

Commutative Moufang loops with maximum conditions for subloops

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Abstract. It is proved that the maximum condition for subloops in a commutative Moufang loop Q is equivalent with conditions of the finite generating of different subloops of the loop Q and different subgroups of the multiplication group of the loop Q . An analogue equivalence is set for commutative Moufang ZA -loops.

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It is said that the maximum condition (respect. minimum condition) for subalgebras with the property α holds in an algebra A if any ascending (respect. descending) system of subalgebras with the property α $A_1 \subseteq A_2 \subseteq \dots$ (respect. $A_1 \supseteq A_2 \supseteq \dots$) breaks, i.e. $A_n = A_{n+1} \dots$ for a certain n . It is well known that the fulfillment of the maximum condition for subalgebras of an arbitrary algebra is equivalent to the fact that both the algebra and any of its subalgebras are finitely generated.

Commutative Moufang loops (CML's) with maximum condition for subloops are considered in this paper. It is proved that for a non-associative CML Q this condition is equivalent to one of the following equivalent conditions: a) if Q contains a centrally nilpotent subloop of class n , then all its subloops of this type are finitely generated; b) if Q contains a centrally solvable subloop of class s , then all its subloops of this type are finitely generated; c) all invariant subloops of Q are finitely generated; d) all non-invariant associative subloops of Q are finitely generated; e) at least one maximal associative subloop of Q is finitely generated. This list is completed with the condition of the finite generating of various subgroups of the multiplication group of Q . If Q is a ZA -loop, then the list a) – e) is completed with the condition of the finite generating of the center of Q , as well with the condition of the finite generating of other subloops of Q and various subgroups of the multiplication group of Q .

It is worth mentioning that the following statement is proved in [1, 2].

Lemma 1. *The following conditions are equivalent for an arbitrary CML Q :*

- 1) Q is finitely generated;
- 2) the maximum condition for subloops holds in Q .

In [2] the list a) – e) is completed with equivalent statements: h) the CML Q satisfies the maximum condition for invariant subloops; i) the CML Q is a subdirect product of a finite CML of exponent 3 and a finitely generated abelian group; j) the

CML Q possesses a finite central series, whose factors are cyclic groups of simple or infinite order.

Let us bring some notions and results on the theory of commutative Moufang loops, needed for further research.

A *commutative Moufang loop* (CML's) is characterized by the identity $x^2 \cdot yz = xy \cdot xz$. The *multiplication group* $\mathfrak{M}(Q)$ of a CML Q is the group generated by all the *translations* $L(x)$, where $L(x)y = xy$. The subgroup $I(Q)$ of the group $\mathfrak{M}(Q)$, generated by all the *inner mappings* $L(x, y) = L(xy)^{-1}L(x)L(y)$ is called the *inner mapping group* of the CML Q . The subloop H of a CML Q is called *normal (invariant)* in Q if $I(Q)H = H$.

Lemma 2 [3]. *Let Q be a commutative Moufang loop with the multiplication group \mathfrak{M} . Then $\mathfrak{M}/Z(\mathfrak{M})$, where $Z(\mathfrak{M})$ is the centre of the group \mathfrak{M} , and $\mathfrak{M}' = (\mathfrak{M}, \mathfrak{M})$ are locally finite 3-groups and will be finite if Q is finitely generated.*

Lemma 3. *The multiplication group \mathfrak{M} of an arbitrary CML is locally nilpotent.*

Proof. Let $\bar{\mathfrak{N}}$ be the image of finitely generated subgroup of group \mathfrak{M} under the homomorphism $\mathfrak{M} \rightarrow \mathfrak{M}/Z(\mathfrak{M})$. It follows from Lemma 2 that $\bar{\mathfrak{N}}$ is a finite 3-group, therefore it is nilpotent. Let us write $\bar{\mathfrak{N}}$ in the form $\mathfrak{N}Z(\mathfrak{M})/Z(\mathfrak{M})$. We have $\mathfrak{N}Z(\mathfrak{M})/Z(\mathfrak{M}) \cong \mathfrak{N}/(\mathfrak{N} \cap Z(\mathfrak{M}))$. It is obvious that $\mathfrak{N} \cap Z(\mathfrak{M}) \subseteq Z(\mathfrak{N})$. Then

$$\mathfrak{N}/Z(\mathfrak{N}) \cong (\mathfrak{N}/(\mathfrak{N} \cap Z(\mathfrak{M}))) / (Z(\mathfrak{N})/(\mathfrak{N} \cap Z(\mathfrak{M}))).$$

Therefore $\mathfrak{N}/Z(\mathfrak{N})$ is nilpotent, as a homomorphic image of the nilpotent group $\mathfrak{N}/(\mathfrak{N} \cap Z(\mathfrak{M}))$. Then the group \mathfrak{N} is nilpotent as well. Consequently, the group \mathfrak{M} is locally nilpotent, as required.

The *center* $Z(Q)$ of a CML Q is an invariant subloop $Z(Q) = \{x \in Q \mid x \cdot yz = xy \cdot z \forall y, z \in Q\}$.

Lemma 4 [3]. *The quotient loop $Q/Z(Q)$ of an arbitrary CML Q by its center $Z(Q)$ has the index three.*

Lemma 5 [3]. *A periodic CML is locally finite.*

The associator (a, b, c) of the elements a, b, c in CML Q is defined by the equality $ab \cdot c = (a \cdot bc)(a, b, c)$. We denote by Q_i (respect. $Q^{(i)}$) the subloop of the CML Q , generated by all associators of the form $(x_1, x_2, \dots, x_{2i+1})$ (respect. $(x_1, \dots, x_{3i})^{(i)}$), where $(x_1, \dots, x_{2i-1}, x_{2i}, x_{2i+1}) = ((x_1, \dots, x_{2i-1}), x_{2i}, x_{2i+1})$ (respect. $(x_1, \dots, x_{3i})^{(i)} = ((x_1, \dots, x_{3i-1})^{(i-1)}, (x_{3i-1+1}, \dots, x_{2 \cdot 3^{i-1}})^{(i-1)}, x_{2 \cdot 3^{i-1}+1}, \dots, x_{3i})^{(i-1)}$), where $(x_1, x_2, x_3)^{(1)} = (x_1, x_2, x_3)$. The series of normal subloops $1 = Q_0 \subseteq Q_1 \subseteq \dots \subseteq Q_i \subseteq \dots$ (respect. $1 = Q^{(0)} \subseteq Q^{(1)} \subseteq \dots \subseteq Q^{(i)} \subseteq \dots$) is called the *lower central series* (respect. *derived series*) of the CML Q . We will also use for associator loop the designation $Q^{(1)} = Q'$.

A CML Q is *centrally nilpotent* (respect. *centrally solvable*) of class n if and only if its lower central series (respect. derived series) has the form $1 \subset Q_1 \subset \dots \subset Q_n = Q$ (respect. $1 \subset Q^{(1)} \subset \dots \subset Q^{(n)} = Q$) [3].

An *ascending central series* of CML Q is a linearly ordered by the inclusion system

$$1 = Q_0 \subseteq Q_1 \subseteq \dots \subseteq Q_\alpha \subseteq \dots \subseteq Q_\gamma = Q$$

of invariant subloops of Q , satisfying the conditions:

- 1) $Q_\alpha = \sum_{\beta < \alpha} Q_\beta$ for limit ordinal α ;
- 2) $Q_{\alpha+1}/Q_\alpha \subseteq Z(Q/Q_\alpha)$.

A CML, possessing an ascending central series is called *ZA-loop*. If the ascending central series of CML is finite, then it is centrally nilpotent [3].

We will often use the following statements in our further proofs.

Lemma 6. *The following statements are equivalent for an arbitrary CML Q :*

- 1) Q satisfies the minimum condition for subloops;
- 2) Q is a direct product of a finite number of quasicyclic groups, belonging to the center CML Q , and a finite CML;
- 3) Q satisfies the minimum condition for invariant subloops;
- 4) Q satisfies the minimum condition for non-invariant associative subloops;
- 5) if Q contains a centrally nilpotent subloop of class n , then it satisfies the minimum condition for centrally nilpotent subloops of class n ;
- 6) if Q contains a centrally solvable subloop of class s , then it satisfies the minimum condition for centrally solvable subloops of class s ;
- 7) at least one maximal associative subloop of Q satisfies the minimum condition for subloops.

The equivalence of conditions 1), 2), 3) is proved in [4], the equivalence of conditions 1), 4), 5), 6) is proved in [5] and the equivalence of conditions 1), 7) is proved in [6].

Lemma 7 [4]. *The following statements are equivalent for an arbitrary non-associative CML Q with a multiplication group \mathfrak{M} :*

- 1) Q satisfies the minimum condition for subloops;
- 2) \mathfrak{M} satisfies the minimum condition for subgroup;
- 3) \mathfrak{M} is a product of a finite number of quasicyclic groups, lying in the center of \mathfrak{M} , and a finite group;
- 4) \mathfrak{M} satisfies the minimum condition for invariant subgroup;
- 5) at least one maximal abelian subgroup of \mathfrak{M} satisfies the minimum condition for subgroups;
- 6) if \mathfrak{M} contains a nilpotent subgroup of class n , then \mathfrak{M} satisfies the minimum condition for nilpotent subgroups of class n .
- 7) if \mathfrak{M} contains a solvable subgroup of class s , then \mathfrak{M} satisfies the minimum condition for solvable subgroups of class s .

Lemma 8 [4]. *If the center $Z(Q)$ of a commutative Moufang ZA-loop Q satisfies the minimum condition for subloops, then Q satisfies the minimum condition for subloop itself.*

Let us now consider an arbitrary non-periodic CML Q . Let $Q^3 = \{x^3 | x \in Q\}$. CML is di-associative [3], then it is easy to show that Q^3 is a subloop. It follows from Lemma 4 that $Q^3 \subseteq Z(Q)$, where $Z(Q)$ is the center of CML Q , therefore Q^3 is an invariant subloop of Q . Let us suppose that the subloop $Z(Q)$ is finitely generated. Then the abelian group Q^3 is also finitely generated. Therefore it decomposes into a direct product of cyclic groups $Q^3 = \langle r_1 \rangle \times \dots \times \langle r_k \rangle \times \langle s_1 \rangle \times \dots \times \langle s_m \rangle = \langle R \rangle \times \langle S \rangle$, where $\langle r_i \rangle$ are cyclic groups of infinite order, $\langle s_j \rangle$ are finite cyclic groups [7]. The group R is free abelian, therefore it is without torsion. It is shown in [3] that the associator loop Q' has the exponent three, then

$$R \cap Q' = \{1\}. \quad (1)$$

Lemma 9. *Let Q be a CML, R be its subloop, which is considered above, and let \overline{H} be a subloop of CML $Q/R = \overline{Q}$. The subloop \overline{H} satisfies one of the properties: 1) \overline{H} is centrally nilpotent of class n ; 2) \overline{H} is centrally solvable of class s ; 3) \overline{H} is a maximal associative subloop of CML \overline{Q} ; 4) \overline{H} is the center of CML \overline{Q} ; 5) \overline{H} is a non-invariant subloop of CML Q ; 6) \overline{H} is an invariant subloop of CML \overline{Q} if and only if the inverse image H of subloop \overline{H} has the same property as the subloop \overline{H} , under the homomorphism $\varphi : Q \rightarrow Q/R$.*

Proof. Let us suppose that the subloop \overline{H} is centrally nilpotent of class n . Let $h_1, h_2, \dots, h_{2n+1}$ be arbitrary elements from H . Let us denote $\varphi(h_i) = \overline{h}_i, \varphi(1) = \overline{1}$. Then $\overline{h}_i = h_i R, \overline{1} = R$. We have $(\overline{h}_1, \overline{h}_2, \dots, \overline{h}_{2n+1}) = \overline{1}, (h_1 R, h_2 R, \dots, h_{2n+1} R) = R$. But $R \subseteq Z(Q)$. Therefore, if $u \in R$, then $(au, b, c) = (a, b, c)$ for any elements $a, b, c \in Q$. Then $(h_1, h_2, \dots, h_{2n+1}) = r$, where $r \in R$. It follows from (1) that $r = 1$. We have obtained that $(h_1, h_2, \dots, h_{2n+1}) = 1$, i.e. the subloop H is centrally nilpotent of class n .

Conversely, let us suppose that the subloop H is centrally nilpotent of class n . Then there exist such elements $h_1, h_2, \dots, h_{2n-1}$ from H that $(h_1, h_2, \dots, h_{2n-1}) \neq 1$. It follows from (1) that $(h_1, h_2, \dots, h_{2n-1}) \notin R$. Therefore $(\overline{h}_1, \overline{h}_2, \dots, \overline{h}_{2n-1}) \notin \overline{1}$. Consequently \overline{H} , as homomorphic image of subloop H , will be a centrally nilpotent subloop of class n . It proves the statement 1). The statement 2) is proved by analogy.

Let us now suppose that \overline{H} is a maximal associative subloop of CML \overline{Q} and the inverse image H is not a maximal associative subloop of CML Q . Then there exists such an element $a \notin H$ that $(a, h_1, h_2) = 1$ for all $h_1, h_2 \in H$. Obviously $R \subseteq H$. Then $\varphi a = \overline{a} \notin \overline{H}$ and $(\overline{a}, \overline{h}_1, \overline{h}_2) = \overline{1}$ for all $\overline{h}_1, \overline{h}_2 \in \overline{H}$. We have obtained that the non-associative subloop $\langle \overline{a}, \overline{H} \rangle$, generated by the set $\{\overline{a}, \overline{H}\}$, strictly

contains \overline{H} , i.e. \overline{H} is not a maximal associative subloop of CML \overline{Q} . Contradiction. Consequently, H is a maximal associative subloop of CML Q .

Conversely, let us suppose that H is a maximal associative subloop of CML Q and \overline{H} is not a maximal associative subloop of CML \overline{Q} . Then there exists such an element $\overline{a} \notin \overline{H}$ that $(\overline{a}, \overline{h}_1, \overline{h}_2) = \overline{1}$ for all $\overline{h}_1, \overline{h}_2 \in \overline{H}$. We have obtained that $(aR, h_1R, h_2R) = R$ for all $h_1, h_2 \in H$. As $R \subseteq Z(Q)$, then $(a, h_1, h_2) = r$, where $r \in R$. It follows from (1) that $r = 1$, therefore $(a, h_1, h_2) = 1$ for all $h_1, h_2 \in H$ and $a \notin H$. It means that the subloop H is strictly contained in the associative subloop $\langle a, H \rangle$. We have obtained a contradiction with the fact that the subloop H is a maximal associative subloop. This proves statement 3). Statement 4) is proved by analogy.

Statements 5), 6) follow from the fact that the natural homomorphism $Q \rightarrow Q/R$ sets a one-to-one mapping between all non-invariant (respect. invariant) subloops of CML Q , with contained R , and all non-invariant (respect. invariant) subloops of CML Q/R . This completes the proof of Lemma 9.

Theorem 1. *The following statements are equivalent for an arbitrary non-associative CML Q :*

- 1) Q satisfies the maximum condition for subloops;
- 2) if Q contains a centrally nilpotent subloop of class n , then all its subloops of this type are finitely generated;
- 3) if Q contains a centrally solvable subloop of class s , then all its subloops of this type are finitely generated;
- 4) at least one maximal associative subloop of Q is finitely generated;
- 5) non-invariant associative subloops of Q are finitely generated;
- 6) invariant subloops of Q are finitely generated.

Proof. Let us suppose that the CML Q is non-periodic. It follows from Lemma 4 that the subloop Q^3 belongs to the center of CML Q . If H is a centrally nilpotent subloop of class n either a centrally solvable subloop of class s , or a maximal associative subloop, or a non-invariant associative subloop, or an invariant subloop, then the subloop $\langle H, Q^3 \rangle$ will be of this type too. Therefore it follows from the justice of one of the statements 2) – 6) of the theorem that the abelian group Q^3 is finitely generated. Then it decomposes into a direct product $Q^3 = R \times S$, where R is an abelian group without torsion, S is a finite abelian group [7]. It is obvious that CML Q/R is periodic. Then by Lemma 5 it is locally finite.

If the CML Q satisfies one of the conditions 2) – 6) of theorem, then by Lemma 9 CML Q/R satisfies this condition as well. Then all centrally nilpotent subloops of class n either all centrally solvable subloops of class s , or at least one maximal associative subloop, or all non-invariant associative subloops, or all invariant subloops are respectively finite in CML Q/R . Therefore by Lemma 6 the CML Q/R satisfies the minimum condition for subloops in any case. The center of CML Q/R is finite. Then by 2) of Lemma 6 the CML Q/R is finite. Therefore the CML Q is finitely generated and by Lemma 1, the condition 1) holds in it. It proves the implications 2) \rightarrow 1), 3) \rightarrow 1), 4) \rightarrow 1), 5) \rightarrow 1), 6) \rightarrow 1). The case when the

CML Q is periodic is contained in the proof of previous case. As the implications $1) \rightarrow 2), 1) \rightarrow 3), 1) \rightarrow 4), 1) \rightarrow 5), 1) \rightarrow 6)$ are obvious, the theorem is proved.

Theorem 2. *The following statements are equivalent for an arbitrary non-associative CML Q with the multiplication group \mathfrak{M} :*

- 1) Q satisfies the maximum condition for subloops;
- 2) \mathfrak{M} is finitely generated;
- 3) \mathfrak{M} satisfies the maximum condition for subgroups;
- 4) all invariant subgroups of \mathfrak{M} are finitely generated;
- 5) at least one maximal abelian subgroup of \mathfrak{M} is finitely generated;
- 6) if \mathfrak{M} contains a nilpotent subgroup of class n , then all its subgroups of this type are finitely generated;
- 7) if \mathfrak{M} contains a solvable subgroup of class s , then all its subgroups of this type are finitely generated.

Proof. If the CML Q satisfies the condition 1), then it is finitely generated, and by [3] the associator loop Q' is finite. By Lemma 2 the inner mapping group $I(Q)$ of Q is also finite. It is show in [8] that the relation

$$\mathfrak{M}(G/G') \cong \mathfrak{M}(G) / \langle I(G), \mathbf{M}(G') \rangle, \quad (2)$$

holds in an arbitrary CML G , where $\mathbf{M}(G')$ denotes a subgroup of the group $\mathfrak{M}(G)$, generated by the set $\{L(a) | a \in G'\}$. It is obvious that the group $\langle I(Q), \mathbf{M}(Q') \rangle$ is finitely generated in our case. As the abelian group $\mathfrak{M}(Q/Q')$ is finitely generated, then it follows from (2) that the group \mathfrak{M} is finitely generated as well. Consequently, $1) \rightarrow 2)$.

If the group \mathfrak{M} is finitely generated, then by Lemma 3 it is nilpotent. It is known (for instance, see [7]) that the maximum condition for subgroups holds in such groups.

Let $Z(Q)$ be the center of an arbitrary CML Q , $\{Z(\mathfrak{M})\}$ be the upper central series of its multiplication group $\mathfrak{M}(Q)$. Then

$$Z(Q) \cong Z(\mathfrak{M}). \quad (3)$$

Indeed, if $\varphi \in Z(\mathfrak{M})$, then $\varphi L(x) = L(x)\varphi$ for any $x \in Q$. Further, $\varphi L(x)y = L(x)\varphi y, \varphi(xy) = x\varphi y$. Let $y = 1$. Then $\varphi x = x\varphi 1, \varphi x = L(\varphi 1)x, \varphi = L(\varphi 1)$. Now, using the equality $\varphi(xy) = x\varphi y$ we obtain that $xy \cdot \varphi 1 = \varphi(xy) = x \cdot \varphi y = x \cdot \varphi(y \cdot 1) = x \cdot y\varphi 1$. Consequently, if $\varphi \in Z(\mathfrak{M})$, then $\varphi = L(a)$ and $a \in Z(Q)$. Conversely, let $a \in Z(Q)$. Then $a \cdot xy = ax \cdot y, L(a)L(y)x = L(y)L(a)x, L(a)L(y) = L(y)L(a)$. It follows from the definition of group \mathfrak{M} that $L(a) \in Z(\mathfrak{M})$. Finally, if $a, b \in Z(Q)$, then the homomorphism (3) follows from the equalities $a \cdot bx = ab \cdot x, L(a)L(b)x = L(ab)x, L(a)L(b) = L(ab)$.

In order to prove the implication $3) \rightarrow 1)$ we use the relation

$$\mathfrak{M}/Z_2(\mathfrak{M}) \cong \mathfrak{M}(Q/Z(Q)) \quad (4)$$

which takes place in an arbitrary CML [3]. By Lemma 2 the group $\mathfrak{M}/Z(\mathfrak{M})$ is periodic. Then the group $\mathfrak{M}/Z_2(\mathfrak{M})$, as an homomorphic image of the group $\mathfrak{M}/Z(\mathfrak{M})$, is also periodic. If the group \mathfrak{M} satisfies the maximum condition for subgroups, then the center $Z(\mathfrak{M})$ and by (3), also the center $Z(Q)$, are finitely generated. By Lemma 3 the group \mathfrak{M} is nilpotent. Then the group $\mathfrak{M}/Z_2(\mathfrak{M})$ is also nilpotent and, as it is periodic, then is finite. Hence it follows from (4) that the CML $Q/Z(Q)$ is also finite. Therefore the CML Q is finitely generated and by Lemma 1, the condition 1) holds in it. Consequently, 3) \rightarrow 1).

Let us now suppose that the group \mathfrak{M} is non-periodic. By Lemma 2 the group $\mathfrak{M}/Z(\mathfrak{M})$ is locally finite. It α is an element of infinite order in \mathfrak{M} , then $\alpha^n \in Z(\mathfrak{M})$ for a certain natural number n . We denote by \mathfrak{N} the subgroup of group \mathfrak{M} , generated by all elements of the form α^n . It is obvious that the abelian group $Z(\mathfrak{M})$ is finitely generated if the group \mathfrak{M} satisfies one of the conditions 4) – 7). Then $Z(\mathfrak{M}) = \mathfrak{N} \times \mathfrak{S}$, where \mathfrak{N} is a finitely generated abelian group without torsion, \mathfrak{S} is a finite abelian group [7] and $\mathfrak{N} = \mathfrak{N}$. As $\mathfrak{N} \cap \mathfrak{S} = \{1\}$, then $Z(\mathfrak{M})/\mathfrak{N} = (\mathfrak{N} \times \mathfrak{S})/\mathfrak{N} \cong \mathfrak{S}$. By Lemma 2 the group $\mathfrak{M}/Z(\mathfrak{M})$ is locally finite. It follows from the relation $\mathfrak{M}/Z(\mathfrak{M}) \cong (\mathfrak{M}/\mathfrak{N})/(Z(\mathfrak{M})/\mathfrak{N})$ that the group $\mathfrak{M}/\mathfrak{N}$ is the extension of the finite group $Z(\mathfrak{M})/\mathfrak{N}$ by locally finite group $\mathfrak{M}/Z(\mathfrak{M})$. Therefore the group $\mathfrak{M}/\mathfrak{N}$ is locally finite.

By Lemma 2 the commutator group \mathfrak{M}' is locally finite and as the group \mathfrak{N} is without torsion, then

$$\mathfrak{N} \cap \mathfrak{M}' = \{1\}.$$

Let either the condition 4), or 5), or 6), or 7) hold in group \mathfrak{M} . By analogy with the proof of Lemma 9 we can show that in the group $\mathfrak{M}/\mathfrak{N}$ a condition analogue with either conditions 4), or 5), or 6), or 7) holds. We have already shown that group $\mathfrak{M}/\mathfrak{N}$ is locally finite. Then either all invariant subgroups, or at least one maximal abelian subgroup, or all nilpotent of class n subgroups, or all solvable of class s subgroups are finite respectively in $\mathfrak{M}/\mathfrak{N}$. The group $Z(\mathfrak{M})/\mathfrak{N}$ is finite, then it follows from the relation

$$\mathfrak{M}/Z(\mathfrak{M}) \cong (\mathfrak{M}/\mathfrak{N})/(Z(\mathfrak{M})/\mathfrak{N})$$

that in the group $\mathfrak{M}/Z(\mathfrak{M})$ the same condition as in group $\mathfrak{M}/\mathfrak{N}$ holds. Further, it follows from the relation

$$\mathfrak{M}/Z_2(\mathfrak{M}) \cong (\mathfrak{M}/Z_1(\mathfrak{M}))/(Z_2(\mathfrak{M})/Z_1(\mathfrak{M})) = (\mathfrak{M}/Z_1(\mathfrak{M}))/Z(\mathfrak{M}/Z_1(\mathfrak{M}))$$

that $\mathfrak{M}/Z_2(\mathfrak{M})$ satisfies the same condition as the group $\mathfrak{M}/Z_1(\mathfrak{M})$, and it follows from (4) that $\mathfrak{M}(Q/Z(Q))$ satisfies this condition as well, i.e. either all its invariant subgroups are finite, or at least one maximal abelian subgroup is finite, or all its nilpotent subgroups of class n are finite, or all its solvable subgroups of class s are finite. In such a case, by Lemma 7 the group $\mathfrak{M}(Q/Z(Q))$ satisfies the minimum condition for subgroups. It is obvious that the center of the group $\mathfrak{M}(Q/Z(Q))$ is

finite. Then by 2) of Lemma 7 the group $\mathfrak{M}(Q)/Z(Q)$ is finite, and consequently, the CML $Q/Z(Q)$ is also finite. The center $Z(\mathfrak{M})$ of the group \mathfrak{M} is finitely generated, then it follows from (3) that the center $Z(Q)$ of Q is finitely generated, too. Then the CML Q is finitely generated and by Lemma 1 it satisfies the condition 1). Consequently, if the group \mathfrak{M} is non-periodic the implications 4) \rightarrow 1), 5) \rightarrow 1), 6) \rightarrow 1), 7) \rightarrow 1) hold.

The case when the group \mathfrak{M} is periodic is proved by analogy for $\mathfrak{N} = \{1\}$. Further, as the implications 3) \rightarrow 4), 3) \rightarrow 5), 3) \rightarrow 6), 3) \rightarrow 7) are obvious, the theorem is proved.

Theorem 3. *The following conditions are equivalent for an arbitrary non-associative commutative Moufang ZA-loop Q with the multiplication group \mathfrak{M} :*

- 1) Q satisfies the maximum condition for subloops;
- 2) if Q contains a non-invariant (respect. invariant) centrally nilpotent subloop of class n , then at least one maximal non-invariant (respect. invariant) centrally nilpotent subloop of class n is finitely generated;
- 3) if Q contains a non-invariant (respect. invariant) centrally solvable subloop of class s , then at least one maximal non-invariant (respect. invariant) centrally solvable subloop of class s is finitely generated;
- 4) if Q contains a non-invariant (respect. invariant) centrally nilpotent subloop of class n , then it satisfies the maximum condition for non-invariant (respect. invariant) centrally nilpotent subloops of class n ;
- 5) if Q contains a non-invariant (respect. invariant) centrally solvable subloop of class s , then it satisfies the maximum condition for non-invariant (respect. invariant) centrally solvable subloops of class s ;
- 6) the center $Z(Q)$ of CML Q is finitely generated;
- 7) the group \mathfrak{M} is finitely generated;
- 8) if \mathfrak{M} contains a non-invariant (respect. invariant) nilpotent subgroup of class n , then at least one maximal non-invariant (respect. invariant) nilpotent subgroup of class n is finitely generated;
- 9) if \mathfrak{M} contains a non-invariant (respect. invariant) solvable subgroup of class s , then at least one maximal non-invariant (respect. invariant) solvable subgroup of class s is finitely generated;
- 10) if \mathfrak{M} contains a non-invariant (respect. invariant) nilpotent subgroup of class n , then it satisfies the maximum condition for non-invariant (respect. invariant) nilpotent subgroups of class n ;
- 11) if \mathfrak{M} contains a non-invariant (respect. invariant) solvable subgroup of class s , then it satisfies the maximum condition for non-invariant (respect. invariant) solvable subgroups of class s ;
- 12) the center $Z(\mathfrak{M})$ of group \mathfrak{M} is finitely generated.

Proof. The implications 1) \rightarrow 2), 1) \rightarrow 3), 1) \rightarrow 4), 1) \rightarrow 5), 1) \rightarrow 6) are obvious. If H is a non-invariant (respect. invariant) centrally nilpotent of class n (or centrally solvable of class s) subloop of the CML Q , then the subloop $\langle N, Z(Q) \rangle$ will be of this type too. Therefore by Lemma 1 the implications 2) \rightarrow 6), 3) \rightarrow 6),

4) \rightarrow 6), 5) \rightarrow 6) hold. Let us now suppose that the condition 6) holds in the CML Q and let R be an invariant subloop, defined in Lemma 9. By 4) of Lemma 9 the center $Z(Q/R)$ is finitely generated and periodic. It follows from Lemma 5 that $Z(Q/R)$ is finite, and it follows from Lemma 8 that the CML Q/R satisfies the minimum condition for subloops. As the center $Z(Q/R)$ is finite, then by 2) of Lemma 6, the CML Q/R is finite. By its construction, the subloop L is finitely generated, therefore the CML Q is also finitely generated and the justice of condition 1) follows from Lemma 1. Consequently, the conditions 1), 2), 3), 4), 5), 6) are equivalent.

The equivalence of conditions 7), 8), 9), 10), 11), 12) is proved by analogy, using 1), 2) of theorem 2. Finally, the equivalence of conditions 6), 12) follows from (3). Therefore the theorem is fully proved.

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