Numerical treatment of the Kendall equation in the analysis of priority queueing systems

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Abstract. We investigate here how to treat numerically the Kendall functional equation occuring in the theory of branching processes and queueing theory. We discuss this question in the context of priority queueing systems with switchover times. In numerical analysis of such systems one deals with functional equations of the Kendall type and efficient numerical treatment of these is necessary in order to estimate important system performance characteristics.

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1 Introduction

1.1 Preliminary notes

It is well known that the Kendall equation is one of fundamental functional equations which appear in the queueing theory and the theory of branching processes.

Consider a system $M|G|1$ with Poisson(λt) input flow of requests and random service time B with c.d.f. $B(t)$. The busy periods in such system are independent and identically distributed (i.i.d.) random variables (r.v.'s) with some cumulative distribution function (c.d.f.) $\Pi(t)$. How does $\Pi(t)$ depend on $B(t)$ and λ? Let $\beta(s)$ be the Laplace-Stieltjes transform of $B(t)$ and $\pi(s)$—the Laplace-Stieltjes transforms of $\Pi(t)$. It is well-known result that $\pi(s)$ satisfies the following functional equation:

$$\pi(s) = \beta(s + \lambda(1 - \pi(s))). \tag{1}$$

The equation (1) is known as *Kendall equation* due to Kendall (1951). It is not easy generally to obtain the analytical solution to this equation. Even in simple cases (choice of $B(t)$) the solution $\pi(s)$ may be analytically intractable. This, with the fact that $\pi(s)$ should be inverted in order to obtain full information on busy periods' distribution, leads to the necessity in numerical method for obtaining the solution of (1) and providing the absolute error of the evaluation. Such method is known and it is based on the fact that the right side of Kendall equation, being considered as a functional operator, has a fixed point, see Abate and Whitt (1992).

The Kendall equation makes part of many analytical results regarding distribution of busy periods in priority queueing systems $M_r|G_r|1$—queueing systems with

Poisson input flows of requests distinguished by importance and one server. Often such results are stated in the form of the systems of functional equations expressed in terms of Laplace-Stieltjes transforms.

For example, the LST $\pi(s)$ of the busy periods' c.d.f. $\Pi(t)$ in the priority queueing system $M_r|G_r|1$ with switchover times under the scheme "with losses" (for the description of such systems appeal to Mishkoy et al. (2006)) is determined by the following system of functional equations (see Klimov and Mishkoy (1979)):

$$
\begin{cases}
\pi_k(s) = \frac{\sigma_{k-1}}{\sigma_k}\{\pi_{k-1}(s+\lambda_k) + \Delta_{k-1}(s)\nu_k(s+\lambda_k[1-\overline{\pi_k}(s)])\} + \frac{\lambda_k}{\sigma_k}\pi_{kk}(s), \\
\Delta_{k-1}(s) = \pi_{k-1}(s+\lambda_k[1-\overline{\pi_k}(s)]) - \pi_{k-1}(s+\lambda_k), \\
\pi_{kk}(s) = \nu_k(s+\lambda_k[1-\overline{\pi_k}(s)])\overline{\pi_k}(s), \\
\overline{\pi_k}(s) = h_k(s+\lambda_k[1-\overline{\pi_k}(s)]), \\
\nu_k(s) = c_k(s+\sigma_{k-1})\{1 - \frac{\sigma_{k-1}}{s+\sigma_{k-1}}[1-c_k(s+\sigma_{k-1})]\pi_{k-1}(s)\}^{-1}, \\
h_k(s) = \beta_k(s+\sigma_{k-1}) + \frac{\sigma_{k-1}}{s+\sigma_{k-1}}[1-\beta_k(s+\sigma_{k-1})]\pi_{k-1}(s)\nu_k(s),\ k=1,\ldots,r, \\
\pi_0(s) \equiv 0.
\end{cases}
$$

Here $\pi_r(s) \equiv \pi(s)$, λ_i is the parameter of the ith Poisson input flow, and σ_i stands for $\sum_{i=1}^{k} \lambda_i$.

For the same model with zero switchover times as an immediate corollary the following result follows:

$$
\begin{cases}
\sigma_k\pi_k(s) = \sigma_{k-1}\pi_{k-1}(s+\lambda_k[1-\pi_{kk}(s)]) + \lambda_k\pi_{kk}(s), \\
\pi_{kk}(s) = h_k(s+\lambda_k[1-\pi_{kk}(s)]), \\
h_k(s) = \beta_k(s+\sigma_{k-1}) + \frac{\sigma_{k-1}}{s+\sigma_{k-1}}[1-\beta_k(s+\sigma_{k-1})]\pi_{k-1}(s),\ k=1,\ldots,r, \\
\pi_0(s) \equiv 0.
\end{cases}
$$

One can notice the presence of the Kendall equations in such systems.

In the light of these facts, is suggested an acceleration of the scheme of obtaining a numerical solution to the Kendall equation which gives a gain in the number of operations (iterations) needed to solve one-dimensional problem (the case of $M|G|1$ system) as well as multidimensional problem (the case of $M_r|G_r|1$ system).

1.2   Notations and supporting results

1.2.1   Laplace and Laplace-Stieltjes transforms

Definition 1. Let $f(t)$ be a complex-valued function of real argument satisfying the following conditions: (i) $f(t) = 0 \ \forall t < 0$, (ii) it is a function of bounded variation on any segment $[0,T]$, (iii) $\exists s_0, A \in \mathbb{R}$ s.t. $|f(t)| \le Ae^{s_0 t}$. The Laplace transform of a function $f(t)$ is denoted by $\hat{f}(s)$ and is given by

$$
\hat{f}(s) \overset{\partial ef}{=} \int_0^\infty e^{-st}f(t)dt, \tag{2}
$$

where $s \in \mathbb{C} : \Re s > s_0$.

This is a general definition. Conditions (i)-(iii) are required for correctness and existence of the transform. The Laplace transform is analytical in the right half-plane $\Re s > s_0$. The infimum of all such s for which the Laplace transform exists is called the *abscissa of convergence* and is denoted by σ_0. The Laplace transform for a real-valued function is defined automatically by Definition 1.

The Laplace transform is unique, in the sense that, given two functions $f_1(t)$ and $f_2(t)$ with the same transform, i.e. $\hat{f}_1(t) \equiv \hat{f}_2(t)$, the integral

$$\int_0^T N(t)dt$$

of the null function $N(t) \overset{def}{=} f_1(t) - f_2(t)$ vanishes for all $T > 0$ (Lerch's theorem for integral transforms). The Laplace transform is linear:

$$a\hat{f}(t) + b\hat{g}(t) = a\hat{f}(t) + b\hat{g}(t) \tag{3}$$

The Laplace transform of a convolution $h(t) = f(t) * g(t) = \int_0^\infty f(t-\tau)g(\tau)d\tau$ is given by

$$\hat{h}(t) = \hat{f}(t)\hat{g}(t) \tag{4}$$

Definition 2. The Laplace-Stieltjes transform of a real-valued function $F(t)$ of real argument is denoted by $\check{F}(s)$ and is given by

$$\check{F}(s) \overset{def}{=} \int_0^\infty e^{-st}dF(t), \tag{5}$$

where $s \in \mathbb{C}$, whenever this integral exists. The integral here is the *Lebesgue-Stieltjes integral.*

Often s is a real variable, and then the LST is a real-valued function. If $F(t)$ is differentiable, i.e. $dF(t) = f(t)dt$, then the Laplace-Stieltjes transform of $F(t)$ is just a Laplace transform of its derivative.

Theorem 1. *(Uniqueness) Two different probability distributions have different Laplace-Stieltjes transforms.*

Theorem 2. *(Continuity) Let $F_n(t)$ be a cumulative distribution function with LST $\varphi_n(s)$, $n = 1, 2, \ldots$. If $F_n \to F$, where F is a possibly improper distribution with LST $\varphi(s)$, then $\varphi_n(s) \to \varphi(s)$ $\forall s > 0$. Conversely, if a sequence $\{\varphi_n(s)\}$ converges to $\varphi(s)$ for any $s > 0$, then φ is an LST of a possibly improper distribution F, s.t. $F_n \to F$. The limiting distribution F will be proper (or a probability distribution indeed) iff $\varphi(s) \to 1$ when $s \downarrow 0$.*

For the proofs of Theorem 1 and Theorem 2 see Feller (1971).

The nth moment of the non-negative random variable X with p.d.f. $f_X(t)$ may be obtained via its Laplace transform in the following way:

$$\mathbb{E}[X^n] = (-1)^n \hat{f}_X^{(n)}(s)|_{s=0} \tag{6}$$

Abate and Whitt (1996) investigated functional operators that map one or more probability distributions on the positive real line into another using their Laplace-Stieltjes transforms.

1.2.2   Complete monotonicity and Bernstein theorem

We introduce now an important notion for our further considerations—a notion of *complete monotonicity*.

Definition 3. A real valued function φ on $[0, \infty)$ is said to be *completely monotone* (c.m.) if

$$(-1)^n \varphi^{(n)}(s) \geq 0 \ \forall n \in \mathbb{N} \cup \{0\} \ \forall s \in (0, \infty). \tag{7}$$

Example 1. Functions $s^{-\alpha}$, $e^{-\alpha s}$ ($\alpha \geq 0$), $\frac{1}{1+s}$, $\frac{1}{s}$ are completely monotone functions. Function $\varphi(a+bs)$ is completely monotone when $\varphi(s)$ is completely monotone ($a \geq 0$, $b > 0$).

We will make use of the following important properties of complete monotone functions in our further exposition. Functions φ and ψ are considered to be defined on $\mathbb{R}^+$.

Property 1. *If φ and ψ are two complete monotone functions, then their linear combination $\alpha\varphi + \beta\psi$ is a complete monotone function ($\alpha^2 + \beta^2 > 0$).*

Proof. The affirmation follows directly from the definition of complete monotone function. $\square$

Property 2. *If φ and ψ are two complete monotone functions, then their product $\varphi\psi$ is a complete monotone function.*

Proof. We use the method of mathematical induction to show that

$$(-1)^n (\varphi\psi)^{(n)} \geq 0 \ \forall n = 0, 1, 2, ... \tag{8}$$

Assume that first n derivatives of $\varphi\psi$ alternate in sign. We first not that $\varphi\psi$ is nonnegative and $-\varphi'$ and $-\psi'$ are complete monotone functions. Therefore, (i) $-(\varphi\psi)' = -\varphi'\psi - \varphi\psi' \geq 0$ and this corresponds to the case $n = 1$ (the basis of induction), and (ii) we can apply the induction hypothesis for the products $-\varphi'\psi$ and $-\varphi\psi'$. But this immediately means that $\varphi\psi$ alternates in sign $n + 1$ times. By the principle of mathematical induction the property is proven. $\square$

Property 3. *If φ is complete monotone and ψ is a non-negative function, s.t. ψ' is a complete monotone function, then $\varphi(\psi)$ is complete monotone.*

Proof. First, note that $\varphi(\psi)$ is a nonnegative function on $\mathbb{R}^+$. Then, notice that $-\varphi'(\psi)$ and ψ' are complete monotone functions. This makes $-\varphi'(\psi(s)) = -\varphi'(\psi)\psi'$ to be complete monotone (Property 2). We have proven that $-[\varphi(\psi)]'$ is a complete monotone function, i.e. $\varphi(\psi)$ is necessarily complete monotone. The property is proven. $\square$

Laplace transforms of positive Borel measures on $\mathbb{R}^+$ are completely characterized by the Bernstein theorem in terms of complete monotonicity.

Theorem 6. (Bernstein (1928)) *Function φ is complete monotone iff there exists a unique nonnegative Borel measure μ on $[0, \infty]$, s.t. $\mu([0, \infty]) = \varphi(0^+)$ and $\forall s > 0$*
$$\varphi(s) = \int\limits_0^\infty e^{-sx}\mu(dt). \text{ Here } [0, \infty] \text{ is a one-point compactification of } [0, \infty).$$

Remark 1. The theorem says that the class of complete monotone functions $\varphi(s)$ on half-line $\mathbb{R}^+$, such that $\varphi(0^+) \leq 1$ coincides with Laplace-Stieltjes transforms of cumulative distribution functions.

Example 2. It was mentioned above that the LST of the busy period in the system $M|G|1$ satisfies the Kendall equation. It can be shown (Feller (1971), Gnedenko et al (1971)), that Kendall equation determines a unique function $\pi(s)$ which is analytic in the right-half complex plane $\Re s > 0$. More of this, if the system workload $\rho = -\frac{\beta(0)}{\lambda} < 1$, then $\pi(0) = 1$, and the c.d.f. $\Pi(t)$ is a proper cumulative distribution function. The moments of the busy period Π in the system $M|G|1$ can be easily calculated using (6). For example, evaluating the first and second derivatives of $\pi(s)$ at zero in (1), one gets:

$$\mathbb{E}[\Pi] = -\pi'(0) = -\frac{\beta'(0)}{1-\rho} = \frac{\mathbb{E}[B]}{1-\rho},$$

$$\mathbb{E}[\Pi^2] = \pi''(0) = \frac{\mathbb{E}[B^2]}{(1-\rho)^3},$$

so that

$$\text{var}[\Pi] = \mathbb{E}[\Pi^2] - \mathbb{E}[\Pi]^2 = \frac{\mathbb{E}[B^2] - \mathbb{E}[B]^2 + \rho\mathbb{E}[B]^2}{(1-\rho)^3}.$$

However, in order to obtain full information on the busy period Π one needs to invert $\pi(s)$. This should be found either analytically or evaluated numerically first.

In the case when $\rho > 1$, the following takes place: $\pi(0) < 1$, and $\Pi(t)$ is an improper c.d.f., i.e. $\lim\limits_{t\to\infty} \Pi(t) < 1$, that means that a busy period is of indefinite length with a positive probability. The case $\rho = 1$ is very special one.

A thorough discussion on Kendall equation treatment follows next.

2 Kendall equation: numerical treatment

2.1 Kendall fixed point operator

We discuss here in more detail the treatment of the Kendall equation

$$\hat{\varphi}(s) = \hat{g}(s + \rho - \rho\hat{\varphi}(s)), \tag{9}$$

where $\rho > 0$ and $\hat{g}(s)$ is a Laplace transform of some p.d.f. $g(s)$ associated with some proper c.d.f. $G(s)$. In other words, in virtue of Bernstein theorem (Theorem 6 and Remark 1), we suppose $\hat{g}(s)$ to be a complete monotone function, such that $\hat{g}(0^+) \leq 1$. The coefficient ρ is some non-negative real number.

Denote by $\mathcal{CM}$ the set of all complete monotone functions, by $\mathcal{CM}_1$ the set $\{\varphi(s) \in \mathcal{CM} \,|\, \varphi(0^+) \leq 1\}$, i.e., $\mathcal{CM}_1$ is the subset of complete monotone functions which correspond to proper or improper c.d.f.'s on $\mathbb{R}^+$.

To analyze (9) we introduce the following operator, which we will call *the Kendall operator*:

$$K_{\hat{g}}[\varphi](s) : \mathcal{CM}_1 \mapsto \mathcal{CM}_{\hat{g}} := \mathrm{Im}(K_{\hat{g}}) \subseteq \mathcal{CM}_1, \tag{10}$$

$$K_{\hat{g}}[\varphi](s) \stackrel{def}{=} \hat{g}(s + \rho - \rho\hat{\varphi}(s))), \text{ for some } \hat{g} \in \mathcal{CM}_1, \text{ and } \rho > 0. \tag{11}$$

First, we show that indeed $\mathrm{Im}(K_{\hat{g}}) \subseteq \mathcal{CM}_1$. To see this, note that

$$\psi_\varphi(s) = s + \rho(1 - \hat{\varphi}(s))$$

is a positive function with complete monotone derivative:

$$\psi'_\varphi(s) = 1 - \rho\hat{\varphi'}(s) > 1 > 0,$$
$$\psi''_\varphi(s) = -\rho\hat{\varphi}''(s) \leq 0,$$

$$\dots$$

We can apply Property 3 for $\hat{g}(\psi_\varphi(s))$ to see that $K_{\hat{g}}[\varphi](s)$ is complete monotone. Furthermore, this function is continuous as composition of continuous functions, therefore

$$K_{\hat{g}}[\varphi](0^+) = \hat{g}(\rho(1 - \hat{\varphi}(0^+))) \leq \hat{g}(0^+) \leq 1.$$

We conclude that $\mathrm{Im}(K_{\hat{g}}) \subseteq \mathcal{CM}_1$ for any given $\hat{g} \in \mathcal{CM}_1$.

The following theorem is well-known. Its proof is constructive and it is important for us.

Theorem 7. *The Kendall equation (9) has unique solution $\hat{\varphi}(s) \leq 1$, s.t. $\hat{\varphi}(s)$ is a Laplace-Stieltjes transform of some (probability) distribution Φ which is proper (and then it is a probability distribution) when $-\rho\hat{g}'(0) \leq 1$, and improper otherwise.*

Proof. Consider the following equation

$$Q_s(x) = \hat{g}(s + \rho - \rho x) - x = 0 \tag{12}$$

for a fixed $s > 0$ and for some $x \in [0, 1]$. The function $Q_s(x)$ is a convex function in respect to x (for any given $s > 0$), since its second derivative $\rho^2 \hat{g}''(s + \rho(1 - x)) > 0$. Moreover, $Q_s(0) > 0$ and $Q_s(1) < 0$, therefore $Q_s(x)$ has a unique root x^* in $[0, 1]$. Allowing s to take any value from $\mathbb{R}^+$ one obtains the existence and uniqueness of the solution to the Kendall equation.

We want to show now that this solution is a c.m. function. Denote by $0(s)$ and $1(s)$ functions identical to zero and one on $\mathbb{R}^+$, corresp.

Consider two functional sequences: $\mathcal{U} = \{K_{\hat{g}}^n[0](s)\}_{n=0}^{\infty}$ with $K_{\hat{g}}^0[0](s) \equiv 0$ and $\mathcal{D} = \{K_{\hat{g}}^n[1](s)\}_{n=0}^{\infty}$ with $K_{\hat{g}}^0[1](s) \equiv 1 \; \forall s > 0$. We show that these sequences converge in a point-wise sense to the solution $\hat{\varphi}(s)$ to the Kendall equation (9).

It is obvious that $K_{\hat{g}}^0[0](s) \leq K_{\hat{g}}^1[0](s) \; \forall s > 0$, since $\hat{g}$ is a c.m. function. Suppose that $K_{\hat{g}}^{n-1}[0](s) \leq K_{\hat{g}}^n[0](s) \; \forall s > 0$. Then,

$$K_{\hat{g}}^{n+1}[0](s) = \hat{g}(s + \rho - \rho K_{\hat{g}}^n[0](s)) \geq \hat{g}(s + \rho - \rho K_{\hat{g}}^{n-1}[0](s)) = K_{\hat{g}}^n[0](s) \; \forall s > 0.$$

By induction

$$K_{\hat{g}}^n[0](s) \leq K_{\hat{g}}^{n+1}[0](s) \leq \hat{g}(0) \leq 1(s) \; \forall s > 0.$$

Thus, the sequence $\mathcal{U}$ is a monotone sequence of c.m. functions and is bounded from above—it has a limit which is a c.m. function (by Theorem 2) and which satisfies the Kendall equation, i.e. $\lim_{n \to \infty} K_{\hat{g}}^n[0](s) = \hat{\varphi}(s)$. This limit $\hat{\varphi}(s)$ is an LST of some probability distribution Φ with a total mass $\Phi([0, \infty]) = \varphi(0) \leq 1$.

One can show in a full analogy that $\mathcal{D}$ is a non-increasing sequence bounded from below by $0(s)$ and its limit is the unique solution to the Kendall equation:

$$\lim_{n \to \infty} K_{\hat{g}}^n[1](s) = \hat{\varphi}(s).$$

It remains to show that the probability distribution corresponding to the solution of the Kendall equation is proper iff $-\rho \hat{g}'(0) \leq 1$. To show this, consider

$$Q_0(x) = \hat{g}(\rho(1 - x)) - x = 0, \tag{13}$$

as in (12), for $x = \varphi(0) = \lim_{n \to \infty} K_{\hat{g}}^n[0](s)$, and, thus, $x = \varphi(0)$ is the smallest root of (13) on $[0, 1]$. Moreover, since $\hat{g}$ is an LST of a probability distribution, $Q_0(0) > 0$ and $Q_0(1) = 0$. So, the only possibility for (13) to have 2 roots on $[0, 1]$ can be realized when $Q_0'(1) > 0$ (note that $Q_0''(x) > 0 \; \forall x \in (0, 1)$), see Figure 1. The inequality $Q_0'(1) > 0$ is equivalent to the condition $-\rho \hat{g}'(0) > 1$. If $-\rho \hat{g}'(0) \leq 1$ then $\varphi(0) = 1$ and Φ is a proper probability distribution. $\square$

We might also wish to consider operator $K_{\hat{g}}[\varphi](s)$ as well as the Kendall equation for complex argument s, s.t. $\Re s \geq 0$. Once obtained a c.m. function of real argument we can consider it as a function of complex arguments—this is justified by the principle of analytic continuation.

Theorem 8. *The sequence $\{K_{\hat{g}}^n[\hat{f}](s)\}_{n=0}^{\infty}$, where $\hat{f}$ is an LST of some possibly improper c.d.f. F, converges to the unique solution $\varphi(s)$ of the corresponding Kendall equation. The claim holds for complex s, s.t. $\Re s > 0$.*

Proof. Let first s be a real argument. It was proven in Theorem 7 that the Kendall equation has a unique solution. Moreover, the evidence of convergence of $\{\hat{\varphi}_n(s) := K_{\hat{g}}^n[\hat{f}](s)\}_{n=0}^{\infty}$ to the solution $\varphi(s)$ for any real $s > 0$ can be obtained in a constructive way exactly as it was made for the functions $\hat{f}(s) \equiv 0(s)$ and $\hat{f}(s) \equiv$
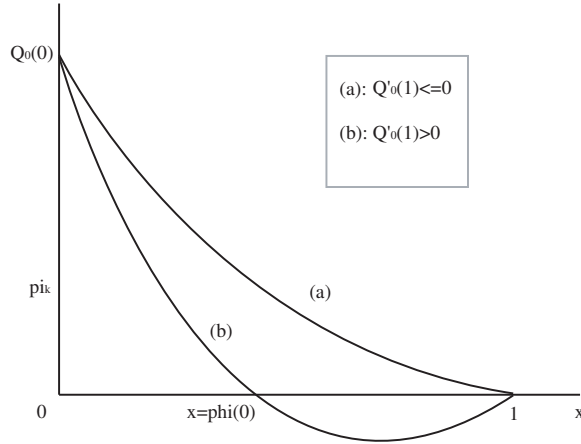
Figure 1. Alternative for $x = \varphi(0)$.

$1(s)$. Restricting attention to real s suffices to imply the point-wise convergence of corresponding c.d.f.'s Φ_n to Φ—the c.d.f. of the solution φ (Theorem 2):

$$\hat{\varphi}_n(s) \to \hat{\varphi}(s) \Rightarrow \Phi_n(t) \to \Phi(t).$$

However, the point-wise convergence for c.d.f.'s implies the transform convergence for all complex $s = x + iy$, s.t. $\Re s = x > 0$, since

$$\Re\hat{\phi}_n(s) = \int\limits_0^\infty \Re e^{-st} d\Phi_n(t) = \int\limits_0^\infty e^{-st} \cos yt \, d\Phi_n(t) \to \int\limits_0^\infty e^{-st} \cos yt \, d\Phi(t), \qquad (14)$$

and

$$\Im\hat{\phi}_n(s) = \int\limits_0^\infty \Im e^{-st} d\Phi_n(t) = \int\limits_0^\infty e^{-st} \sin yt \, d\Phi_n(t) \to \int\limits_0^\infty e^{-st} \sin yt \, d\Phi(t), \qquad (15)$$

so that

$$\Re\hat{\phi}_n(s) \to \Re\hat{\phi}(s),$$
$$\Im\hat{\phi}_n(s) \to \Im\hat{\phi}(s). \qquad (16)$$

We have the following result: $\hat{\phi}_n(s) \to \hat{\phi}(s)$ for all complex s with $\Re s > 0$.

The existence and uniqueness of the solution to the Kendall equation for complex s follows from the existence and uniqueness of the solution for non-negative real s and the principle of analytic continuity. $\qquad\square$

The result and construction similar to that given in Theorem 7 was well-known for decades now. The fact that the iterations in the Kendall equation also work for complex values of the argument s was empirically found by a few authors (e.g. Doshi (1983)). The first proof of this result can be found in Abate and Whitt (1992). Our proof is similar, although some of its parts are different. It is heavily based on the notion of complete monotone function and the Bernstein theorem.

2.2   Adjustments in iterations of the Kendall operator

We know that if $\beta(s)$ is a complete monotone function and $\pi(s)$ satisfies the Kendall equation

$$\pi(s) = \beta(s + \rho(1 - \pi(s))),$$

then $\pi(s)$ is also a complete monotone function and can be numerically estimated at the point $s = s^* \geq 0$ using the following iteration procedure:

$$\pi(s^*) = \lim_{n \to \infty} \pi^{(n)}(s^*), \text{ where}$$

$$\pi^{(n)}(s^*) = \beta(s^* + \rho(1 - \pi^{(n-1)}(s^*))), \text{ with } \pi^{(0)}(s^*) \in [0, 1]. \tag{17}$$

Here, the sequence $\{\pi^{(n)}(s^*)\}_{n=0}^{\infty}$ is non-increasing if $\pi^{(0)}(s^*) > \pi(s^*)$, and non-decreasing if $\pi^{(0)}(s^*) < \pi(s^*)$.

Remark 2. Note that the convergence of $\{\pi^{(n)}(s)\}_{n=0}^{\infty}$ to the solution $\pi(s)$ for complex s is assured by Theorem 8. However, there is no general result on the monotonicity of such convergence unless s is real. Abate and Whitt (1992) provided an example with no monotonicity (in any sense) of iterations in the complex case.

Using the property of complete monotonicity of the solution to the Kendall equation one can decrease the number of iterations to obtain the solution with a given precision. The idea is to set initial guess $\pi^{(0)}(s^*)$ of $\pi(s^*)$ closer to this value (s^* is a real non-negative number).

Suppose we are evaluating the solution to the Kendall equation on a regular grid $\{0, h, 2h, \ldots, (k-2)h, (k-1)h, kh, \ldots\}$. Suppose that π_{k-2} and π_{k-1} are the values which approximate $\pi(s)$ at the points $s = (k-2)h$ and $s = (k-1)h$. Then, let

$$\pi^{(0)}(s^* = kh) := \frac{\pi_{k-1} + \max(0, 2\pi_{k-1} - \pi_{k-2})}{2} \tag{18}$$

be the initial value for the process of iterations to the Kendall equation to evaluate $\pi(s)$ at $s = s^*$. The simple rationale behind this setting can easily be seen from Figure 2.

Alternatively, setting

$$\tilde{\pi}^{(0)}(s^*) := \pi_{k-1} \approx \pi(s^* - h),$$

and

$$\underset{\sim}{\pi}^{(0)}(s^*) := \max(0, 2\pi_{k-1} - \pi_{k-2}) \approx \max(0, 2\pi(s^* - h) - \pi(s^* - 2h)),$$

one may produce two sequences $\{\tilde{\pi}^{(n)}(s^*)\}_{n=0}^{\infty} \searrow \pi(s^*)$ and $\{\underset{\sim}{\pi}^{(n)}(s^*)\}_{n=0}^{\infty} \nearrow \pi(s^*)$ by iterating the Kendall equation:

$$\tilde{\pi}^{(n)}(s^*) = \beta(s^* + \rho(1 - \tilde{\pi}^{(n-1)}(s^*))),$$
$$\underset{\sim}{\pi}^{(n)}(s^*) = \beta(s^* + \rho(1 - \underset{\sim}{\pi}^{(n-1)}(s^*))), \tag{19}$$
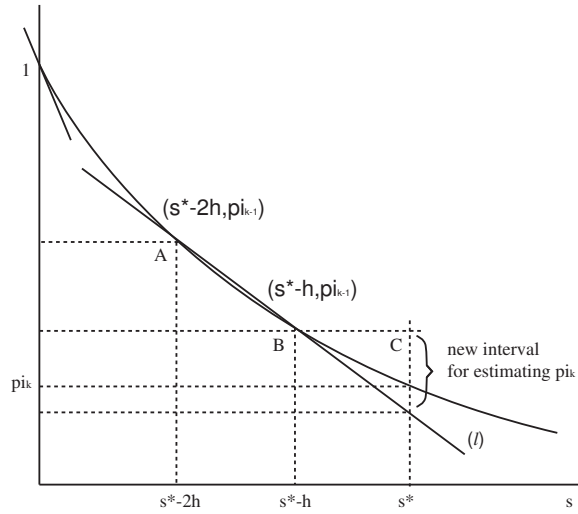
Figure 2. Improvement for iterations in Kendall equation

until

$$\epsilon_n = \frac{\tilde{\pi}^{(n)}(s^*) - \underline{\pi}^{(n)}(s^*)}{2} < \epsilon,$$

where ϵ is a given precision. Finally, evaluate $\pi(s^*) \approx \frac{\tilde{\pi}^{(n)}(s^*) + \underline{\pi}^{(n)}(s^*)}{2}$. The dependence of the number of iterations on the order of accuracy in iterations of the Kendall equation (for a particular value of s) is shown in Figure 3. The comparison between improved and not improved iteration process for solving the Kendall equation (for different types of service distribution) is shown in Figure 4.

3   Concluding remarks

It might seem the adjustments in iterations of the Kendall equation give non-essential gain in the number of operations needed to perform the iterations in order to achieve a solution with a certain level of precision in the case of $M|G|1$. However, in the context of the study of priority queueing systems with switchover times one needs to perform the iteration process quite many times. In the light of this, the adjustment process can be efficiently used for the acceleration of the numerical scheme's performance. Using this acceleration procedure the algorithms of busy period determination for priority queueing systems (e.g., the algorithm BPLST in Bejan (2004)) can be reviewed and improved.
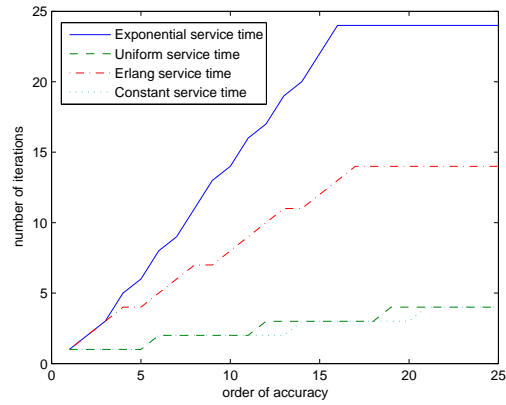
Figure 3. Number of iterations needed to solve the Kendall equation with corresponding accuracy (for different types of service distribution). The plot was obtained evaluating LST at $s = 2$—different values of s will produce different plots! This can be seen from Figure 4.
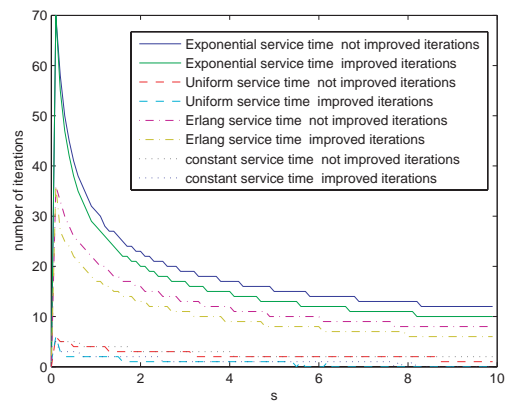

Figure 4. Comparison between improved and not improved iteration process for solving the Kendall equation; order of accuracy is 15.

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