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Quotient rings of pseudonormed rings

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Abstract. The present article is devoted to the study of the connection between the restriction of a pseudonorm of a pseudonormed ring on various subrings and the pseudonorm of quotient ring. The basic results of this article were announced in [2].

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1 Introduction

1.1 Definition. A real function ξ on a ring R is called a pseudonorm if the following conditions are satisfied:

1. $\xi(x) \geqslant 0$ for all $x \in R$;
2. $\xi(x) = 0$ iff $x = 0$;
3. $\xi(x - y) \leqslant \xi(x) + \xi(y)$ for all $x, y \in R$;
4. $\xi(x \cdot y) \leqslant \xi(x) \cdot \xi(y)$ for all $x, y \in R$.

1.2 Remark. The condition 3 is equivalent to the following conditions: $\xi(x + y) \leqslant \xi(x) + \xi(y)$ and $\xi(-x) = \xi(x)$ for all $x, y \in R$.

1.3 Definition. The pseudonorm ξ is called a norm if the condition $\xi(x \cdot y) = \xi(x) \cdot \xi(y)$ is satisfied for all $x, y \in R$.

1.4 Remark. It is clear that any pseudonorm ξ defines some separated topology on a ring R. However, the same topology can be defined by various pseudonorms.

1.5 Definition. Let (R, ξ) and $(\bar{R}, \bar{\xi})$ be pseudonormed rings. A homomorphism $\varphi : R \to \bar{R}$ is called an isometric homomorphism if $\bar{\xi}(\varphi(x)) = \inf\{\, \xi(x + a) \mid a \in Ker\varphi\}$ for all $x \in R$.

If φ is also an isomorphism then the concept of isometric homomorphism coincides with the concept of isometric isomorphism in usual sense.

The following isomorphism theorem is frequently applied in algebra.

1.6 Theorem. *Let R be a ring and B be a subring of the ring R. If N is an ideal of the ring R then the quotient rings $B/(B \cap N)$ and $(B + N)/N$ are isomorphic.*

1.7 Remark. In particular, if the condition $B \cap N = \{0\}$ is satisfied in the theorem 1.6 then the rings B and $(B + N)/N$ are isomorphic.

1.8 Remark. Let (R, ξ) be a topological or pseudonormed ring. In order to formulate analogues of this theorem it is natural to demand that the isomorphism preserves the topology or the pseudonorm, respectively. So:

– if ξ is a topology then the isomorphism should be a homeomorphism;

– if ξ is a norm or a pseudonorm then the isomorphism should be an isometric isomorphism.

Therefore situation is more difficult in this case.

First, it is necessary to define the corresponding structure $\bar{\xi}$ (the topology or the pseudonorm, respectively) on the quotient ring R/A.

We shall consider one of the most natural definitions of $\bar{\xi}$ for the topology or the pseudonorm ξ.

A. If ξ is a topology then the topology $\bar{\xi}$ is defined by $\bar{\xi} = \sup\{\tau \mid \tau$ is a ring topology on R/A and the canonical homomorphism $f_A : (R, \xi) \to (R/A, \tau)$ is a continuous homomorphism $\}$ in topological algebra.

In this case $f_A : (R, \xi) \to \left(R/A, \bar{\xi}\right)$ is a surjective, continuous and open homomorphism. Such homomorphisms are called topological homomorphisms.

B. If ξ is a pseudonorm then the pseudonorm $\bar{\xi}$ is defined by the equality $\bar{\xi}\,(x + A) = \inf\{\, \xi\,(x + a) \mid a \in A\,\}$ in the theory of the normed rings, i.e. the canonical homomorphism $f_A : (R, \xi) \to \left(R/A, \bar{\xi}\right)$ is an isometric homomorphism (see Definition 1.5).

If ξ is a topology or a pseudonorm then the ring $\left(R/A, \bar{\xi}\right)$ is designated by $(R, \xi)/A$ hereinafter.

Second, theorem 1.6 is not always true for the above mentioned topology or the pseudonorm on the quotient rings R/A.

This article is devoted to the study of analogues of Theorem 1.6 for pseudonormed rings. (Analogues of Theorem 1.6 for topological rings have been investigated in [1]).

1.9 Remark. If (R, ξ) and $\left(\bar{R}, \bar{\xi}\right)$ are the pseudonormed rings, $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ is a surjection and an isometric homomorphism then the mapping $\tilde{f} : (R, \xi)/(Kerf) \to$ $\to \left(\bar{R}, \bar{\xi}\right)$ defined by the equality $\tilde{f}\,(r + Kerf) = f\,(r)$ is an isometric isomorphism.

1.10 Remark. The topology or the pseudonorm $\bar{\xi}$ defined above on the quotient ring R/A is a separated topology or a separated pseudonorm if and only if A is a closed ideal in the topological ring (R, ξ) or (R, τ_ξ), respectively.

If ξ is a pseudonorm then the topology $\tau_{\bar{\xi}}$ coincides with the topology on the topological ring $(R, \tau_\xi)/A$.

2 Basic results

2.1 Theorem. *Let (R, ξ) and $\left(\tilde{R}, \tilde{\xi}\right)$ be pseudonormed rings, $\varphi : R \mapsto \tilde{R}$ be a ring isomorphism. The inequality $\tilde{\xi}\,(\varphi\,(x)) \leqslant \xi\,(x)$ is satisfied for all $x \in R$ iff there exists:*

– *A pseudonormed ring* $\left(\hat{R}, \hat{\xi}\right)$ *such that the pseudonormed ring* (R, ξ) *is a subring of the pseudonormed ring* $\left(\hat{R}, \hat{\xi}\right)$*;*

– *An isometric homomorphism* $\hat{\varphi} : \left(\hat{R}, \hat{\xi}\right) \to \left(\tilde{R}, \tilde{\xi}\right)$ *such that* $\hat{\varphi}$ *is an extension of the isomorphism* φ*, i.e.*

$\hat{\varphi}(x) = \varphi(x)$ *and* $\tilde{\xi}(\varphi(x)) = \inf\left\{ \hat{\xi}(x+a) \mid a \in Ker\hat{\varphi} \right\}$ *for all* $x \in R$*.*

Proof. Necessity. Let the inequality $\tilde{\xi}(\varphi(x)) \leqslant \xi(x)$ be valid for all $x \in R$. We shall consider the ring $\hat{R}$ which is the direct product of rings R and $\tilde{R}$, i.e. $\hat{R} = \left\{ \hat{r} = \left(a, \tilde{b}\right) \,\middle|\, a \in R,\ \tilde{b} \in \tilde{R} \right\}$ is a ring with operations of addition $\hat{r}_1 + \hat{r}_2 = \left(a_1 + a_2, \tilde{b}_1 + \tilde{b}_2\right)$ and multiplication $\hat{r}_1 \cdot \hat{r}_2 = \left(a_1 \cdot a_2,\ \tilde{b}_1 \cdot \tilde{b}_2\right)$, where $\hat{r}_1 = \left(a_1, \tilde{b}_1\right)$ and $\hat{r}_2 = \left(a_2, \tilde{b}_2\right)$.

Let's define the pseudonorm $\hat{\xi}$ on the ring $\hat{R}$ as follows: $\hat{\xi}(\hat{r}) = \max\left\{ \xi(a), \tilde{\xi}\left(\tilde{b}\right) \right\}$, where $\hat{r} = \left(a, \tilde{b}\right)$. It is clear that the function $\hat{\xi}$ satisfies the axioms of pseudonorm.

Let's consider the subring $R' = \{ a' = (a, \varphi(a)) \mid a \in R \}$ of the ring $\hat{R}$. It follows from the inequality $\tilde{\xi}(\varphi(a)) \leqslant \xi(a)$ that

$$\xi'\left(a'\right) = \hat{\xi}((a, \varphi(a))) = \max\left\{ \xi(a), \tilde{\xi}(\varphi(a)) \right\} = \xi(a).$$

If we put in correspondence to an element $a \in R$ the element $(a, \varphi(a)) \in R'$ then the mapping defined by this rule is an isometric isomorphism of the pseudonormed rings (R, ξ) and (R', ξ'). Therefore we shall identify any element $a \in R$ with the element $(a, \varphi(a)) \in R'$. Hence, we shall not distinguish the pseudonormed rings (R, ξ) and (R', ξ'), i.e. we can assume that the pseudonormed ring (R, ξ) is a subring of the pseudonormed ring $\left(\hat{R}, \hat{\xi}\right)$.

We shall consider as mapping $\hat{\varphi} : \left(\hat{R}, \hat{\xi}\right) \to \left(\tilde{R}, \tilde{\xi}\right)$ the mapping defined by the equality $\hat{\varphi}\left(\left(a, \tilde{b}\right)\right) = \tilde{b}$. Then $\hat{\varphi}(a) = \hat{\varphi}((a, \varphi(a))) = \varphi(a)$ for any $a \in R$, i.e. the mapping $\hat{\varphi}$ is an extension of the isomorphism φ.

Then $Ker\hat{\varphi} = \left\{ \hat{r} \in \hat{R} \,|\, \hat{\varphi}(\hat{r}) = 0 \right\} = \left\{ \left(a, \tilde{b}\right) \in \hat{R} \,\middle|\, \tilde{b} = 0 \right\} = \{ (a, 0) \mid a \in R \}$ is an ideal of the ring $\hat{R}$ and

$$\begin{aligned} &\inf\left\{ \hat{\xi}(\hat{r} + \hat{a}) \mid \hat{a} \in Ker\hat{\varphi} \right\} = \inf\left\{ \hat{\xi}((r, \tilde{r}) + (a, 0)) \mid a \in R \right\} = \\ &= \inf\left\{ \hat{\xi}((r + a, \tilde{r})) \mid a \in R \right\} = \inf_{a \in R}\left\{ \max\left\{ \xi(r+a), \tilde{\xi}(\tilde{r}) \right\} \right\} \leqslant \\ &\leqslant \max\left\{ \xi(0), \tilde{\xi}(\tilde{r}) \right\} = \tilde{\xi}(\tilde{r}) = \tilde{\xi}(\hat{\varphi}((r, \tilde{r}))) = \tilde{\xi}(\hat{\varphi}(\hat{r})). \end{aligned}$$

Thus,

$$\tilde{\xi}(\hat{\varphi}(\hat{r})) \geqslant \inf\left\{ \hat{\xi}(\hat{r} + \hat{a}) \mid \hat{a} \in Ker\hat{\varphi} \right\}. \tag{1}$$

On the other hand, for any $a \in Ker\varphi$ and $\hat{r} = (r, \tilde{r}) \in \hat{R}$ the inequality

$$\max\left\{\xi(r+a), \tilde{\xi}(\tilde{r})\right\} \geqslant \tilde{\xi}(\tilde{r})$$

also takes place.

The set $\left\{\max\left\{\xi(r+a), \tilde{\xi}(\tilde{r})\right\} \mid a \in Ker\hat{\varphi}\right\}$ is bounded below by the number $\tilde{\xi}(\tilde{r})$, therefore $\inf\left\{\max\left\{\xi(r+a), \tilde{\xi}(\tilde{r})\right\} \mid a \in Ker\hat{\varphi}\right\} \geqslant \tilde{\xi}(\tilde{r})$. We have

$$\inf\left\{\hat{\xi}(\hat{r}+\hat{a}) \mid \hat{a} \in Ker\hat{\varphi}\right\} = \inf\left\{\hat{\xi}((r+a, \tilde{r})) \mid a \in R\right\} =$$

$$= \inf_{a \in Ker\hat{\varphi}}\left\{\max\left\{\xi(r+a), \tilde{\xi}(\tilde{r})\right\}\right\} \geqslant \tilde{\xi}(\tilde{r}) = \tilde{\xi}(\hat{\varphi}(\hat{r})).$$

Hence,

$$\inf\left\{\hat{\xi}(\hat{r}+\hat{a}) \mid \hat{a} \in Ker\hat{\varphi}\right\} \geqslant \tilde{\xi}(\hat{\varphi}(\hat{r})). \tag{2}$$

From inequalities (1) and (2) we shall receive the required equality:

$$\tilde{\xi}(\hat{\varphi}(\hat{r})) = \inf\left\{\hat{\xi}(\hat{r}+\hat{a}) \mid \hat{a} \in Ker\hat{\varphi}\right\}, \tag{3}$$

i.e. $\hat{\varphi} : \left(\hat{R}, \hat{\xi}\right) \to \left(\tilde{R}, \tilde{\xi}\right)$ is an isometric homomorphism.

Sufficiency. Let $(\hat{R}, \hat{\xi})$ be a pseudonormed ring and $\hat{\varphi} : \left(\hat{R}, \hat{\xi}\right) \to \left(\tilde{R}, \tilde{\xi}\right)$ be an isometric homomorphism such that the pseudonormed ring (R, ξ) is a subring of the pseudonormed ring $(\hat{R}, \hat{\xi})$ and the homomorphism $\hat{\varphi}$ is an extension of the isomorphism φ. Then

$$\xi(x) = \hat{\xi}(x) \geqslant \inf\left\{\hat{\xi}(x+a) \mid a \in Ker\hat{\varphi}\right\} = \tilde{\xi}(\varphi(x)),$$

i.e. the inequality $\tilde{\xi}(\varphi(x)) \leqslant \xi(x)$ is valid for any $x \in R$.

The theorem is proved.

2.2 Definition. Let (R, ξ) and $\left(\bar{R}, \bar{\xi}\right)$ be pseudonormed rings. An isomorphism $f : R \to \bar{R}$ is said to be a semi-isometric isomorphism if there exists a pseudonormed ring $\left(\hat{R}, \hat{\xi}\right)$ such that the following conditions are valid:

– the ring R is an ideal in the ring $\hat{R}$;

– $\hat{\xi}\Big|_R = \xi$;

– the isomorphism f can be extended up to an isometric homomorphism $\hat{f}$: $\left(\hat{R}, \hat{\xi}\right) \to (\bar{R}, \bar{\xi})$ of the pseudonormed rings, i.e.

$$\bar{\xi}\left(\hat{f}(\hat{r})\right) = \inf\left\{\hat{\xi}(\hat{r}+i) \,\Big|\, i \in Ker\hat{f}\right\} \quad \text{for all } \hat{r} \in \hat{R}.$$

2.3 Theorem. *Let (R,ξ) and $\left(\bar{R},\bar{\xi}\right)$ be pseudonormed rings and $f: R \to \bar{R}$ be a ring isomorphism. Then the following statements are equivalent:*

I. The isomorphism $f:(R,\xi) \to \left(\bar{R},\bar{\xi}\right)$ is a semi-isometric isomorphism of the pseudonormed rings;

II. $\dfrac{\xi(a\cdot b)}{\xi(b)} \leqslant \bar{\xi}(f(a)) \leqslant \xi(a)$ *and* $\dfrac{\xi(b\cdot a)}{\xi(b)} \leqslant \bar{\xi}(f(a)) \leqslant \xi(a)$ *for any* $a \in R$ *and* $b \in R\backslash\{0\}$;

III. There exist a pseudonormed ring $\left(\tilde{R},\tilde{\xi}\right)$ and a homomormism $\tilde{f}:\tilde{R}\to\bar{R}$ such that:

a) R is an ideal in the ring $\tilde{R}$, $\tilde{\xi}\Big|_R = \xi$ and $\tilde{f}\Big|_R = f$;

b) $\bar{\xi}(f(r)) = \min\left\{\tilde{\xi}(r+a)\;\middle|\; a \in Ker\tilde{f}\right\}$ for every $r \in R$, i.e. for every $r \in R$ there exists an element $a_r \in Ker\tilde{f}$ such that $\bar{\xi}(f(r)) = \tilde{\xi}(r+a_r)$.

Proof $I \Rightarrow II$**.**

1. Let $f:(R,\xi)\to\left(\bar{R},\bar{\xi}\right)$ be a semi-isometric isomorphism. Then it follows from Definition 2.2 that there exist a pseudonormed ring $\left(\hat{R},\hat{\xi}\right)$ and an isometric homomormism $\hat{f}:\left(\hat{R},\hat{\xi}\right)\to\left(\bar{R},\bar{\xi}\right)$ such that R is an ideal of the ring $\hat{R}$, $\hat{\xi}\Big|_R = \xi$ and $\hat{f}\Big|_R = f$.

Since $\hat{f}$ is an isometric homomorphism then $\bar{\xi}(\hat{f}(\hat{r})) = \inf\{\hat{\xi}(\hat{r}+i) \mid i \in Ker\hat{f}\}$ for any $\hat{r}\in\hat{R}$. It means that this equality is valid also for $r \in R$, i.e.

$$\bar{\xi}\left(\hat{f}(r)\right) = \inf\left\{\hat{\xi}(r+i)\;\middle|\; i \in Ker\hat{f}\right\}.$$

Since $\hat{\xi}\Big|_R = \xi$ and $\hat{f}\Big|_R = f$ then we have

$$\bar{\xi}(f(r)) = \bar{\xi}\left(\hat{f}(r)\right) = \inf\left\{\hat{\xi}(r+i)\;\middle|\; i\in Ker\hat{f}\right\} \leqslant \hat{\xi}(r+0) = \hat{\xi}(r) = \xi(r).$$

Thus the inequality $\bar{\xi}(f(r)) \leqslant \xi(r)$ is valid for any $r \in R$.

2. Let's show in the beginning that $R \cap Ker\hat{f} = \{0\}$.

Since $\hat{f}\Big|_R = f$ and $f: R \to \bar{R}$ is a ring isomorphism then $R\cap Ker\hat{f} = \left\{ i \in R \;\middle|\; \hat{f}(i) = 0\right\} = \{i \in R \mid f(i) = 0\} = \{0\}$.

3. Let's verify the inequality $\dfrac{\xi(r\cdot a)}{\xi(a)} \leqslant \bar{\xi}(f(r))$ for any $r\in R$, $a \in R\backslash\{0\}$. Let $j \in Ker\hat{f}$ and $\hat{r} = r + j \in \hat{R}$. Then $\hat{r}\cdot a = (r+j)\cdot a = r\cdot a + j\cdot a$.

Since $R\cap Ker\hat{f} = \{0\}$ then $(a\cdot j) \in R \cap Ker\hat{f} = \{0\}$. It means that $\hat{r}\cdot a = r\cdot a + 0 = r\cdot a \in R$. Then

$$\xi(r\cdot a) = \hat{\xi}(r\cdot a) = \hat{\xi}(\hat{r}\cdot a) \leqslant \hat{\xi}(\hat{r})\cdot\hat{\xi}(a) = \hat{\xi}(\hat{r})\cdot\xi(a) = \hat{\xi}(r+j)\cdot\xi(a).$$

Hence

$$\frac{\xi(r\cdot a)}{\xi(a)} \leqslant \hat{\xi}(r+j) \quad \text{for any} \quad j \in Ker\hat{f}.$$

The set $\left\{ \hat{\xi}(r+j) \,\middle|\, j \in Ker\hat{f} \right\}$ is bounded below by the number $\dfrac{\xi(r\cdot a)}{\xi(a)}$. It means that the number $\dfrac{\xi(r\cdot a)}{\xi(a)}$ is one of the lower bounds of that set. Therefore

$$\frac{\xi(r\cdot a)}{\xi(a)} \leqslant \inf\left\{ \hat{\xi}(r+i) \,\middle|\, i \in Ker\hat{f} \right\} = \bar{\xi}\left(\hat{f}(r)\right) = \bar{\xi}(f(r)).$$

The inequality $\dfrac{\xi(a\cdot r)}{\xi(a)} \leqslant \bar{\xi}\left(\hat{f}(r)\right) = \bar{\xi}(f(r))$ is similarly proved.

Hence $I \Rightarrow II$ is proved.

Proof $II \Rightarrow III$.

Let the mapping $f:(R,\xi) \to \left(\bar{R},\bar{\xi}\right)$ possesses the following properties:

f is an isomorphism;

$\bar{\xi}(f(a)) \leqslant \xi(a)$ for any $a \in R$;

$\dfrac{\xi(a\cdot b)}{\xi(b)} \leqslant \bar{\xi}(f(a))$ and $\dfrac{\xi(b\cdot a)}{\xi(b)} \leqslant \bar{\xi}(f(a))$ for any $a \in R$ and $b \in R\backslash\{0\}$.

Let's prove that the statement III is valid.

Let's consider the ring $\tilde{R} = R \oplus \bar{R} = \left\{ (r,\bar{r}) \,\middle|\, r \in R, \bar{r} \in \bar{R} \right\}$ which is the direct sum of the rings R and $\bar{R}$. Let's define the real-valued function $\tilde{\xi}$ on $\tilde{R}$ as follows:

$$\tilde{\xi}((r,\bar{r})) = \xi\left(r - f^{-1}(\bar{r})\right) + \bar{\xi}(\bar{r}).$$

Let's define the mapping $\tilde{f}:\left(\tilde{R},\tilde{\xi}\right) \to \left(\bar{R},\bar{\xi}\right)$ by the equality $\tilde{f}((r,\bar{r})) = f(r)$.

1. Let's show that $\tilde{\xi}$ is a pseudonorm on the ring $\tilde{R}$.

1.1. It is obvious that $\tilde{\xi}((r,\bar{r})) \geqslant 0$ for all $r \in R$ and $\bar{r} \in \bar{R}$ because the pseudonorms ξ and $\bar{\xi}$ accept non-negative values, i.e. the condition 1 of the definition of a pseudonorm is valid.

1.2. Since $\xi(x) = 0 \Leftrightarrow x = 0$ and $\bar{\xi}(\bar{y}) = 0 \Leftrightarrow \bar{y} = 0$ then $\tilde{\xi}((r,\bar{r})) = = 0 \Leftrightarrow \xi\left(r - f^{-1}(\bar{r})\right) + \bar{\xi}(\bar{r}) = 0 \Leftrightarrow \begin{cases} \xi\left(r - f^{-1}(\bar{r})\right) = 0 \\ \bar{\xi}(\bar{r}) = 0 \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} r - f^{-1}(\bar{r}) = 0 \\ \bar{r} = 0 \end{cases} \Leftrightarrow \begin{cases} r = 0 \\ \bar{r} = 0 \end{cases} \Leftrightarrow (r,\bar{r}) = 0.$$

Thus, the condition 2 of the definition of a pseudonorm is valid, i.e. $\tilde{\xi}((r,\bar{r})) = 0$ iff $(r,\bar{r}) = 0$.

1.3. Since the inequalities $\xi(x_1 - x_2) \leqslant \xi(x_1) + \xi(x_2)$ and $\bar{\xi}(\bar{y}_1 - \bar{y}_2) \leqslant \bar{\xi}(\bar{y}_1) + $ $+ \bar{\xi}(\bar{y}_2)$ are valid for any $x_1,\, x_2 \in R$ and $\bar{y}_1,\, \bar{y}_2 \in \bar{R}$ then

$$\tilde{\xi}\left((r - q, \bar{r} - \bar{q})\right) = \xi\left(r - q - f^{-1}(\bar{r} - \bar{q})\right) + \bar{\xi}(\bar{r} - \bar{q}) =$$

$$= \xi\left(r - q - f^{-1}(\bar{r}) + f^{-1}(\bar{q})\right) + \bar{\xi}(\bar{r} - \bar{q}) =$$
$$= \xi\left(\left(r - f^{-1}(\bar{r})\right) - \left(q - f^{-1}(\bar{q})\right)\right) + \bar{\xi}(\bar{r} - \bar{q}) \leqslant$$
$$\leqslant \xi\left(r - f^{-1}(\bar{r})\right) + \xi\left(q - f^{-1}(\bar{q})\right) + \bar{\xi}(\bar{r}) + \bar{\xi}(\bar{q}) =$$
$$= \left(\xi\left(r - f^{-1}(\bar{r})\right) + \bar{\xi}(\bar{r})\right) + \left(\xi\left(q - f^{-1}(\bar{q})\right) + \bar{\xi}(\bar{q})\right) = \tilde{\xi}\left((r, \bar{r})\right) + \tilde{\xi}\left((q, \bar{q})\right).$$

We have shown that the condition 3 of the definition of a pseudonorm is valid, i.e. $\tilde{\xi}\left((r - q, \bar{r} - \bar{q})\right) \leqslant \tilde{\xi}\left((r, \bar{r})\right) + \tilde{\xi}\left((q, \bar{q})\right)$ for all $r,\, q \in R$ and $\bar{r},\, \bar{q} \in \bar{R}$.

1.4. Let's verify the inequality $\tilde{\xi}\left((r \cdot q, \bar{r} \cdot \bar{q})\right) \leqslant \tilde{\xi}\left((r, \bar{r})\right) \cdot \tilde{\xi}\left((q, \bar{q})\right)$ for any $r,\, q \in R$ and $\bar{r},\, \bar{q} \in \bar{R}$.

Really,

$$\tilde{\xi}\left((r \cdot q, \bar{r} \cdot \bar{q})\right) = \xi\left(r \cdot q - f^{-1}(\bar{r} \cdot \bar{q})\right) + \bar{\xi}(\bar{r} \cdot \bar{q}) =$$
$$= \xi\left(r \cdot q - f^{-1}(\bar{r}) \cdot f^{-1}(\bar{q})\right) + \bar{\xi}(\bar{r} \cdot \bar{q}) =$$
$$= \xi\left(\left(r \cdot q - r \cdot f^{-1}(\bar{q})\right) + \left(r \cdot f^{-1}(\bar{q}) - f^{-1}(\bar{r}) \cdot f^{-1}(\bar{q})\right)\right) + \bar{\xi}(\bar{r} \cdot \bar{q}).$$

Since the inequality $\xi(x_1 + x_2) \leqslant \xi(x_1) + \xi(x_2)$ is valid for any $x_1,\, x_2 \in R$ then

$$\xi\left(\left(r \cdot q - r \cdot f^{-1}(\bar{q})\right) + \left(r \cdot f^{-1}(\bar{q}) - f^{-1}(\bar{r}) \cdot f^{-1}(\bar{q})\right)\right) + \bar{\xi}(\bar{r} \cdot \bar{q}) \leqslant$$
$$\leqslant \xi\left(r \cdot q - r \cdot f^{-1}(\bar{q})\right) + \xi\left(r \cdot f^{-1}(\bar{q}) - f^{-1}(\bar{r}) \cdot f^{-1}(\bar{q})\right) + \bar{\xi}(\bar{r} \cdot \bar{q}) =$$
$$= \xi\left(r \cdot \left(q - f^{-1}(\bar{q})\right)\right) + \xi\left(\left(r - f^{-1}(\bar{r})\right) \cdot f^{-1}(\bar{q})\right) + \bar{\xi}(\bar{r} \cdot \bar{q}).$$

Since the inequalities $\xi(x_1 \cdot x_2) \leqslant \bar{\xi}(f(x_1)) \cdot \xi(x_2)$ and $\xi(x_1 \cdot x_2) \leqslant$ $\leqslant \xi(x_1) \cdot \bar{\xi}(f(x_2))$ are valid for any $x_1,\, x_2 \in R$ then

$$\xi\left(r \cdot \left(q - f^{-1}(\bar{q})\right)\right) + \xi\left(\left(r - f^{-1}(\bar{r})\right) \cdot f^{-1}(\bar{q})\right) + \bar{\xi}(\bar{r} \cdot \bar{q}) \leqslant$$
$$\leqslant \bar{\xi}(f(r)) \cdot \xi\left(q - f^{-1}(\bar{q})\right) + \xi\left(r - f^{-1}(\bar{r})\right) \cdot \bar{\xi}\left(f\left(f^{-1}(\bar{q})\right)\right) + \bar{\xi}(\bar{r} \cdot \bar{q}).$$

The inequality $\bar{\xi}(\bar{y}_1 \cdot \bar{y}_2) \leqslant \bar{\xi}(\bar{y}_1) \cdot \xi(\bar{y}_2)$ is valid for any $\bar{y}_1,\, \bar{y}_2 \in \bar{R}$. Therefore

$$\bar{\xi}(f(r)) \cdot \xi\left(q - f^{-1}(\bar{q})\right) + \xi\left(r - f^{-1}(\bar{r})\right) \cdot \bar{\xi}(\bar{q}) + \bar{\xi}(\bar{r} \cdot \bar{q}) \leqslant$$

$$\leqslant \bar{\xi}(f(r)) \cdot \xi\left(q - f^{-1}(\bar{q})\right) + \xi\left(r - f^{-1}(\bar{r})\right) \cdot \bar{\xi}(\bar{q}) + \bar{\xi}(\bar{r}) \cdot \bar{\xi}(\bar{q}) =$$
$$= \bar{\xi}\left((f(r) - \bar{r}) + \bar{r}\right) \cdot \xi\left(q - f^{-1}(\bar{q})\right) + \xi\left(r - f^{-1}(\bar{r})\right) \cdot \bar{\xi}(\bar{q}) + \bar{\xi}(\bar{r}) \cdot \bar{\xi}(\bar{q}).$$

Since the inequality $\bar{\xi}(\bar{y}_1 + \bar{y}_2) \leqslant \bar{\xi}(\bar{y}_1) + \xi(\bar{y}_2)$ is valid for any $\bar{y}_1,\, \bar{y}_2 \in \bar{R}$ then

$$\bar{\xi}\left((f(r) - \bar{r}) + \bar{r}\right) \cdot \xi\left(q - f^{-1}(\bar{q})\right) + \xi\left(r - f^{-1}(\bar{r})\right) \cdot \bar{\xi}(\bar{q}) + \bar{\xi}(\bar{r}) \cdot \bar{\xi}(\bar{q}) \leqslant$$

$$\leqslant \left(\bar{\xi}(f(r) - \bar{r}) + \bar{\xi}(\bar{r})\right) \cdot \xi\left(q - f^{-1}(\bar{q})\right) + \xi\left(r - f^{-1}(\bar{r})\right) \cdot \bar{\xi}(\bar{q}) + \bar{\xi}(\bar{r}) \cdot \bar{\xi}(\bar{q}) =$$

$$= \left(\bar{\xi}\left(f\left(r\right)-\bar{r}\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\xi\left(q-f^{-1}\left(\bar{q}\right)\right)+\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\bar{\xi}\left(\bar{q}\right).$$

Since the inequality $\bar{\xi}\left(f\left(x\right)\right)\leqslant\xi\left(x\right)$ is valid for any $x\in R$ then

$$\left(\bar{\xi}\left(f\left(r\right)-\bar{r}\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\xi\left(q-f^{-1}\left(\bar{q}\right)\right)+\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\bar{\xi}\left(\bar{q}\right)\leqslant$$

$$\leqslant\left(\xi\left(f^{-1}\left(f\left(r\right)-\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\xi\left(q-f^{-1}\left(\bar{q}\right)\right)+\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\bar{\xi}\left(\bar{q}\right)=$$

$$=\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\xi\left(q-f^{-1}\left(\bar{q}\right)\right)+\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\bar{\xi}\left(\bar{q}\right)=$$

$$=\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\left(\xi\left(q-f^{-1}\left(\bar{q}\right)\right)+\bar{\xi}\left(\bar{q}\right)\right)=\tilde{\xi}\left(\left(r,\bar{r}\right)\right)\cdot\tilde{\xi}\left(\left(q,\bar{q}\right)\right).$$

Thus, the condition 4 of the definition of a pseudonorm is valid, i.e. $\tilde{\xi}\left(\left(r\cdot q,\bar{r}\cdot\bar{q}\right)\right)\leqslant\tilde{\xi}\left(\left(r,\bar{r}\right)\right)\cdot\tilde{\xi}\left(\left(q,\bar{q}\right)\right)$ for any $r,\, q\in R$ and $\bar{r},\,\bar{q}\in\bar{R}$.

We have shown that the function $\tilde{\xi}\left(\left(r,\bar{r}\right)\right)=\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)$ defines a pseudonorm on the ring $\tilde{R}$.

2. Let's identify the ring R with the set of pairs $\{\,(r,0)\mid r\in R\}$. It is obvious that R is an ideal of the ring $\hat{R}$.

Let's consider the restrictions of the pseudonorm $\tilde{\xi}$ and the homomorphism $\tilde{f}$ on the ring $R=\{\,(r,0)\mid r\in R\}$, i.e. $\tilde{\xi}\left(\left(r,0\right)\right)=\xi\left(r-f^{-1}\left(0\right)\right)+\bar{\xi}\left(0\right)=\xi\left(r-0\right)+0=$ $=\xi\left(r\right)$ and $\tilde{f}\left(\left(r,0\right)\right)=f\left(r\right)$.

We have that $\tilde{\xi}\Big|_{R}=\xi$ and $\tilde{f}\Big|_{R}=f$.

3. Let's show that $\tilde{f}:\left(\tilde{R},\tilde{\xi}\right)\to\left(\bar{R},\bar{\xi}\right)$ is an isometric homomorphism.

3.1. Since f is an isomorphism and $\tilde{f}\Big|_{R}=f$ then

$$Ker\tilde{f}=\left\{\tilde{r}\in\tilde{R}\,\middle|\,\tilde{f}\left(\tilde{r}\right)=0\right\}=\left\{\left(r,\bar{r}\right)\in\tilde{R}\,\middle|\,\tilde{f}\left(\left(r,\bar{r}\right)\right)=0\right\}=$$

$$=\left\{\left(r,\bar{r}\right)\in\tilde{R}\,|\,f\left(r\right)=0\right\}=\left\{\left(r,\bar{r}\right)\in\tilde{R}\,|\,r=0\right\}=\left\{\,\left(0,\bar{r}\right)\,\middle|\,\bar{r}\in\bar{R}\right\}.$$

It means that the kernel of the homomorphism is $Ker\tilde{f}=\left\{\,(0,\bar{r})\,\middle|\,\bar{r}\in\bar{R}\right\}$.

3.2. Let's take any $(r,\bar{r})\in\tilde{R}$ and $(0,\bar{j})\in Ker\tilde{f}$. Then

$$\tilde{\xi}\left(\left(r,\bar{r}\right)+\left(0,\bar{j}\right)\right)=\tilde{\xi}\left(\left(r,\bar{r}+\bar{j}\right)\right)=\xi\left(r-f^{-1}\left(\bar{r}+\bar{j}\right)\right)+\bar{\xi}\left(\bar{r}+\bar{j}\right)=$$

$$=\xi\left(r-f^{-1}\left(\bar{r}\right)-f^{-1}\left(\bar{j}\right)\right)+\bar{\xi}\left(\bar{r}+\bar{j}\right)\geqslant\bar{\xi}\left(f\left(r\right)-\bar{r}-\bar{j}\right)+\bar{\xi}\left(\bar{r}+\bar{j}\right)\geqslant\bar{\xi}\left(f\left(r\right)\right).$$

Thus, the inequality $\bar{\xi}\left(f\left(r\right)\right)\leqslant\tilde{\xi}\left(\left(r,\bar{r}\right)+\left(0,\bar{j}\right)\right)$ is valid for the element $(r,\bar{r})\in\tilde{R}$ and any element $(0,\bar{j})\in Ker\tilde{f}$. It means that $\bar{\xi}\left(f\left(r\right)\right)$ is one of the lower bounds of the set $\left\{\tilde{\xi}\left(\left(r,\bar{r}\right)+\left(0,\bar{j}\right)\right)\,\middle|\,\left(0,\bar{j}\right)\in Ker\tilde{f}\right\}$. Therefore, the inequality

$$\bar{\xi}\left(f\left(r\right)\right)\leqslant\inf\left\{\tilde{\xi}\left(\left(r,\bar{r}\right)+\left(0,\bar{j}\right)\right)\,\middle|\,\left(0,\bar{j}\right)\in Ker\tilde{f}\right\}\tag{4}$$

is valid for any $(r,\bar{r})\in\tilde{R}$.

3.3. Let's take any element $(r, \bar{r}) \in \tilde{R}$. Let $\bar{j}_0 = f(r) - \bar{r} \in \bar{R}$. Then $(r, \bar{r}) + (0, \bar{j}_0) = (r, \bar{r}) + (0, f(r) - \bar{r}) = (r, f(r))$, that is

$$\tilde{\xi}\left((r, \bar{r}) + (0, \bar{j}_0)\right) = \tilde{\xi}\left((r, f(r))\right) = \xi\left(r - f^{-1}(f(r))\right) + \bar{\xi}(f(r)) =$$

$$\xi(r - r) + \bar{\xi}(f(r)) = \xi(0) + \bar{\xi}(f(r)) = 0 + \bar{\xi}(f(r)) = \bar{\xi}(f(r)).$$

Thus, for any $(r, \bar{r}) \in \tilde{R}$ there exists $\bar{j}_0 = (0, f(r) - \bar{r}) \in Ker\tilde{f}$ such that $\tilde{\xi}\left((r, \bar{r}) + (0, \bar{j}_0)\right) = \bar{\xi}(f(r))$.

From here the inequality

$$\inf\left\{ \tilde{\xi}(\tilde{r} + j) \,\middle|\, j \in Ker\tilde{f} \right\} \leqslant \tilde{\xi}\left((r, \bar{r}) + (0, \bar{j}_0)\right) = \bar{\xi}(f(r)) \tag{5}$$

follows. From inequalities (4) and (5) the equality $\bar{\xi}(f(r)) = \inf\{\tilde{\xi}((r, \bar{r}) + +(0, \bar{j})) \mid (0, \bar{j}) \in Ker\tilde{f}\}$ follows. Besides it follows from the equality $\tilde{\xi}((r, \bar{r}) + +(0, \bar{j}_0)) = \bar{\xi}(f(r))$ that

$$\bar{\xi}(f(r)) = \min\left\{ \tilde{\xi}\left((r, \bar{r}) + (0, \bar{j})\right) \,\middle|\, (0, \bar{j}) \in Ker\tilde{f} \right\} =$$

$$= \min\{\tilde{\xi}((r, 0) + (0, \bar{i})) \mid (0, \bar{i}) \in Ker\tilde{f}\}, \text{ where } (0, \bar{i}) = (0, \bar{r}) + (0, \bar{j}) \in Ker\tilde{f}.$$

We have shown that there exist a pseudonormed ring $\left(\tilde{R}, \tilde{\xi}\right)$ and a homomorphism $\tilde{f} : \tilde{R} \to \bar{R}$ such that:

R is an ideal of the ring $\tilde{R}$, $\tilde{\xi}\Big|_R = \xi$ and $\tilde{f}\Big|_R = f$;

$\bar{\xi}(f(r)) = \min\{\tilde{\xi}((r, 0) + (0, \bar{i})) \mid (0, \bar{i}) \in Ker\tilde{f}\}$, for every $r \in R$, i.e. for any $r \in R$ there exists an element $(0, \bar{a}_r) \in Ker\tilde{f}$ such that $\bar{\xi}(f(r)) = \tilde{\xi}((r, 0) + (0, \bar{a}_r))$.

Hence $II \Rightarrow III$ is proved.

Proof $III \Rightarrow I$.

From the condition 3 of the theorem there exist a pseudonormed ring $\left(\tilde{R}, \tilde{\xi}\right)$ and a homomorphism $\tilde{f} : \tilde{R} \to \bar{R}$ such that:

R is an ideal of the ring $\tilde{R}$;

$\tilde{\xi}\Big|_R = \xi$, $\tilde{f}\Big|_R = f$;

$\bar{\xi}(f(r)) = \min\left\{ \tilde{\xi}(r + a) \,\middle|\, a \in Ker\tilde{f} \right\}$ for every $r \in R$.

Let $\tilde{r} \in \tilde{R}$. As $f : R \to \bar{R}$ is an isomorphism then there exists a unique element $r \in R$ such that $f(r) = \tilde{f}(\tilde{r})$. Since the isomorphism f is the restriction of the homomorphism $\tilde{f}$ on the ring R then $f(r) = \tilde{f}(r)$. It means that $\tilde{f}(r) = \tilde{f}(\tilde{r})$. Then $\tilde{f}(r - \tilde{r}) = 0$. Hence, the element $r - \tilde{r}$ belongs to the kernel of the homomorphism $\tilde{f}$.

Then

$$\bar{\xi}\left(\tilde{f}(\tilde{r})\right) = \bar{\xi}(f(r)) = \min\left\{ \tilde{\xi}(r + a) \,\middle|\, a \in Ker\tilde{f} \right\} =$$

$$= \min\left\{ \tilde{\xi}(r + a + (\tilde{r} - \tilde{r})) \,\middle|\, a \in Ker\tilde{f} \right\} =$$

$$= \min\left\{ \tilde{\xi}\left(\tilde{r} + (a + (r - \tilde{r}))\right) \;\middle|\; a \in Ker\tilde{f} \right\} = \min\left\{ \tilde{\xi}\left(\tilde{r} + j\right) \;\middle|\; j \in Ker\tilde{f} \right\}.$$

Since for any set of real numbers S having the least element this element coincides with $\inf S$ then

$$\bar{\xi}\left(\tilde{f}\left(\tilde{r}\right)\right) = \inf\left\{ \tilde{\xi}\left(\tilde{r} + j\right) \;\middle|\; j \in Ker\tilde{f} \right\}.$$

Thus, f can be extended up to the isometric homomorphism $\tilde{f} : \left(\tilde{R}, \tilde{\xi}\right) \to \left(\bar{R}, \bar{\xi}\right)$, and $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ is a semi-isometric isomorphism by Definition 2.2.

The theorem is proved.

2.4 Corollary. *If (R, ξ) is a pseudonormed ring with the unit e and $\xi(e) = 1$ then any semi-isometric isomorphism of (R, ξ) is isometric.*

Let's consider the inequality $\dfrac{\xi(a \cdot b)}{\xi(b)} \leqslant \bar{\xi}(f(a)) \leqslant \xi(a)$ *for* $b = e$. *We have* $\xi(a) = \dfrac{\xi(a \cdot e)}{1} = \dfrac{\xi(a \cdot e)}{\xi(e)} \leqslant \bar{\xi}(f(a)) \leqslant \xi(a)$. *Therefore* $\bar{\xi}(f(a)) = \xi(a)$.

2.5 Corollary. *If (R, ξ) is a normed ring then any semi-isometric isomorphism of (R, ξ) is isometric.*

Really, in normed rings the equality $\xi(a \cdot b) = \xi(a) \cdot \xi(b)$ *is valid. From this equality it follows that* $\xi(a) = \dfrac{\xi(a) \cdot \xi(b)}{\xi(b)} = \dfrac{\xi(a \cdot b)}{\xi(b)} \leqslant \bar{\xi}(f(a)) \leqslant \xi(a)$. *It means that* $\bar{\xi}(f(a)) = \xi(a)$.

2.6 Corollary. *Let R and $\bar{R}$ be rings with zero multiplication (i.e. $a \cdot b = 0$ for all $a, b \in R$ and $\bar{a} \cdot \bar{b} = 0$ for all $\bar{a}, \bar{b} \in \bar{R}$). If ξ and $\bar{\xi}$ are pseudonorms on R and $\bar{R}$, accordingly, and $f : R \to \bar{R}$ is a ring isomorphism such that $\bar{\xi}(f(r)) \leqslant \xi(r)$ for every $r \in R$ then the isomorphism f is semi-isometric.*

Really, since $\dfrac{\xi(r \cdot q)}{\xi(q)} = 0 \leqslant \bar{\xi}(f(r)) \leqslant \xi(q)$ *then from Theorem 2.3 it follows that* $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ *is a semi-isometric isomorphism.*

2.7 Corollary. *Let (R, ξ) and $\left(\bar{R}, \bar{\xi}\right)$ be pseudonormed rings and $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ be a semi-isometric isomorphism. If $\tilde{\xi}$ is a pseudonorm on $\bar{R}$ such that $\bar{\xi}(f(r)) \leqslant \tilde{\xi}(f(r)) \leqslant \xi(r)$ for every $r \in R$ then $f : (R, \xi) \to \left(\bar{R}, \tilde{\xi}\right)$ is a semi-isometric isomorphism.*

Really, $\dfrac{\xi(r \cdot q)}{\xi(q)} \leqslant \bar{\xi}(f(r)) \leqslant \tilde{\xi}(f(r)) \leqslant \xi(q)$. *It means that* $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ *is a semi-isometric isomorphism.*

2.8 Theorem. *Let (R, ξ) and $\left(\bar{R}, \bar{\xi}\right)$ be pseudonormed rings and $f : R \to \bar{R}$ be a ring isomorphism. Then the following statements are equivalent:*

I. $\xi(a) \geqslant \bar{\xi}(f(a))$ *and* $\xi(a \cdot b) \leqslant \bar{\xi}(f(a)) \cdot \bar{\xi}(f(b))$ *for any* $a, b \in R$.

II. There exist a pseudonormed ring $\left(\hat{R}, \hat{\xi}\right)$ and a homomormism $\hat{f} : \hat{R} \to \bar{R}$ such that:

R is an ideal in the ring $\hat{R}$, $\hat{\xi}\Big|_R = \xi$ and $\hat{f}\Big|_R = f$;

$\bar{\xi}(f(r)) = \min\left\{ \hat{\xi}(r+a) \;\Big|\; a \in Ker\hat{f} \right\}$ for every $r \in R$, i.e. for every $r \in R$ there exists an element $a_r \in Ker\hat{f}$ such that $\bar{\xi}(f(r)) = \hat{\xi}(r+a_r)$;

The annihilator of the ring $\hat{R}$ contains $Ker\hat{f}$, i.e. $Ker\hat{f} \subseteq \left\{ a \in \hat{R} \;\Big|\; a \cdot \hat{R} = \hat{R} \cdot a = 0 \right\}$ (in particular, $\left(Ker\hat{f}\right)^2 = 0$).

Proof $I \Rightarrow II$**.** Let the mapping $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ possesses the following properties:

f is an isomorphism;

$\bar{\xi}(f(a)) \leqslant \xi(a)$ for any $a \in R$;

$\xi(a \cdot b) \leqslant \bar{\xi}(f(a)) \cdot \bar{\xi}(f(b))$ for any $a \in R$, $b \in R \backslash \{0\}$.

Let's prove that the statement II is valid.

Let $\left(\bar{R}', \bar{\xi}\right)$ be a ring with zero multiplication which elements belong to $\bar{R}$. We shall consider the ring $\hat{R} = R \oplus \bar{R}' = \left\{ (r, \bar{r}) \;\big|\; r \in R, \bar{r} \in \bar{R}' \right\}$ which is the direct sum of the rings R and $\bar{R}'$. Let's define the real-valued function $\tilde{\xi}$ on $\tilde{R}$ as in Theorem 2.3, i.e. $\hat{\xi}((r, \bar{r})) = \xi\left(r - f^{-1}(\bar{r})\right) + \bar{\xi}(\bar{r})$.

Let's define the mapping $\tilde{f} : \left(\tilde{R}, \tilde{\xi}\right) \to \left(\bar{R}, \bar{\xi}\right)$ by the equality $\tilde{f}((r, \bar{r})) = f(r)$.

Let's show that $\hat{\xi}$ is a pseudonorm on the ring $\hat{R}$.

The conditions 1 – 3 of Definition 1.1 can be verified by analogy with Theorem 2.3.

Let's verify the condition 4 of the definition of a pseudonorm. Since $\bar{R}'$ is a ring with zero multiplication then $\hat{\xi}((r, \bar{r}) \cdot (q, \bar{q})) = \hat{\xi}((r \cdot q, \bar{r} \cdot \bar{q})) = \hat{\xi}((r \cdot q, 0)) = \xi(r \cdot q)$ for any $r, q \in R$ and $\bar{r}, \bar{q} \in \bar{R}'$.

It follows from the inequality $\xi(a \cdot b) \leqslant \bar{\xi}(f(a)) \cdot \bar{\xi}(f(b))$ that

$$\xi(r \cdot q) \leqslant \bar{\xi}(f(r)) \cdot \bar{\xi}(f(q)) = \bar{\xi}((f(r) - \bar{r}) + \bar{r}) \cdot \bar{\xi}((f(q) - \bar{q}) + \bar{q}).$$

Since the inequality $\bar{\xi}(\bar{y}_1 + \bar{y}_2) \leqslant \bar{\xi}(\bar{y}_1) + \xi(\bar{y}_2)$ is valid for any $\bar{y}_1, \bar{y}_2 \in \bar{R}'$ then

$$\bar{\xi}((f(r) - \bar{r}) + \bar{r}) \cdot \bar{\xi}((f(q) - \bar{q}) + \bar{q}) \leqslant \left(\bar{\xi}(f(r) - \bar{r}) + \bar{\xi}(\bar{r})\right) \cdot \left(\bar{\xi}(f(q) - \bar{q}) + \bar{\xi}(\bar{q})\right).$$

It follows from the inequality $\bar{\xi}(f(a)) \leqslant \xi(a)$ that

$$\left(\bar{\xi}(f(r) - \bar{r}) + \bar{\xi}(\bar{r})\right) \cdot \left(\bar{\xi}(f(q) - \bar{q}) + \bar{\xi}(\bar{q})\right) \leqslant$$

$$\leqslant \left(\xi\left(r - f^{-1}(\bar{r})\right) + \bar{\xi}(\bar{r})\right) \cdot \left(\xi\left(q - f^{-1}(\bar{q})\right) + \bar{\xi}(\bar{q})\right) = \hat{\xi}((r, \bar{r})) \cdot \hat{\xi}((q, \bar{q})).$$

Thus, the condition 4 of the definition of a pseudonorm is valid, i.e. $\hat{\xi}((r \cdot q, \bar{r} \cdot \bar{q})) \leqslant \hat{\xi}((r, \bar{r})) \cdot \hat{\xi}((q, \bar{q}))$ for any $r, q \in R$ and $\bar{r}, \bar{q} \in \bar{R}'$.

We have shown that the function $\hat{\xi}((r, \bar{r})) = \xi\left(r - f^{-1}(\bar{r})\right) + \bar{\xi}(\bar{r})$ defines a pseudonorm on the ring $\hat{R}$.

Like in Theorem 2.3 let's identify the ring R with the set of pairs $\{(r,0)\mid r\in R\}$ which is an ideal in the ring $\hat{R}$. Let's consider the restrictions of the pseudonorm $\hat{\xi}$ and homomorphism $\hat{f}$ on the ring $R=\{(r,0)\mid r\in R\}$, i.e. $\hat{\xi}((r,0)) = = \xi\left(r-f^{-1}(0)\right)+\bar{\xi}(0)=\xi(r-0)+0=\xi(r)$ and $\hat{f}((r,0))=f(r)$.

We have that $\hat{\xi}\Big|_R=\xi$ and $\hat{f}\Big|_R=f$.

Let's show that the annihilator of the ring $\hat{R}$ contains $Ker\hat{f}$.

Since $f:R\to\bar{R}$ is an isomorphism then $Ker\hat{f}=\Big\{(r,\bar{r})\in\hat{R}|\ \hat{f}((r,\bar{r}))=0\Big\}= =\Big\{(r,\bar{r})\in\hat{R}|\ f(r)=0\Big\}=\Big\{(0,\bar{r})\in\hat{R}|\ \bar{r}\in\bar{R}'\Big\}$.

Since $(0,\bar{r})\cdot(a,\bar{a})=(0\cdot a,\bar{r}\cdot\bar{a})=(0,0)$ and $(a,\bar{a})\cdot(0,\bar{r})=(a\cdot 0,\bar{a}\cdot\bar{r})=(0,0)$ for any $(0,\bar{r})\in Ker\hat{f}$ and $(a,\bar{a})\in\hat{R}$ then $Ker\hat{f}\subset Ann\hat{R}$.

Let's show that $\hat{f}:\left(\hat{R},\hat{\xi}\right)\to\left(\bar{R},\bar{\xi}\right)$ is an isometric homomorphism by analogy with Theorem 2.3. Let $(r,\bar{r})\in\hat{R}$ and $(0,\bar{j})\in Ker\hat{f}$. Then

$$\hat{\xi}((r,\bar{r})+(0,\bar{j}))=\hat{\xi}((r,\bar{r}+\bar{j}))=\xi\left(r-f^{-1}(\bar{r}+\bar{j})\right)+\bar{\xi}(\bar{r}+\bar{j})=$$

$$=\xi\left(r-f^{-1}(\bar{r})-f^{-1}(\bar{j})\right)+\bar{\xi}(\bar{r}+\bar{j})\geqslant\bar{\xi}(f(r)-\bar{r}-\bar{j})+\bar{\xi}(\bar{r}+\bar{j})\geqslant\bar{\xi}(f(r)).$$

Thus, the inequality $\bar{\xi}(f(r))\leqslant\hat{\xi}((r,\bar{r})+(0,\bar{j}))$ is valid for the element $(r,\bar{r})\in\hat{R}$ and any element $(0,\bar{j})\in Ker\hat{f}$. It means that $\bar{\xi}(f(r))$ is one of the lower bounds of the set $\left\{\hat{\xi}((r,\bar{r})+(0,\bar{j}))\ \Big|\ (0,\bar{j})\in Ker\hat{f}\right\}$. Therefore, the inequality

$$\bar{\xi}(f(r))\leqslant\inf\left\{\hat{\xi}((r,\bar{r})+(0,\bar{j}))\ \Big|\ (0,\bar{j})\in Ker\hat{f}\right\}\tag{6}$$

is valid for any $(r,\bar{r})\in\hat{R}$.

Let $(r,\bar{r})\in\hat{R}$ and $\bar{j}_0=f(r)-\bar{r}\in\bar{R}$. Then $(r,\bar{r})+(0,\bar{j}_0)=(r,\bar{r})+(0,f(r)-\bar{r})= =(r,f(r))$. It means that

$$\widehat{\xi}((r,\bar{r})+(0,j_0))=\widehat{\xi}((r,f(r)))=\xi\left(r-f^{-1}(f(r))\right)+\bar{\xi}(f(r))=$$

$$=\xi(r-r)+\bar{\xi}(f(r))=\xi(0)+\bar{\xi}(f(r))=0+\bar{\xi}(f(r))=\bar{\xi}(f(r)).$$

Thus for any $(r,\bar{r})\in\hat{R}$ there exists $\bar{j}_0=(0,f(r-\bar{r}))\in Ker\hat{f}$ such that $\hat{\xi}((r,\bar{r})+(0,\bar{j}_0))=\bar{\xi}(f(r))$.

From here the inequality

$$\inf\left\{\hat{\xi}(\hat{r}+j)\ \Big|\ j\in Ker\hat{f}\right\}\leqslant\hat{\xi}((r,\bar{r})+(0,\bar{j}_0))=\bar{\xi}(f(r))\tag{7}$$

follows.

From inequalities (6) and (7) the equality

$$\bar{\xi}(f(r))=\inf\left\{\hat{\xi}((r,\bar{r})+(0,\bar{j}))\ \Big|\ (0,\bar{j})\in Ker\hat{f}\right\}$$

follows.

Besides it follows from the equality $\hat{\xi}\left((r,\bar{r})+(0,\bar{j}_0)\right)=\bar{\xi}\left(f\left(r\right)\right)$ that

$$\bar{\xi}\left(f\left(r\right)\right)=\min\left\{\hat{\xi}\left((r,\bar{r})+(0,\bar{j})\right)\ \middle|\ (0,\bar{j})\in Ker\hat{f}\right\}=$$

$$=\min\left\{\hat{\xi}\left((r,0)+(0,\bar{i})\right)\ \middle|\ (0,\bar{i})\in Ker\hat{f}\right\},$$

where $(0,\bar{i})=((0,\bar{r})+(0,\bar{j}))\in Ker\hat{f}$.

We have shown that there exist a pseudonormed ring $\left(\hat{R},\hat{\xi}\right)$ and a homomorphism $\hat{f}:\hat{R}\to\bar{R}$ such that:

R is an ideal of the ring $\hat{R}$, $\hat{\xi}\Big|_R=\xi$ and $\hat{f}\Big|_R=f$;

$\bar{\xi}\left(f\left(r\right)\right)=\min\left\{\hat{\xi}\left((r,0)+(0,\bar{i})\right)\ \middle|\ (0,\bar{i})\in Ker\hat{f}\right\}$ for every $r\in R$, i.e. for any $r\in R$ there exist an element $(0,\bar{a}_r)\in Ker\hat{f}$ such that $\bar{\xi}\left(f\left(r\right)\right)=\hat{\xi}\left((r,0)+(0,\bar{a}_r)\right)$;

The annihilator of the ring $\hat{R}$ contains $Ker\hat{f}$.

Hence $I\Rightarrow II$ is proved.

Proof $II\Rightarrow I$**.** Let (R,ξ) and $\left(\bar{R},\bar{\xi}\right)$ be pseudonormed rings, $f:R\to\bar{R}$ be a ring isomorphism. Let $\left(\hat{R},\hat{\xi}\right)$ be a pseudonormed ring and $\hat{f}:\hat{R}\to\bar{R}$ be a homomorphism such that the following conditions are valid:

a) R is an ideal in the ring $\hat{R}$; $\hat{\xi}\Big|_R=\xi$ and $\hat{f}\Big|_R=f$;

b) $\bar{\xi}\left(f\left(r\right)\right)=\min\left\{\hat{\xi}\left(r+a\right)\ \middle|\ a\in Ker\hat{f}\right\}$ for every $r\in R$;

c) The annihilator of a ring $\hat{R}$ contains $Ker\hat{f}$.

It follows from Theorem 2.3 that the inequality $\xi\left(a\right)\geqslant\bar{\xi}\left(f\left(a\right)\right)$ is valid for any $a\in R$.

Let's show that the inequality $\xi\left(a\cdot b\right)\leqslant\bar{\xi}\left(f\left(a\right)\right)\cdot\bar{\xi}\left(f\left(b\right)\right)$ is valid for any $a,b\in R$ as well.

The equalities $\bar{\xi}\left(f\left(a\right)\right)=\min\left\{\hat{\xi}\left(a+i\right)\ \middle|\ i\in Ker\hat{f}\right\}=\hat{\xi}\left(a+i_a\right)$ and $\bar{\xi}\left(f\left(b\right)\right)=$ $=\min\left\{\hat{\xi}\left(b+j\right)\ \middle|\ j\in Ker\hat{f}\right\}=\hat{\xi}\left(b+j_b\right)$ are valid for any $a,b\in R$, where i_a, j_b are some elements from $Ker\hat{f}$. It means that

$$\bar{\xi}\left(f\left(a\right)\right)\cdot\bar{\xi}\left(f\left(b\right)\right)=\hat{\xi}\left(a+i_a\right)\cdot\hat{\xi}\left(b+j_b\right)\geqslant\hat{\xi}\left((a+i_a)\cdot(b+j_b)\right)=$$

$$=\hat{\xi}\left(a\cdot b+a\cdot j_b+i_a\cdot b+i_a\cdot j_b\right).$$

Sinse $i_a,j_b\in Ker\hat{f}\subset Ann\hat{R}$ then $a\cdot j_b=i_a\cdot b=i_a\cdot j_b=0$. Hence $\bar{\xi}\left(f\left(a\right)\right)\cdot\bar{\xi}\left(f\left(b\right)\right)\geqslant\hat{\xi}\left(a\cdot b+a\cdot j_b+i_a\cdot b+i_a\cdot j_b\right)=\hat{\xi}\left(a\cdot b\right)$ for any $a,b\in R$.

Thus, $\xi\left(a\right)\geqslant\bar{\xi}\left(f\left(a\right)\right)$ and $\xi\left(a\cdot b\right)\leqslant\bar{\xi}\left(f\left(a\right)\right)\cdot\bar{\xi}\left(f\left(b\right)\right)$ for any $a,b\in R$.

The theorem is proved.

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