BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(50), 2006, Pages 85–91
ISSN 1024–7696

Unsteady flow of an Oldroyd-B fluid induced by a constantly accelerating plate

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Abstract. We study the start-up flow of an Oldroyd-B fluid between two infinite parallel plates, one of them at rest and the other one being subject, after time zero, to a constant acceleration A. The solutions that are obtained satisfy both the associate partial differential equations and all imposed initial and boundary conditions. They reduce to those for a Maxwell, Second grade or Navier-Stokes fluid as a limiting case.

Mathematics subject classification: 76A05.
Keywords and phrases: Oldroyd-B fluid, velocity field, tangential tension, constantly accelerating plate.

1 Introduction

In a recent paper [1], we established exact solutions for the motion of a second grade fluid and of a Maxwell one between two infinite parallel plates, one of them being subject to a constant acceleration A. It is the goal of this work to extend these results to a larger class of non-Newtonian fluids, namely Oldroyd-B fluids. The constitutive equations of an incompressible Oldroyd-B fluid, as they were presented by Rajagopal [2], are

$$\mathbf{T} = -\,\mathrm{p}\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}) = \mu[\mathbf{A} + \lambda_r(\dot{\mathbf{A}} - \mathbf{LA} - \mathbf{AL}^{\mathrm{T}})], \tag{1.1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{S}$ the extra-stress tensor, $\mathbf{L}$ the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}$ is the first Rivlin-Ericksen tensor, $-\mathrm{p}\mathbf{I}$ denotes the indeterminate spherical stress, μ is the dynamic viscosity, λ and $\lambda_r (< \lambda)$ are relaxation and retardation times and the superposed dot indicates the material time derivative.

This model includes as special cases the Maxwell model and linearly viscous fluid model. Consequently, their solutions will appear as special cases of our solutions. Furthermore, the solutions for a second grade fluid can be also obtained. Recently, the Oldroyd-B fluids have received a lot of attention from both the theoreticians and the experimentalists in rheology. They can describe many of the non-Newtonian characteristics exhibited by polymeric materials such as stress-relaxation, normal stress differences in simple shear flows and non-linear creep.

2 Governing equations

In the following we shall consider unidirectional motions of the form [1, 3, 4]

$$\mathbf{v} = \mathbf{v}(y,\, t) = v(y,\, t)\, i, \tag{2.1}$$

where i is the unit vector along the $x-$coordinate direction of the system of Cartesian coordinates x, y and z. For this velocity field, the constraint of incompressibility is automatically satisfied. We shall also suppose that the extra-stress tensor $\mathbf{S}$ depends on y and t only, i.e., $\mathbf{S} = \mathbf{S}(\mathrm{y}, \mathrm{t})$. If the fluid has been at rest till the moment $t = 0$, then the initial condition

$$\mathbf{S}(y,\, 0) = \mathbf{0}, \tag{2.2}$$

together with $(1.1)_2$ lead to $S_{xz} = S_{yz} = S_{yy} = S_{zz} = 0$ [5] and

$$(1 + \lambda\partial_t)\tau = \mu(1 + \lambda_r\partial_t)\partial_y v, \quad (1 + \lambda\partial_t)\sigma - 2\lambda\tau\partial_y v = -2\mu\lambda_r(\partial_y v)^2, \tag{2.3}$$

where $\tau = S_{xy}$ and $\sigma = S_{xx}$.

The balance of linear momentum, in the absence of body forces and of a pressure gradient in the x-direction, reduce to

$$\partial_y\tau = \rho\partial_t v, \quad \partial_y p = \partial_z p = 0\,. \tag{2.4}$$

Eliminating τ between Eqs. $(2.3)_1$ and $(2.4)_1$, we attain to the linear partial differential equation

$$\lambda\partial_t^2 v(y,t) + \partial_t v(y,t) = \nu(1 + \lambda_r\partial_t)\partial_y^2 v(y,t), \tag{2.5}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid and ρ its constant density.

3 Couette flow induced by a constantly accelerating plate

Consider an incompressible Oldroyd-B fluid at rest between two infinite parallel plates at a distance h apart. Suppose that the lower plate is subject, after time zero, to a constant acceleration A in a direction parallel to the upper one, which is stationary. The governing equation is (2.5) while the initial and boundary conditions are [5]

$$v(y,0) = \partial_t v(y,0) = 0; \quad y \in [0,\, h)\,, \tag{3.1}$$

respectively,

$$v(0,t) = At, \quad v(h,t) = 0; \quad t > 0\,. \tag{3.2}$$

Multiplying both sides of Eq. (2.5) by $\sin(\lambda_n y)$, integrating between the limits $y = 0$ and $y = h$ and having (3.1) and (3.2) in mind, we find that [6]

$$\lambda\ddot{v}_{sn}(t) + (1 + \alpha\lambda_n^2)\dot{v}_{sn}(t) + \nu\lambda_n^2 v_{sn}(t) = \lambda_n A(\nu t + \alpha); \quad t > 0\,, \tag{3.3}$$

where $\alpha = \nu\lambda_r$, $\lambda_n = n\pi/h$ and the finite Fourier sine transforms $v_{sn}(t)$, $(n = 1, 2, 3, \dots)$, of $v(y, t)$ have to satisfy the conditions

$$v_{sn}(0) = \partial_t v_{sn}(0) = 0; \quad n = 1, 2, 3, \dots \tag{3.4}$$

The solutions of the ordinary differential equations (3.3) with the initial conditions (3.4) are

$$v_{sn}(t) = \frac{A}{\lambda_n}\begin{cases} t - \dfrac{1}{\nu\lambda_n^2}\left[1 - \dfrac{r_{2n}r_{3n}\exp(r_{1n}t) - r_{1n}r_{4n}\exp(r_{2n}t)}{r_{2n} - r_{1n}}\,\lambda\right] \\ \qquad\qquad \text{if } \lambda_n \in (0, \infty)\backslash\{a, b\} \\ t - \dfrac{1}{\nu\lambda_n^2}\left\{1 - \left[\dfrac{1-\alpha^2\lambda_n^4}{4\lambda}\,t + 1\right]\exp\left(-\dfrac{1+\alpha\lambda_n^2}{2\lambda}\,t\right)\right\} \\ \qquad\qquad \text{if } \lambda_n \in \{a, b\}\,, \end{cases} \tag{3.5}$$

where

$$r_{1n},\, r_{2n} = \frac{-(1+\alpha\lambda_n^2) \pm \sqrt{(1+\alpha\lambda_n^2)^2 - 4\nu\lambda\lambda_n^2}}{2\lambda},$$

$$r_{3n},\, r_{4n} = \frac{1-\alpha\lambda_n^2 \pm \sqrt{(1+\alpha\lambda_n^2)^2 - 4\nu\lambda\lambda_n^2}}{2\lambda},$$

$$a = \frac{1}{\sqrt{\nu}(\sqrt{\lambda} + \sqrt{\lambda - \lambda_r})}$$

and

$$b = \frac{1}{\sqrt{\nu}(\sqrt{\lambda} - \sqrt{\lambda - \lambda_r})}.$$

Now, using the Fourier's sine formula [6], we find that

$$v(y, t) = \left(1 - \frac{y}{h}\right) At - \frac{2A}{\nu h}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n y)}{\lambda_n^3} +$$

$$+ \frac{2A}{\nu h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\Phi_n(t)\frac{\sin(\lambda_n y)}{\lambda_n^3}\,, \tag{3.6}$$

where

$$\Phi_n(t)=\exp\left(-\frac{\alpha\lambda_n^2}{2\lambda}\,t\right)\begin{cases} ch\left(\dfrac{\beta_n t}{2\lambda}\right)+\dfrac{1+\nu(\lambda_r-2\lambda)\lambda_n^2}{\beta_n}sh\left(\dfrac{\beta_n t}{2\lambda}\right) & \text{if } \lambda_n\in(0,\,a)\cup(b,\,\infty)\\[2mm] \dfrac{1-\alpha^2\lambda_n^4}{4\lambda}\,t+1 & \text{if } \lambda_n\in\{a,\,b\}\\[2mm] \cos\left(\dfrac{\gamma_n t}{2\lambda}\right)+\dfrac{1+\nu(\lambda_r-2\lambda)\lambda_n^2}{\gamma_n}\sin\left(\dfrac{\gamma_n t}{2\lambda}\right) & \text{if } \lambda_n\in(a,\,b),\end{cases}$$

where

$$\beta_n=\sqrt{(1+\alpha\lambda_n^2)^2-4\nu\lambda\lambda_n^2}$$

and

$$\gamma_n=\sqrt{4\nu\lambda\lambda_n^2-(1+\alpha\lambda_n^2)^2}.$$

From $(2.3)_1$ and (2.2) it easily results that

$$\tau(y,t)=\frac{\mu}{\lambda}\exp\left(-\frac{t}{\lambda}\right)\int_0^t\exp\left(\frac{\tau}{\lambda}\right)(1+\lambda_r\partial_\tau)\partial_y v(y,\tau)d\tau. \tag{3.7}$$

By substituting (3.6) into (3.7) we find that

$$\tau(y,\,t)=-\frac{\mu A}{h}\left\{t+(\lambda_r-\lambda)\left[1-\exp\left(-\frac{t}{\lambda}\right)\right]\right\}-$$
$$-\frac{2\rho A}{h}\sum_{n=1}^{\infty}\frac{\cos(\lambda_n y)}{\lambda_n^2}+\frac{2\rho A}{h}\exp\left(-\frac{t}{\lambda}\right)\sum_{n=1}^{\infty}\Psi_n(t)\frac{\cos(\lambda_n y)}{\lambda_n^2}\,, \tag{3.8}$$

where

$$\Psi_n(t)=\exp\left(-\frac{\alpha\lambda_n^2}{2\lambda}\,t\right)\begin{cases} ch\left(\dfrac{\beta_n t}{2\lambda}\right)+\dfrac{1-\alpha\lambda_n^2}{\beta_n}sh\left(\dfrac{\beta_n t}{2\lambda}\right) & if\ \ \lambda_n\in(0,\,a)\cup(b,\,\infty)\\[2mm] \dfrac{1+\alpha\lambda_n^2}{2\lambda}\left[1-\lambda_r\dfrac{1+\alpha\lambda_n^2}{2\lambda}\right]t+1 & \text{if } \lambda_n\in\{a,\,b\}\\[2mm] \cos\left(\frac{\gamma_n t}{2\lambda}\right)+\dfrac{1-\alpha\lambda_n^2}{\gamma_n}\sin\left(\dfrac{\gamma_n t}{2\lambda}\right) & \text{if } \lambda_n\in(a,\,b).\end{cases}$$

As soon as the velocity field $v(y,\,t)$ and the tangential tension $\tau(y,\,t)$ have been determined, we can find the normal tension $\sigma(y,\,t)$ using $(2.3)_2$ and (2.2). The hydrostatic pressure as it results from (2.4), is an arbitrary function of t.

4 Limiting cases

1. Taking the limit of Eqs. (3.6) and (3.8) as $\lambda_r \to 0$, we attain to the similar solutions for a Maxwell fluid (see [1], Eq. (4.8), for the velocity field)

$$v(y,\, t) = \left(1 - \frac{y}{h}\right)\, At - \frac{2A}{\nu h}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n y)}{\lambda_n^3} +$$

$$+\frac{2A}{\nu h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}V_n(t)\frac{\sin(\lambda_n y)}{\lambda_n^3} \tag{4.1}$$

and

$$\tau(y,\, t) = -\frac{\mu A}{h}\left\{t + \lambda\left[\exp\left(-\frac{t}{\lambda}\right) - 1\right]\right\} - \frac{2\rho A}{h}\sum_{n=1}^{\infty}\frac{\cos(\lambda_n y)}{\lambda_n^2} +$$

$$+\frac{2\rho A}{h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}T_n(t)\frac{\cos(\lambda_n y)}{\lambda_n^2}, \tag{4.2}$$

where

$$V_n(t) = \begin{cases} ch\left(\dfrac{\alpha_n t}{2\lambda}\right) + \dfrac{1 - 2\nu\lambda\lambda_n^2}{\alpha_n} sh\left(\dfrac{\alpha_n t}{2\lambda}\right) & \text{if } \lambda_n \in \left(0,\, \dfrac{1}{2\sqrt{\nu\lambda}}\right) \\[2ex] \dfrac{1}{4\lambda}\, t + 1 & \text{if } \lambda_n = \dfrac{1}{2\sqrt{\nu\lambda}} \\[2ex] \cos\left(\dfrac{\delta_n t}{2\lambda}\right) + \dfrac{1 - 2\nu\lambda\lambda_n^2}{\delta_n}\sin\left(\dfrac{\delta_n t}{2\lambda}\right) & \text{if } \lambda_n \in \left(\dfrac{1}{2\sqrt{\nu\lambda}},\, \infty\right) \end{cases},$$

$$T_n(t) = \begin{cases} ch\left(\dfrac{\alpha_n t}{2\lambda}\right) + \dfrac{1}{\alpha_n} sh\left(\dfrac{\alpha_n t}{2\lambda}\right) & \text{if } \lambda_n \in \left(0,\, \dfrac{1}{2\sqrt{\nu\lambda}}\right) \\[2ex] \dfrac{1}{2\lambda}\, t + 1 & \text{if } \lambda_n = \dfrac{1}{2\sqrt{\nu\lambda}} \\[2ex] \cos\left(\dfrac{\delta_n t}{2\lambda}\right) + \dfrac{1}{\delta_n}\sin\left(\dfrac{\delta_n t}{2\lambda}\right) & \text{if } \lambda_n \in \left(\dfrac{1}{2\sqrt{\nu\lambda}},\, \infty\right) \end{cases},$$

$\alpha_n = \sqrt{1 - 4\nu\lambda\lambda_n^2}$ and $\delta_n = \sqrt{4\nu\lambda\lambda_n^2 - 1}$.

2. In the special case when both λ_r and $\lambda \to 0$ into Eqs. (3.6) and (3.8), or $\lambda \to 0$ in Eqs. (4.1) and (4.2), we get the simple solutions (see [1], Eq. (3.10) for the velocity field)

$$v(y,\,t) = \left(1 - \frac{y}{h}\right)\,At - \frac{2A}{h}\sum_{n=1}^{\infty}\left[1 - e^{-\nu\lambda_n^2 t}\right]\frac{\sin(\lambda_n y)}{\lambda_n^3}\,, \tag{4.3}$$

respectively,

$$\tau(y,\,t) = -\frac{\mu A}{h}t - \frac{2\rho A}{h}\sum_{n=1}^{\infty}\left[1 - e^{-\nu\lambda_n^2 t}\right]\frac{\cos(\lambda_n y)}{\lambda_n}\,, \tag{4.4}$$

for a Navier-Stokes fluid.

3. Finally, by formally letting $\lambda_r \to 0$ into (3.6) and (3.8) (but using only the first lines of $\Phi_n(\cdot)$ and $\Psi_n(\cdot)$), we also attain to the solutions (see [1], Eq. (3.9) for the velocity field)

$$v(y,\,t) = \left(1 - \frac{y}{h}\right)\,At - \frac{2A}{\nu h}\sum_{n=1}^{\infty}\left[1 - \exp\left(-\frac{\nu\lambda_n^2}{1+\alpha\lambda_n^2}\,t\right)\right]\frac{\sin(\lambda_n y)}{\lambda_n^3} \tag{4.5}$$

and

$$\tau(y,\,t) = -\frac{\mu A}{h}(t+\lambda_r) - \frac{2\rho A}{h}\sum_{n=1}^{\infty}\left[1 - \frac{1}{1+\alpha\lambda_n^2}\exp\left(-\frac{\nu\lambda_n^2}{1+\alpha\lambda_n^2}\,t\right)\right]\frac{\cos(\lambda_n y)}{\lambda_n^2}, \tag{4.6}$$

corresponding to a second grade fluid.

5 Conclusions

In the present paper we have established the velocity filed and the associated tangential tension, corresponding to an unsteady lineal flow of an incompressible Oldroyd-B fluid between two infinite parallel plates. One of plates is held fixed and other one is subject, after time zero, to a constant acceleration A. Direct computations show that $v(y,t)$ and $\tau(y,t)$, given by (3.6) and (3.8), satisfy both the associate partial differential equations (2.5) and $(2.3)_1$ and all imposed initial and boundary conditions, the differentiation term by term in sums being clearly permissible. These solutions reduce to those for Maxwell, Navier-Stokes and Second grade fluids as limiting cases.

The solutions (3.6) and (3.8) as well as those for a Maxwell fluid (4.1) and (4.2), contain sine and cosine terms. This indicates that in contrast with the Navier-Stokes and second grade fluids, whose solutions (4.3)-(4.6) do not contain such terms, oscillations are set up in the fluid. The amplitudes of these oscillations decay exponentially in time, the damping being proportional to $\exp(-t/2\lambda)$.

Finally, it is important to underline that the Oldroyd-B model does not contain as a special case the second grade fluid model. However, for some special classes of motions, as that considered here, the governing equations also include the equations of motion for the second grade fluid. Consequently, in all these cases, the similar solutions corresponding to second grade fluids can be also obtained as limiting cases of those for Oldroyd-B fluids.

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Received October 28, 2005

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