The property of universality for some monoid algebras over non-commutative rings

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Abstract. We define on an arbitrary ring $A$ a family of mappings $(\sigma_{x,y})$ subscripted with elements of a multiplicative monoid $G$. The assigned properties allow to call these mappings derivations of the ring $A$. A monoid algebra of $G$ over $A$ is constructed explicitly, and the universality property of it is shown.

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In this note we consider monoid algebras over non-commutative rings. First, we introduce axiomatically a family of mappings $\sigma_{x,y}$ defined on a ring $A$ and subscripted with elements of a multiplicative monoid $G$. Due to their assigned properties these mappings can be called derivations of $A$. Next, we construct a monoid algebra $\langle G \rangle$ over $A$ by means of the family $\sigma_{x,y}$, and the universality of it is shown.

1. Let $A$ be a ring (in general non-commutative) and $G$ a multiplicative monoid. Throughout the paper we consider $1 \neq 0$ (where 0 is the null element of $A$, and 1 is the unit element for multiplication), the unit element of $G$ is denoted by $e$. We introduce a family of mappings of $A$ into itself by the following assumption.

(A) For each $x \in G$ there exists a unique family $\sigma_x = (\sigma_{x,y})_{y \in G}$ of mappings $\sigma_{x,y} : A \to A$ such that $\sigma_{x,y} = 0$ for almost all $y \in G$ (here and thereafter, almost all will mean all but a finite number, that is, $\sigma_{x,y} \neq 0$ only for a finite set of $y \in G$) and for which the following properties are fulfilled:

(i) $\sigma_{x,y}(a + b) = \sigma_{x,y}(a) + \sigma_{x,y}(b)$ ($a, b \in A; x, y \in G$);
(ii) $\sigma_{x,y}(ab) = \sum_{z \in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$ ($a, b \in A; x, y \in G$);
(iii) $\sigma_{x,y,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$ ($x, y, z \in G$);
(iv$_1$) $\sigma_{x,y}(1) = 0$ ($x \neq y; x, y \in G$);
(iv$_2$) $\sigma_{x,x}(1) = 1$ ($x \in G$);
(iv$_3$) $\sigma_{e,x}(a) = 0$ ($x \neq e; x \in G$);
(iv$_4$) $\sigma_{e,e}(a) = a$ ($a \in A$).

In (ii) the elements are multiplied as in the ring $A$, but in (iii) the symbol $\circ$ means the composition of maps.

Examples. 1. Let $A$ be a ring and let $G$ be a multiplicative monoid, and let $\sigma$ be a monoid-homomorphism of $G$ into $\text{End}(A)$, i.e. $\sigma(xy) = \sigma(x) \circ \sigma(y)$ ($x, y \in G$) and $\sigma(e) = 1_A$. We define $\sigma_{x,y} : A \to A$ such that $\sigma_{x,x} = \sigma(x)$ for $x \in G$ and $\sigma_{x,y} = 0$ for $y \neq x$. The properties (i) -- (iv$_4$) of (A) are verified at once.
2. Let \( A \) be a ring, and let \( \alpha \) be an endomorphism of \( A \) and \( \delta \) be an \( \alpha \)-differentiation of \( A \), i.e.

\[
\delta(a + b) = \delta(a) + \delta(b), \quad \delta(ab) = \delta(a)b + \alpha(a)\delta(b)
\]

for every \( a, b \in A \). Denote by \( G \) the monoid of elements \( x_n \) \((n = 0, 1, \ldots)\) endowed with the law of composition defined by \( x_n x_m = x_{n+m} \) \((n, m = 0, 1, \ldots; x_0 := e)\). We write \( \sigma_{nm} \) instead of \( \sigma_{x_n x_m} \) by defining \( \sigma_{nm} : A \to A \) as the following mappings \( \sigma_{00} = 1_A, \sigma_{10} = \delta, \sigma_{20} = \alpha, \sigma_{nm} = 0 \) for \( m > n \) and \( \sigma_{nm} = \sum_{j_1 + \ldots + j_n = m} \sigma_{1j_1} \circ \ldots \circ \sigma_{1j_n} \) \((m = 0, 1, \ldots, n; n = 1, 2, \ldots)\), where \( j_k = 0, 1 \) \((k = 1, \ldots, n)\). The family \( \sigma = (\sigma_{nm}) \) satisfies the axioms \((i)-(iv_4)\) of \( A \).

2. Next, we consider an algebra \( A\langle G \rangle \) connected with the structure of differentiation \( \sigma = (\sigma_{x,y}) \). Let \( A\langle G \rangle \) be the set of all mappings \( \alpha : G \to A \) such that \( \alpha(x) = 0 \) for almost all \( x \in G \). We define the addition in \( A\langle G \rangle \) to be the ordinary addition of mappings into the additive group of \( A \) and define the operation of \( A \) on \( A\langle G \rangle \) by the map \((a, \alpha) \to a\alpha \) \((a \in A)\), where \((a\alpha)(x) = a\alpha(x) \) \((x \in G)\). Note that, in respect to these operations, \( A\langle G \rangle \) forms a left module over \( A \). Following notations made in [1] we write an element \( \alpha \in A\langle G \rangle \) as a sum \( \alpha = \sum_{x \in G} a_x \cdot x \), where by \( a \cdot x \) \((a \in A, x \in G)\) is denoted the mapping whose value at \( x \) is \( a \) and \( 0 \) at elements different from \( x \). Certainly, the above sum is taken over almost all \( x \in G \). \( A\langle G \rangle \) becomes a ring if for elements of the form \( a \cdot x \) \((a \in A; x \in G)\) we define their product by the rule

\[
(a \cdot x)(b \cdot y) = \sum_{z \in G} a_x \sigma_{x,z}(b) \cdot zy \quad (a, b \in A; x, y \in G)
\]

and then extend for \( \alpha, \beta \in A\langle G \rangle \) by the property of distributivity. We let

\[
\alpha \alpha = \sum_{x \in G} \left( \sum_{y \in G} a_y \sigma_{y,x}(a) \right) \cdot x, \quad (a \in A, \alpha \in A\langle G \rangle)
\]

for \( a \in A \) and \( \alpha \in A\langle G \rangle \), and thus we obtain an operation of \( A \) on \( A\langle G \rangle \) and in such a way we make \( A\langle G \rangle \) into a right \( A \)-module. Thus, we may view \( A\langle G \rangle \) as an algebra over \( A \).

**Remark.** Let us consider the situation described in Example 1. Then the law of multiplication in \( A\langle G \rangle \) is given as follows

\[
\left( \sum_{x \in G} a_x \cdot x \right) \left( \sum_{x \in G} b_x \cdot x \right) = \sum_{x \in G} \sum_{y \in G} a_x \sigma_{x,y}(b_y) \cdot xy.
\]

In this case, the monoid algebra \( A\langle G \rangle \) represents a crossed product \([2, 3]\) of the multiplicative monoid \( G \) over the ring \( A \) with respect to the factors \( \rho_{x,y} = 1 \) \((x, y \in G)\). If \( G \) is a group, and \( \sigma : G \to End(A) \) is such that \( \sigma(x) = 1_A \) for all \( x \in G \), we evidently obtain an ordinary group ring \([4]\) (the commutative case see also \([5]\)).
3. In this subsection we show that $\langle A; f \rangle$ is a free $G$-algebra over $A$. Let $B$ be another ring. Given a ring-homomorphism $f : A \rightarrow B$ it can be defined on the ring $B$ a structure of $A$-module, defining the operation of $A$ on $B$ by the map $(a, b) \rightarrow f(a)b$ for all $a \in A$ and $b \in B$. We denote this operation by $a \ast b$. The axioms for a module are trivially verified. Let now $\varphi : G \rightarrow B$ be a multiplicative monoid-homomorphism. Denote by $\langle B; f, \varphi \rangle$ the module formed by all linear combinations of elements $\varphi(x)$ $(x \in G)$ over $A$ in respect to the operation $\ast$. The axioms for a left $A$-module are trivially verified.

We assume that the homomorphisms $f$ and $\varphi$ satisfy the following assumption.

(B) $\varphi(G)f(A) \subseteq \langle B; f, \varphi \rangle$.

Thus, it is postulated that an element $\varphi(x)f(a)$ $(a \in A, x \in G)$ can be written as a linear combination of the form $\sum_{b \in B, y \in G} b \varphi(y)$. The coefficients $b$ depend on $\varphi(x), \varphi(y)$ and $f(a)$. To designate this fact we denote the corresponding coefficients by $\sigma_{\varphi(x), \varphi(y)}(f(a))$. Therefore, it can be considered that there are defined a family of mappings $\sigma_{\varphi(x), \varphi(y)} : B \rightarrow B$ such that

$$\varphi(x)f(a) = \sum_{y \in G} \sigma_{\varphi(x), \varphi(y)}(f(a))\varphi(y) \ (a \in A, x \in G).$$

By these considerations, we may view $\langle B; f, \varphi \rangle$ as a right $A$-module. In order to make the module $\langle B; f, \varphi \rangle$ to be a ring we require the following additional assumption.

(C) The homomorphisms $f$ and $\varphi$ are such that the following diagram

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\sigma_{x,y} & & \sigma_{\varphi(x), \varphi(y)} \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}$$

is commutative for every $x, y \in G$, i.e. $\sigma_{\varphi(x), \varphi(y)} \circ f = f \circ \sigma_{x,y}$ $(x, y \in G)$.

We define multiplication in $\langle B; f, \varphi \rangle$ by the rules

$$(\sum_{x \in G} a_x \ast \varphi(x)) (\sum_{x \in G} b_x \ast \varphi(x)) = \sum_{x \in G, y \in G} (a_x \ast \varphi(x))(b_y \ast \varphi(y)),$$

$$(a_x \ast \varphi(x))(b_y \ast \varphi(y)) = f(a_x) \sum_{z \in G} \sigma_{\varphi(x), \varphi(z)}(f(b_y))\varphi(zy).$$

The verification that $\langle B; f, \varphi \rangle$ is a ring under the above laws of composition is direct. Thus, we have made $\langle B; f, \varphi \rangle$ into an algebra over $A$ (in general, non-commutative).

Next, we define a category $\mathcal{C}$ whose objects are algebras $\langle B; f, \varphi \rangle$ constructed as above, and whose morphisms between two objects $\langle B; f, \varphi \rangle$ and $\langle B'; f', \varphi' \rangle$ are ring-homomorphisms $h : B \rightarrow B'$ making the diagrams commutative:

$$\begin{array}{ccc}
G & \xrightarrow{=} & G \\
\varphi & & \varphi' \\
B & \xrightarrow{h} & B' \\
f & \xrightarrow{=} & f' \\
A & \xrightarrow{=} & A
\end{array}$$
The axioms for a category are trivially satisfied. We call a universal object in the category $C$ a free $G$-algebra over $A$, or a free $(A, G)$-algebra. It turns out that the monoid algebra $A\langle G \rangle$ represents a free $(A, G)$-algebra. To this end, we observe that the mapping $\varphi_0 : G \to A\langle G \rangle$ given by $\varphi_0(x) = 1 \cdot x$ ($x \in G$) is a monoid-homomorphism. The mapping $\varphi_0$ is embedding of $G$ into $A\langle G \rangle$. In addition, we have a ring-homomorphism $f_0 : A \to A\langle G \rangle$ given by $f_0(a) = a \cdot e$ ($a \in A$). Obviously, $f_0$ is also an embedding. We identify $A\langle G \rangle$ with the triple $\langle A\langle G \rangle; f_0, \varphi_0 \rangle$ and in this sense we treat $A\langle G \rangle$ as an object of the category $C$. The property of the universality of $A\langle G \rangle$ is formulated by the following assertion.

**Theorem 1.** Let $A$ be a ring, and $G$ a multiplicative monoid for which the assumptions $(A), (B)$ and $(C)$ are satisfied. Then for every object $\langle B; f, \varphi \rangle$ of the category $C$ there exists a unique ring-homomorphism $h : A\langle G \rangle \to B$ making the following diagram commutative

$$
\begin{array}{ccc}
G & \longrightarrow & G \\
\varphi_0 \downarrow & & \downarrow \varphi \\
A\langle G \rangle & \stackrel{h}{\longrightarrow} & B \\
f_0 \uparrow & & \uparrow f \\
A & \longrightarrow & A
\end{array}
$$

The relation with the theory of skew polynomial rings [6–8] and with those obtained by Yu. M. Ryabukhin [9] (see also [10]), and further properties of the general derivation mappings $\sigma_{x,y}$ ($x, y \in G$) will be given in a subsequent publication.

**References**