\left(\frac{ct + \|x\|}{ct - \|x\|} \right)\, dx, \tag{5}$$

$$x = (x_1, x_2, x_3) \in Int\ B_{ct}, \quad \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad dx = dx_1 dx_2 dx_3.$$

Proof. From (4) we have

$$H_1(\alpha, t) =$$

$$= \frac{1}{t} \int_0^t \frac{\sin(cs\|\alpha\|)}{cs\|\alpha\|} \frac{\sin(c(t-s)\|\alpha\|)}{c(t-s)\|\alpha\|} \, ds =$$

$$= \frac{1}{(ct\|\alpha\|)^2} \int_0^t \sin(cs\|\alpha\|) \ \sin(c(t-s)\|\alpha\|) \ \left[\frac{1}{s} + \frac{1}{t-s} \right] \, ds =$$

$$= \frac{2}{(ct\|\alpha\|)^2} \int_0^t \frac{\sin(cs\|\alpha\|) \ \sin(c(t-s)\|\alpha\|)}{s} \, ds =$$

$$= \frac{\sin(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \int_0^t \frac{2\sin(cs\|\alpha\|) \ \cos(cs\|\alpha\|)}{s} \, ds - \frac{\cos(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \int_0^t \frac{2\sin^2(cs\|\alpha\|)}{s} \, ds =$$

$$= \frac{\sin(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \int_0^t \frac{\sin(2cs\|\alpha\|)}{s} \, ds - \frac{\cos(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \int_0^t \frac{1-\cos(2cs\|\alpha\|)}{s} \, ds =$$

$$= \frac{\sin(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \ \mathrm{Si}(2ct\|\alpha\|) + \frac{\cos(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \ \mathrm{Ci}(2ct\|\alpha\|), \tag{6}$$

where $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ are the modified integral sine and cosine, respectively, given by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin z}{z} \, dz, \qquad \mathrm{Ci}(x) = \int_0^x \frac{\cos z - 1}{z} \, dz. \tag{7}$$

To prove the statement of the theorem, we need to show that the inverse Fourier transform of (6) with respect to $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ leads to the conditional distribution (5). However, it is simpler to show that, inversely, the Fourier transform of function (5) in the ball B_{ct} coincides with (6). Passing to three-dimensional polar coordinates we have

$$\int_{B_{ct}} e^{i(\alpha,x)} \ Pr\{X(t) \in dx \mid N(t) = 1\} =$$

$$= \frac{1}{4\pi(ct)^2} \int_0^{ct} dr \ \left\{ r \ln \left(\frac{ct+r}{ct-r} \right) \times \right.$$

$$\left. \times \int_0^\pi \int_0^{2\pi} e^{ir(\alpha_1 \sin\theta_1 \sin\theta_2 + \alpha_2 \sin\theta_1 \cos\theta_2 + \alpha_3 \cos\theta_1)} \ \sin\theta_1 \ d\theta_1 \ d\theta_2 \right\}.$$

According to formula 4.624 of Gradshteyn and Ryzhik (1980)

$$\int_0^\pi \int_0^{2\pi} e^{ir(\alpha_1 \sin\theta_1 \sin\theta_2 + \alpha_2 \sin\theta_1 \cos\theta_2 + \alpha_3 \cos\theta_1)} \ \sin\theta_1 \ d\theta_1 \ d\theta_2 = 4\pi \ \frac{\sin(r\|\alpha\|)}{r\|\alpha\|}.$$

Therefore, applying the auxiliary Lemma (see below), we obtain

$$\int_{B_{ct}} e^{i(\alpha,x)} \; Pr\{X(t) \in dx \mid N(t) = 1\} =$$

$$= \frac{1}{(ct)^2\|\alpha\|} \int_0^{ct} \sin(r\|\alpha\|) \; \ln\left(\frac{ct+r}{ct-r}\right) \; dr =$$

$$= \frac{1}{ct\|\alpha\|} \int_0^{1} \sin(ct\|\alpha\| z) \; \ln\left(\frac{1+z}{1-z}\right) \; dz =$$

$$= \frac{1}{(ct\|\alpha\|)^2} \left[\sin(ct\|\alpha\|) \; \mathrm{Si}(2ct\|\alpha\|) + \cos(ct\|\alpha\|) \; \mathrm{Ci}(2ct\|\alpha\|) \right],$$

and this coincides with (6). The theorem is proved.

Remark 1. Taking into account the well-known equality

$$\mathrm{Arcth}(z) = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$$

we can rewrite (5) as follows:

$$Pr\{X(t) \in dx \mid N(t) = 1\} = \frac{1}{2\pi(ct)^2\|x\|} \; \mathrm{Arcth}\left(\frac{\|x\|}{ct}\right) \; dx,$$

and, for $c = 1$, this is similar to the second term of formulae (1.3) and (4.21) of Stadje (1989). The first (integral) term of these formulae, obviously, is determined by the conditional distributions corresponding to the case when more than one change of direction occurs.

Remark 2. The function (5) represents the discontinuous term of the distribution of the process $X(t)$. Formula (5) shows that the density near the border of the ball B_{ct} is large and this corresponds to the case when only one change of direction occurs. In other words, if only one change of direction occurs, the conditional density is minimal in the neighbourhood of the origin and increases as we approach to the border. This is similar to the behaviour of the analogous conditional density in the planar case (see Kolesnik and Orsingher (2005, Remark 1)).

Finally, we establish an auxiliary lemma which has been used in the proof of our theorem.

Lemma. *For arbitrary $a > 0$ the following relation holds*

$$\int_0^1 \sin(ax) \; \ln\left(\frac{1+x}{1-x}\right) \; dx = \frac{1}{a}\left[\sin a \; \mathrm{Si}(2a) + \cos a \; \mathrm{Ci}(2a) \right], \tag{8}$$

where the integral is treated in the improper sense.

Proof. We have

$$\int_0^1 \sin(ax)\ \ln\left(\frac{1+x}{1-x}\right)\ dx =$$

$$= \int_0^1 \sin(ax)\ \ln(1+x)\ dx - \int_0^1 \sin(ax)\ \ln(1-x)\ dx. \tag{9}$$

Let's evaluate separately the integrals in the right-hand side of (9). Integrating by parts and applying formula 2.641(2) of Gradshteyn and Ryzhik (1980) we obtain for the first integral of (9):

$$\int_0^1 \sin(ax)\ \ln(1+x)\ dx =$$

$$= -\frac{1}{a}\left(\cos a\ \ln 2 - \int_0^1 \frac{\cos(ax)}{1+x}\ dx\right) =$$

$$= -\frac{\cos a}{a}\ln 2 + \frac{1}{a}\Big[\cos a\,(\mathrm{ci}(2a) - \mathrm{ci}(a)) + \sin a\,(\mathrm{si}(2a) - \mathrm{si}(a))\Big]. \tag{10}$$

Here $\mathrm{si}(x)$ and $\mathrm{ci}(x)$ are the standard integral sine and cosine, respectively, given by

$$\mathrm{si}(x) = -\frac{\pi}{2} + \mathrm{Si}(x), \qquad \mathrm{ci}(x) = \mathbf{C} + \ln x + \mathrm{Ci}(x), \tag{11}$$

and $\mathbf{C} = 0.5772\ldots$ being the Euler constant. From (11) we easily obtain

$$\mathrm{ci}(2a) - \mathrm{ci}(a) = \ln 2 + \mathrm{Ci}(2a) - \mathrm{Ci}(a),$$

$$\mathrm{si}(2a) - \mathrm{si}(a) = \mathrm{Si}(2a) - \mathrm{Si}(a).$$

Substituting these expressions into (10) we obtain the first integral of (9):

$$\int_0^1 \sin(ax)\ \ln(1+x)\ dx =$$

$$= \frac{\cos a}{a}\Big(\mathrm{Ci}(2a) - \mathrm{Ci}(a)\Big) + \frac{\sin a}{a}\Big(\mathrm{Si}(2a) - \mathrm{Si}(a)\Big), \tag{12}$$

where $\mathrm{Ci}(x)$ and $\mathrm{Si}(x)$ are given by (7).

The function in the second integral of (9) is unbounded at the point $x = 1$. Therefore we can evaluate this integral in the improper sense only. Similarly, integrating by parts and applying formula 2.641(2) of Gradshteyn and Ryzhik (1980) we have

$$\int_0^1 \sin(ax)\ \ln(1-x)\ dx = \lim_{\varepsilon\to 0^+}\int_0^{1-\varepsilon} \sin(ax)\ \ln(1-x)\ dx =$$

$$= -\frac{1}{a}\lim_{\varepsilon\to 0^+}\left\{\cos\left(a(1-\varepsilon)\right)\ \ln\varepsilon + \int_0^{1-\varepsilon}\frac{\cos(ax)}{1-x}\ dx\right\} =$$

$$= -\frac{1}{a} \lim_{\varepsilon\to 0^+} \{\cos(a(1-\varepsilon)) \ \ln\varepsilon -$$

$$- [\cos a \ (\mathrm{ci}(-a\varepsilon) - \mathrm{ci}(-a)) - \sin a \ (\mathrm{si}(-a\varepsilon) - \mathrm{si}(-a))]\} . \tag{13}$$

Using (11) and the well-known equalities

$$\mathrm{si}(-x) = -\mathrm{si}(x) - \pi, \qquad \mathrm{Ci}(-x) = \mathrm{Ci}(x),$$

we get

$$\mathrm{ci}(-a\varepsilon) - \mathrm{ci}(-a) = \ln\varepsilon + \mathrm{Ci}(a\varepsilon) - \mathrm{Ci}(a),$$
$$\mathrm{si}(-a\varepsilon) - \mathrm{si}(-a) = \mathrm{Si}(a) - \mathrm{Si}(a\varepsilon).$$

Substituting this into (13) we obtain the second integral in the right-hand side of (9):

$$\int_0^1 \sin(ax) \ \ln(1-x) \, dx =$$

$$= -\frac{1}{a} \lim_{\varepsilon\to 0^+} \{(\cos(a(1-\varepsilon)) - \cos a) \ \ln\varepsilon -$$

$$- \cos a \ (\mathrm{Ci}(a\varepsilon) - \mathrm{Ci}(a)) + \sin a \ (\mathrm{Si}(a) - \mathrm{Si}(a\varepsilon))\} =$$

$$= -\frac{1}{a} \ [\sin a \ \mathrm{Si}(a) + \cos a \ \mathrm{Ci}(a)] . \tag{14}$$

Substituting now (12) and (14) into (9) we obtain (8). The lemma is proved.

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