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On the coproducts of cyclics in commutative modular and semisimple group rings

Peter Danchev

Abstract. We study certain properties of the coproducts (= direct sums) of cyclic groups in commutative modular and semisimple group rings. Our results strengthen a statement due to T. Zh. Mollov (Pliska, Stud. Math. Bulgar., 1981) and also they may be interpreted as a natural continuation of a recent investigation of ours (Serdica Math. J., 2003).

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1 Introduction

Let G be an arbitrary multiplicative abelian group and let R be an arbitrary commutative unitary ring of any characteristic. By using standard abbreviations, we now formally introduce the basic concepts. For such a group G, G_p denotes its p-primary component of torsion which can be represented as $G_p = \cup_{n<\omega} G[p^n]$ where $G[p^n] = \{g \in G : g^{p^n} = 1\}$ is the p^n-socle of G, and G^1 denotes the first Ulm subgroup of G. For such a ring R, $char(R)$ denotes its characteristic which is a non-negative integer m with the property that either $m.1_R = 0$ and $m \neq 0$ is minimal with this property, whence we write $char(R) = m \neq 0$, or otherwise $char(R) = 0$ whenever $m \neq n$ implies $m.1_R \neq n.1_R$.

We recall that the field F is of the first kind with respect to the prime p if the degree $(F(\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots) : F) = \infty$, where ϵ_i are the primitive p^i-roots of unity for $i = 1, 2, \cdots, n, \cdots$. In the remaining variant when $(F(\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots) : F) < \infty$, the field F is called a field of the second kind with respect to p.

An example of a field of the first kind with respect to any prime number is the field $\mathbb{Q}$ of all rationals, while the fields $\mathbb{R}$ and $\mathbb{C}$ of all real and complex numbers, respectively, are examples of fields of the second kind with respect to all primes.

Our attention in the present exploration is concentrated on the Sylow p-group $S(RG)$ consisting of all normalized p-elements in the group ring RG. For $A \leq G$, we denote by $I(RG; A)$ the relative augmentation ideal of RG with respect to the subgroup A.

All other unexplained exclusively notions and notation are standard and are in agreement with the cited in the bibliography research papers.

The work is structured as follows: In the second paragraph, divided into two sections, we study the behavior of coproducts of cyclic groups in a modular and a semisimple aspect.

The first section treats the modular case by finding a criterion for $S(RG)/S(RA)$ to be a coproduct of cyclic groups provided that $char(R) = p$, p is a prime, under some additional restrictions on G and A; for instance such as G is torsion and $\coprod_{q \neq p} G_q \subseteq A$. As a consequence, we obtain a necessary and sufficient condition for $S(RG)$ to be a $C_{\omega+1}$-group in the sense of Megibben (= a *pillared* group in other terms due to P. D. Hill). These criteria expand the corresponding ones of T. Mollov in [12]. We also point out that a counterexample due to Mollov in his reviewer's report [16], concerning our assertion in [4], is an absurdity.

The second section deals with the semisimple case, where we establish criteria for $S(RG)$ to belong to major classes of abelian groups provided R is either a field of the first kind or of the second kind with respect to p, both with $char(R) \neq p$. To this aim, we deduce a new extension of a group-theoretic attainment of Dieudonné [11].

We also eliminate some minor misprints in [4].

We end the paper with concluding discussion and remarks where we list certain unanswered questions and conjectures.

2 Main Results

1. *Modular group rings*

The next affirmation was proved by Mollov [12, Proposition 6 b)] in a slightly modified form for the case when R is a field. In his proof such a limitation on R to be a field is essential. We now generalize this claim to an arbitrary commutative ring R with 1 and prime characteristic p by the usage of a different and more smooth approach.

We are now in a position to restate in an equivalent form the cited above Mollov's result.

Theorem 1. *Suppose G is an abelian group with a subgroup A, suppose M is a p-divisible group with $M_p = 1$ and suppose $char(R) = p$ is a prime. Then $S(R(G_p \times M))/S(R(A_p \times M))$ is a coproduct of cyclic groups if and only if G_p/A_p is a coproduct of cyclic groups.*

Proof. Evidently, $G_p/A_p \cong G_pS(R(A_p \times M))/S(R(A_p \times M)) \subseteq S(R(G_p \times M))/S(R(A_p \times M))$ since $G_p \cap S(R(A_p \times M)) = A_p$. Thus the necessity obviously holds.

In order to prove the sufficiency, we assume that G_p/A_p is a coproduct of cyclic groups. Appealing to the classical Kulikov's criterion, $G_p/A_p = \cup_{n<\omega}(G_n/A_p)$, $A_p \subseteq G_n \subseteq G_{n+1} \leq G_p$ and $G_n \cap G_p^{p^n} \subseteq A_p$ that is equivalent to $G_n \cap G^{p^n} \subseteq A_p$. Therefore, $G_p = \cup_{n<\omega} G_n$ and $G_p \times M = \cup_{n<\omega}(G_n \times M)$, whence $S(R(G_p \times M)) = \cup_{n<\omega} S(R(G_n \times M))$ and $S(R(G_p \times M))/S(R(A_p \times M)) = \cup_{n<\omega}[S(R(G_n \times M))/S(R(A_p \times M))]$. Finally, by exploiting the modular law, we compute that

$S(R(G_n \times M)) \cap S^{p^n}(R(G_p \times M)) = S(R(G_n \times M)) \cap S(R^{p^n}(G_p^{p^n} \times M)) = S(R^{p^n}[(G_n \times M) \cap (G_p^{p^n} \times M)]) \subseteq S(R(M \times A_p))$, where the last inclusion holds because of the relationships $M \subseteq G^{p^n}$ and $(G_n \times M) \cap (G_p^{p^n} \times M) = M((G_n \times M) \cap G_p^{p^n}) = M((G_n \times M) \cap G^{p^n}) = M(G_n \cap G^{p^n}) \subseteq M \times A_p$ which are fulfilled over every natural $n \in \mathbb{N}$. This substantiates the claim that the quotient $S(R(G_p \times M))/S(R(A_p \times M))$ is a coproduct of cyclic groups. Hence the theorem.

Corollary 1. *Suppose G is an abelian group such that $G = G_p \times M$, where M is p-divisible, suppose $A \leq G$ such that $M \subseteq A$ and suppose $char(R) = p$ is a prime. Then $S(RG)/S(RA)$ is a coproduct of cyclic groups $\Longleftrightarrow G/A$ is a coproduct of cyclic groups $\Longleftrightarrow G_p/A_p$ is a coproduct of cyclic groups.*

Proof. It easily follows that $A = A_p \times M$. Therefore Theorem 1 applies to show that $S(RG)/S(RA)$ is a coproduct of cycles only when so is $G/A \cong G_p/A_p$.

Corollary 2. *Suppose that G is a torsion abelian group with $\coprod_{q \neq p} G_q \subseteq A \leq G$, and suppose that $char(R) = p$ is a rational prime. Then $S(RG)/S(RA)$ is a coproduct of cyclic groups $\Longleftrightarrow G/A \cong G_p/A_p$ is a coproduct of cyclic groups.*

Proof. It follows immediately from Theorem 1 because $G = G_p \times \coprod_{q \neq p} G_q$, where the latter direct component is p-divisible.

We come now to the original formulation of the Mollov's statement from [12].

Corollary 3 (Mollov, 1981). *Assume that G is an abelian group, p a prime, $G = G_p \times M, A \leq G_p$, and that R is a field of $char(R) = p$. If M is a p-divisible group, then $S(RG)/S(R(A \times M))$ is a coproduct of cyclic groups $\Longleftrightarrow G_p/A$ is a coproduct of cyclic groups.*

Proof. The assertion follows at the substitutions $G = G_p \times M$ and $A = A_p$.

Remark. Proposition 6 a) in [12] was firstly extended in [2] to arbitrary abelian groups and commutative rings with identity of prime characteristic p without nilpotent elements. When G is a p-group, similar expansions of Propositon 6 b) were given by us in [3]. The real advantage here is that we have considered a more general coefficient ring.

For any ordinal number λ an abelian p-group H is termed by C. Megibben a C_λ-group if H/H^{p^α} is totally projective for all $\alpha < \lambda$. Apparently, every abelian p-group is a C_ω-group.

Our next statement concerns the finding of a criterion when $S(RG)$ is a $C_{\omega+1}$-group, that is, the first Ulm factor $S(RG)/S^1(RG)$ is a coproduct of cyclic groups (these groups are called also *pillared* by P.D. Hill).

Proposition 1. *Let G be a torsion abelian group and R a perfect commutative ring with 1 of prime $char(R) = p$. Then $S(RG)$ is a $C_{\omega+1}$-group $\Longleftrightarrow G_p$ is a $C_{\omega+1}$-group.*

Proof. Follows directly from Theorem 1 by putting $A_p = G_p^1$.

We close the modular case with some special critical commentaries (for a convenience, the symbols are as in [4]): The purported in [16] counterexample of Mollov on the Proof 2 of Proposition 1 from [4] is demonstrably false. In fact, Mollov constructed an abelian p-group A so that $A = H \times L$ where H is an infinite coproduct of cycles and L is finite cyclic of order p. Henceforth, $A^p = H^p \subseteq H$. Besides, the Mollov's choice $H_k = H$ is tendentious since in our proof in [4] $H = \cup_{k=1}^{\infty} H_k, H_k \subseteq H_{k+1}$ and $H_k \cap H^{p^{s_k}} = 1$, for each $k < \omega$ and some $s_k \in \mathbb{N}$, where H_k are **proper** subgroups of H (see, for instance, the well-known criterion of Kulikov for coproducts of torsion cyclic groups).

2. Semi-simple group rings

We start here with the specification that in [4] (e.g. the Abstract of [4]) the letter K denotes the first kind field with respect to p of $char(K) \neq p$ with the extra restriction that its spectrum $s_p(K)$ about p contains all naturals that is valid only in the situations when we consider the purity or the direct factor properties of G in $S(KG)$ (i.e. in Lemma on purity on p.38, Theorem 7, Corollary 8, Claim 13, and the comments after Problem 17).

Moreover, as it was truly noted in [16], the criteria (4) and (5), respectively (4') and (5'), of [4] are true only in the infinite case. The word "infinite" was omitted involuntarily. Also in (5) and (5') the condition that the abelian p-group A is a "direct sum of cyclics" follows at once by the relation $A^{p^i} = 1$, which means that A is bounded, so it is out of use and can be dropped.

As usual, (p^t) designates a cyclic group of order p^t.

Now, we shall prepare the finite case.

Claim 1. *Let G be an abelian p-group and let R be a field of the first kind with respect to p of $char(R) \neq p$. Then $S(RG)$ cannot be non-trivial finite elementary or finite reduced homogeneous no elementary.*

Proof. If G is a finite abelian p-group with exponent $exp(G) = p^j, j \geq 1$ and R is a field of the first kind with respect to p of $char(R) \neq p$, consulting with ([13], [15]), we write $S(RG) \cong \coprod_{\delta_{i_0}-1}(p^{i_0}) \times \coprod_{\delta_{i_1}}(p^{i_1}) \times \cdots \times \coprod_{\delta_{i_r}}(p^{i_r})$, where $i_0, i_1, \cdots, i_r \in s_p(R) = \{i_0, i_1, i_2, \cdots | i_0 < i_1 < i_2 < \cdots\}$, i_r is the minimal number so that $i_r \geq j$, $i' \in s_p(R)$ plus the condition that $i' < i$ is the minimal number with this property if it generally exists and where the numbers δ_i are described in the following manner: $\delta_i = (|G[p^i]| - |G[p^{i'}]|)/(R(\varepsilon_i) : R), i \neq i_0$; $\delta_i = |G[p^{i_0}]|, i = i_0$. Moreover, $\delta_i = 0$ whenever $i > i_r$ and $i \in s_p(R)$, since for any $t \in \mathbb{N} : G[p^t] = G[p^{t+1}] \iff G^{p^t} = 1 \iff G = G[p^t]$ as well as $\delta_{i_0} = 1 \iff G[p^{i_0}] = 1 \iff G = 1$ because $i_0 \geq 1$. This shows that our claim really holds true.

Claim 2. *Let G be a finite abelian p-group and R a field of the second kind with respect to p of $char(R) \neq p$. Then $S(RG)$ is elementary $\iff p = 2, R \neq R(\varepsilon_2)$ and $G^p = 1$.*

Proof. Referring to [13,14] it follows that $S(RG) \cong \coprod_{|G|-1}(p)$, where $p = 2, R \neq R(\varepsilon_2)$ and $|G| = |G[p]|$. Since G is finite and $G[p] \subseteq G$, we have $G = G[p]$, so $G^p = 1$.

Remark. In the criteria (5) and (5') from [4] we observe that $i = exp(G) \notin s_p(K)$, otherwise $S(KA)$ is not, never, reduced homogeneous no elementary. Hence we conclude that $s_p(K)$ does not contain all naturals in that situation.

We strongly emphasize that in assertions (7) and (7') of [4] there is a typos. Below we give the correct formulating.

In fact, "$S(KA)$, respectively $U_p(KA)$, is p^α-projective for some $\alpha \geq \omega \iff A$ is a direct sum of cyclics" should be written and read as "$S(KA)$, respectively $U_p(KA)$, is p^α-projective for some $\alpha \leq \omega \iff A$ is (a bounded or an unbounded) direct sum of cyclics".

For the case when $\alpha \geq \omega + 1$, the problem seems to be very difficult. In the next lines, we obtain some partial advantage in this way. In order to do this, we foremost note the classical fact that in [11] it was firstly constructed by Dieudonné the existence of a separable $p^{\omega+1}$-projective abelian p-group which is not a coproduct of cycles.

We formulate the following conjecture, namely:

Conjecture 1. Assume that G is an abelian p-group and that R is the first kind field with respect to p of $char(R) \neq p$. Then $S(RG)$ is p^α-projective for $\alpha \geq \omega + 1$ $\iff S(RG)$ is separable p^α-projective $\iff G$ is separable p^α-projective.

The first implication follows easily, because by [14] we have that $S^1(RG) = 1$ since $S(RG)$ reduced. The second one is difficult. This difficulty ensues via the following reason, thus contrasting with the modular case (see [2]). Utilizing the Nunke's criterion for $p^{\omega+n}$-projectivity where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (see, for instance, [17]), if G is $p^{\omega+n}$-projective then there is $P \leq G[p^n]$ so that G/P is a coproduct of cyclic groups. On the other hand, $S(RG)/[(1 + I(RG; P)) \cap S(RG)] \cong S(R(G/P))$ is a coproduct of cyclic groups according to [14], whereas $(1+I(RG; P)) \cap S(RG)$ is not always bounded at p^n (i.e. equivalently is not always a subgroup of $S(RG)[p^n]$).

Now, we shall verify the second implication of the foregoing stated conjecture under the truthfulness of another conjecture (see [4, Problem 17]). Indeed, we assume that the *Generalized Direct Factor Conjecture* ([4, Problem 17]), which says that $S(RG)/G$ is a coproduct of cyclic groups whenever $G^1 = 1$ under the same circumstances on R and G as in Conjecture 1, holds in the affirmative and also additionally assume that the spectrum $s_p(R)$ of R about p, defined as in ([13-15]), contains all natural numbers, that is $\mathbb{N} \subseteq s_p(R)$. Invoking [7] or the Lemma from [4], G is pure in $S(RG)$. When there are gaps in the $s_p(R)$, we know that G is not always pure in $S(RG)$.

Furthermore, we distinguish two different approaches.

First Approach. Bearing in mind a classical theorem of L.Ya. Kulikov, G must be a direct factor of $S(RG)$, thus writing $S(RG) \cong G \times S(RG)/G$. Thereby $S(RG)$ is completely characterized by making use of [8].

Consequently, $S(RG)$ is separable p^α-projective for any ordinal $\alpha \iff G$ is separable p^α-projective for that α.

Second Approach. We shall apply successfully the following new version of the Generalized Dieudonné Criterion (for a strengthening in other ways see also [9] and [10]).

Criterion. *Suppose $A \leq G$ so that G/A is a coproduct of cyclic groups. Then, for some $m \geq 0$, G is $p^{\omega+m}$-projective $\iff \exists\, C \leq A[p^m] : A = \cup_{n<\omega} A_n, C \subseteq A_n \subseteq A_{n+1}$*

and for every $n \in \mathbb{N}$*:* $A_n \cap G^{p^n} \subseteq C \iff \exists\, C \leq A[p^m] : A[p] = \cup_{n<\omega} B_n, B_n \subseteq B_{n+1}$ *and for each* $n \in \mathbb{N}$*:* $B_n \cap G^{p^n} \subseteq C$.

Proof. "$\Rightarrow$". With the aid of the Nunke's criterion [17], there is a p-group T such that $T \leq G[p^m], G = \cup_{n<\omega} G_n, T \subseteq G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^n} \subseteq T$. Therefore $A = \cup_{n<\omega} A_n$, where $A_n = G_n \cap A$, and we subsequently compute $A_n \cap G^{p^n} = G_n \cap G^{p^n} \cap A \subseteq A \cap T$. So, by the setting $A \cap T = C$, we are done. For the second equivalent form of the criterion, we observe that $A[p]C/C \subseteq (A/C)[p] = \cup_{n<\omega}(E_n/C)$ where $E_n \subseteq E_{n+1} \subseteq A$ and $E_n \cap G^{p^n} \subseteq C$, hence $A[p] = \cup_{n<\omega} E_n[p]$ where $B_n = E_n[p]$.

"$\Leftarrow$". Under the assumptions, there is $C \leq A[p^m] \leq G[p^m]$ so that A/C is a countable union of an ascending chain of subgroups with bounded in G/C heights. After this, because of the isomorphism $G/C/A/C \cong G/A$, the latter factor-group is a coproduct of cycles. Consequently, with the Dieudonné criterion [11] in hand (see [9] and [10] as well), we derive that G/C must be a coproduct of cyclic groups. Finally, Nunke's criterion in [17] is applicable to obtain the claim.

This terminates the proof of the criterion.

And so, since by hypotheses $S(RG)/G$ is a coproduct of cycles whenever $G^1 = 1$ and G is pure in $S(RG)$, the preceding criterion applies to show that $S(RG)$ is separable $p^{\omega+m}$-projective if and only if so does G; $m \in \mathbb{N}$.

Remark. If G/A is a coproduct of cycles and A is $p^{\omega+m}$-projective, it does not follow in general that G is $p^{\omega+m}$-projective as well.

Major consequences arise for various choices of the subgroup A, specifically when $A = G^p$; or $A = G[p]$ so $G/G[p] \cong G^p$; or $A = L$, a large subgroup of G (see [6]) - we notice that G/L is a coproduct of cyclic groups.

Finally, we remark that when R is a field of the second kind with respect to p of $char(R) \neq p$, $S(RG)$ is p^α-projective for some arbitrary ordinal $\alpha \iff S(RG)$ is p^ω-projective (= a coproduct of cycles). This is so since, by consulting with [14], $S(RG)$ is ever a coproduct of cyclic and quasi-cyclic groups (= a coproduct of cocyclic groups). But, on the other hand, it is reduced as being p^α-projective too.

We continue with two minor specifications of technical character.

Firstly, after "Lemma [9]" in [4], the sentence "... is a direct factor of $S(KA)$, hence of $V(KA)$ by [10]" should be more precise by "... is a direct factor of $S(KA)$, hence of $V(KA)$ by [10] *and [17]*". Moreover, in Proposition 9 of [4] the subgroup H should be pure in the separable p-group A, which condition on "purity" was omitted involuntarily.

Secondly, in [16], Mollov has criticized that we have not showed in Lemma 6 of [4] that $I = (1 + I(RG; C)) \cap S(RG)$ is a group whenever $C \leq G$. Of course, that this intersection I is a group follows plainly either owing to the mentioned after Conjecture 1 fact that I is the kernel of the homomorphism $S(RG) \to S(R(G/C))$ or in the following manner: Given $u_1 \in I$ and $u_2 \in I$. Hence $u_1 = 1 + v_1$ and $u_2 = 1 + v_2$, where $v_1, v_2 \in I(RG; C)$. Furthermore, $u_1u_2 = 1 + v_1 + v_2 + v_1v_2 \in I$ since it is a p-element and $I(RG; C)$ is a ring being an ideal. On the other hand, since $u_1 \in S(RG)$, there exists $k \in \mathbb{N}$ such that $u_1^{p^k} = 1$, whence $u_1^{-1} = u_1^{p^k-1} \in I$ employing inductively the previous step. So, the claim is true.

Finally, we answer two problems posed by us in [4] asked when $S(RG)$ is quasi-pure projective (q. p. p.) and quasi-pure injective (q. p. i.), provided R is the first kind field with respect to p of $char(R) \neq p$.

To this goal, we quote the following necessary and sufficient conditions argued in [1].

Criteria (Berlinghoff–Reid, 1977)

(1) The p-group G is q. p. i. $\Longleftrightarrow$ G is a coproduct of a divisible p-group and a torsion complete p-group.

(2) A non-reduced p-group G is q. p. p. $\Longleftrightarrow$ G is an algebraically compact p-group. A reduced p-group G is q. p. p. $\Longleftrightarrow$ G is a coproduct of cyclic p-groups.

And so, we proceed by proving the following.

Theorem 2. *Suppose G is an abelian p-group and R is a field of the first kind with respect to p of characteristic not equal to p. Then*

(1') $S(RG)$ is q. p. i. $\Longleftrightarrow$ G is algebraically compact, provided $\mathbb{N} \subseteq s_p(R)$.

(2') The non-reduced $S(RG)$ is q. p. p. $\Longleftrightarrow$ G is non-reduced algebraically compact. The reduced $S(RG)$ is q. p. p. $\Longleftrightarrow$ G is a coproduct of cyclic groups.

Proof. Point (1') follows directly from (1) and [5] (see [7] as well). The first part half of (2') holds in virtue of (2) combined with ([4], dependence 12) (see also [5]). The second one follows again by (2) along with [14].

The proof is completed.

3 Concluding Discussion

We conclude with certain questions and conjectures of interest. We mainly discuss here a problem related to [4] and pertaining to finding the explicit form of basic subgroups in semisimple group rings. The isomorphism classification of such basic subgroups was firstly established in ([4], Proposition 11). Nevertheless, the explicit type of these subgroups will definitely be of some importance.

We state the following:

Conjecture 2. Suppose that G is a separable abelian p-group with an arbitrary but fixed basic subgroup B and that R is the first kind field with respect to p of $char(R) \neq p$ so that $\mathbb{N} \subseteq s_p(R)$. Then $B' = [1 + I(RG; B)] \cap S(RG)$ is a basic subgroup of $S(RG)$.

We attack our claim like this: That B' is pure in $S(RG)$ and that $S(RG)/B' \cong S(R(G/B))$ is divisible follow not so hard from [4] and more especially subsequently referring to Lemma 6 and relation (8) of [4].

More complicated is the question how to derive that B' is a coproduct of cyclic groups. We show now that such a construction naturally depends on the foregoing used Generalized Direct Factor Conjecture (Problem 17 of [4]). Indeed, if we have a priori that $S(RG)/G$ is a coproduct of cyclic groups, we observe that $B'/B \cong B'G/G \subseteq S(RG)/G$ possesses this property as well. Moreover, appealing to [7], B being pure in G is pure even in $S(RG)$ whence in B'. Finally, the application of a classical theorem of L. Kulikov, used also above, ensures that $B' \cong B \times B'/B$ is a coproduct of cyclic groups, in fact, as wanted.

However, the complete proof is a theme of another research investigation, where a new simpler approach might work.

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13 General Kutuzov Street
bl. 7, floor 2, apart. 4
4003 Plovdiv, Bulgaria
E-mail: *pvdanchev@yahoo.com*

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