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On determining the minimum cost flows in dynamic networks *

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Abstract. The dynamic minimum cost flow problem that generalizes the static one is studied. We assume that the supply and demand function and capacities of edges depend on time. One very important case of the minimum cost flow problem with nonlinear cost functions, defined on edges, that do not depend on flow but depend on time is studied.

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1 Introduction and Problem Formulation

In this paper we study the dynamic version of the minimum cost flow problem on networks, which generalizes the well-known static minimum cost flow problem. We use dynamic network flow models instead of the static ones due to the fact that dynamic flows are much more closer to reality than static flows that can not properly consider the evolution of the system under study over time. For our problem the time is an essential component: the flows of some commodity take time to pass from one location to another and the structure of network changes over time.

The minimum cost flow problem on networks is of special interest not only from the practical point of view but it also has a great theoretical importance in the investigation and solving of various optimization problems on graphs. It can be used for the research and solving of the distribution problem, the synthesis problem of communication networks or the allocation problem. The field of applications of the considered problem considerably enlarges in the case when the cost functions are nonlinear.

In this paper we consider the flow problem on dynamic networks with nonlinear cost functions, defined on edges. Moreover, we assume that the supply and demand function and capacities of edges also depend on time. We study one very important case of the minimum cost flow problem with cost functions that do not depend on flow but depend on time and derive different approaches for its solving.

Let a dynamic network $N = (V, E, q, c, \tau, \varphi)$, which consists of directed graph $G = (V, E)$ with set of vertices $V = V_+ \cup V_- \cup V_0$, where V_+, V_- and V_0 are sets of sources, sinks and intermediate nodes, respectively, and set of arcs E, be given. Without loosing generality, we assume that no edges enter sources or exit sinks. We

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consider the discrete time model, in which all times are integral and bounded by horizon T, which defines the set $\mathbb{T} = \{0, 1, \dots, T\}$ of time moments we consider. The functions in network N are defined as follows: demand and supply function q: $V \times \mathbb{T} \to R$, capacity function c: $E \times \mathbb{T} \to R_+$, transit time function τ: $E \to R_+$, and cost function φ: $E \times \mathbb{T} \times R_+ \to R_+$. The demand and supply function $q_v(t)$ satisfies the following conditions:

a) there exists $v \in V$ with $q_v(0) < 0$;

b) if $q_v(t) < 0$ for a node $v \in V$ then $q_v(t) = 0$, $t = 1, 2, \dots, T$;

c) $\sum\limits_{t \in \mathbb{T}} \sum\limits_{v \in V} q_v(t) = 0$.

Nodes $v \in V$ with $\sum_{t \in \mathbb{T}} q_v(t) < 0$ are called sources, nodes $v \in V$ with $\sum_{t \in \mathbb{T}} q_v(t) > 0$ are called sinks and nodes $v \in V$ with $\sum_{t \in \mathbb{T}} q_v(t) = 0$ are called intermediate.

A dynamic flow on N is a function x: $E \times \mathbb{T} \to R_+$ that satisfies the following conditions:

$$\sum_{\substack{e \in E^+(v) \\ t-\tau_e \geq 0}} x_e(t - \tau_e) - \sum_{e \in E^-(v)} x_e(t) = q_v(t), \ \forall\, t \in \mathbb{T}, \ \forall\, v \in V; \tag{1}$$

$$x_e(t) = 0, \ \forall\, e \in E, \ t = \overline{T - \tau_e + 1, T}; \tag{2}$$

where $E^+(v) = \{(u, v) \,|\, (u, v) \in E\}$, $\quad E^-(v) = \{(v, u) \,|\, (v, u) \in E\}$.

Here the function x defines the value $x_e(t)$ of flow entering edge e at time t. It is easy to observe that the flow does not enter edge e at time t if it has to leave the edge after time T; this is ensured by condition (2). Conditions (1) represent flow conservation constraints.

Feasible dynamic flow also has to verify the following capacity constraints:

$$0 \leq x_e(t) \leq c_e(t), \quad \forall\, t \in \mathbb{T}, \ \forall\, e \in E. \tag{3}$$

Hereinafter we will show that the problem with capacity constraints can be reduced to the one without restrictions on edge capacities.

The considered problem consists in minimizing the integral cost F of transporting all the flow on N:

$$F = \sum_{e \in E} \sum_{t \in \mathbb{T}} \varphi_e(x_e(t), t) \to \min. \tag{4}$$

In the case when $\tau_e = 0, \forall\, e \in E$ and $T = 0$ the formulated problem becomes the classical problem on a static network.

2 The Time-Expanded Network Method

To solve the formulated dynamic problem we reduce it to a static one on an auxiliary time-expanded network N^T. The essence of such a network is that it contains copies of the vertices of the dynamic network for each moment of time, and the transit times and flows are implicit in the edges linking those copies. The time-expanded network $N^T = (V^T, E^T, c^T, q^T, \varphi^T)$ is defined as follows:

1. $V^T\colon\ = \{v(t) \,|\, v \in V,\ t \in \mathbb{T}\}$;
2. $E^T\colon\ = \{(v(t), w(t+\tau_e)) \,|\, e = (v,w) \in E,\ 0 \leq t \leq T - \tau_e\}$;
3. $c^T_{e(t)}\colon\ = c_e(t)$ for $e(t) \in E^T$;
4. $\varphi^T_{e(t)}(x^T_{e(t)})\colon\ = \varphi_e(x_e(t), t)$ for $e(t) \in E^T$;
5. $q^T_{v(t)}\colon\ = q_v(t)$ for $v(t) \in V^T$.

If we define a flow correspondence to be $x^T_{e(t)}\colon\ = x_e(t)$, the minimum-cost flow problem on dynamic networks can be solved by using the solution of the static minimum cost flow problem on the time-expanded network. It is shown in [2] that for each minimum-cost flow in the dynamic network there is a corresponding minimum-cost flow in the static network and vice versa. In such a way, to solve the considered problem, we have to build the time-expanded network N^T for the given dynamic network N, to solve the classical minimum-cost flow problem on the static network N^T and to reconstruct the solution of the static problem on N^T to the dynamic problem on N.

Remark 1. In the case of the acyclic network the constructed time-expanded network can be reduced to the network of the smaller size, using the following algorithm, based on the process of elimination of irrelevant nodes from the time-expanded network [5]:

Algorithm

1. To build the time-expanded network N^{T^*} for the given dynamic network N.

2. To perform a breadth-first parse of the nodes for each source from the time-expanded network. The result of this step is the set $V_-(V_-^{T^*})$ of the nodes that can be reached from at least a source in V^{T^*}.

3. To perform a breadth-first parse of the nodes beginning with the sink for each sink and to parse the edges in the direction opposite to their normal orientation. The result of this step is the set $V_+(V_+^{T^*})$ of nodes from which at least a sink in V^{T^*} can be reached.

4. The reduced network will consist of a subset of nodes V^{T^*} and edges from E^{T^*} determined in the following way
$$V'^{T^*} = V^{T^*} \cap V_-(V_-^{T^*}) \cap V_+(V_+^{T^*}), \qquad E'^{T^*} = E^{T^*} \cap (V'^{T^*} \times V'^{T^*}).$$

5. ${q'}^{T^*}_{v(t)}\colon\ = q_v(t)$ for $v(t) \in V'^{T^*}$;

6. ${c'}^{T^*}_{e(t)}\colon\ = c_e(t)$ for $e(t) \in E'^{T^*}$;

7. ${\varphi'}^{T^*}_{e(t)}(x^T_{e(t)})\colon\ = \varphi_e(x_e(t), t)$ for $e(t) \in E^T$. $\square$

In the next section we derive the procedure of the reduction of the considered problem (1)–(4) to the one without condition (3).

3 Reduction of the Considered Problem to the One without Restrictions on Edge Capacities

The procedure of the reduction of the linear minimum-cost flow problem with restrictions on edge capacities to the one without restrictions on edge capacities was proposed in [1]. In this paper we derive this procedure for the minimum-cost flow problem with nondecreasing and nonnegative cost functions.

It is more optimal to reduce the considered dynamic problem to a static one and after that to reduce it to the problem without restrictions on edge capacities. Therefore let us consider problem (1)-(4) on the static network and let us show that this problem can be reduced to a problem on a new network H. For facility we will use the same notations as in the dynamic network but will discard all time information. The new network H will consist of set of vertices W, $|W| = n + m$, and set of arcs F, $|F| = 2m$. The graph (W, F) is a bipartite graph with two parts E and V, i.e. $W = E \bigcup V$ and there are only arcs leaving from vertices of set E and entering vertices of set V. By $[u, v]$ we denote a vertex which corresponds to arc (u, v) in G. If there is an arc (u, v) in G, then there are arcs $([u, v], u)$ and $([u, v], v)$ in F and $\varphi_{([u,v],u)}(x_{([u,v],u)}) = 0$ and $\varphi_{([u,v],v)}(x_{([u,v],v)}) = \varphi_{(u,v)}(x_{(u,v)})$. We associate value $c_{(u,v)}$ with vertices $[u, v]$ and value $\sum_{(u,v)\in E} c_{(u,v)} - q_v$ with vertices $v \in V$. In a new problem we have to find flows $x_{([u,v],w)}$ that solve the following problem:

$$\sum_{(u,v)\in E} \varphi_{([u,v],v)}(x_{([u,v],v)}) \to \min \tag{5}$$

$$x_{([u,v],u)} + x_{([u,v],v)} = c_{(u,v)} \tag{6}$$

$$\sum_{v\in V} [x_{([u,v],u)} + x_{([v,u],u)}] = \sum_{v\in V} c_{(u,v)} - q_u \tag{7}$$

$$x_{([u,v],w)} \geq 0 \tag{8}$$

Now we show that these problems are equivalent. Let us consider that flow $x_{(u,v)}$ is a feasible one for the initial problem. Set

$$x_{([u,v],v)} = x_{(u,v)} \tag{9}$$

$$x_{([u,v],u)} = c_{(u,v)} - x_{(u,v)} \tag{10}$$

In such a way flows in the new problem are nonnegative, so condition (8) is true. Besides, as

$$x_{([u,v],v)} + x_{([u,v],u)} = c_{(u,v)},$$

$$\sum_{v\in V} [x_{([u,v],u)} + x_{([v,u],u)}] = \sum_{v\in V} c_{(u,v)} - \sum_{v\in V} x_{(u,v)} + \sum_{v\in V} x_{(v,u)},$$

then conditions (6) and (7) are true.

Vice versa let us consider that the new problem is feasible. If we define the flow by formula (9), then it is obvious that condition (2) is true. Further in view of (6)

$$\sum_{v\in V} x_{(u,v)} - \sum_{v\in V} x_{(v,u)} = \sum_{v\in V} [x_{([u,v],v)} - x_{([v,u],u)}] =$$

$$= \sum_{v\in V} [c_{(u,v)} - x_{([u,v],u)}] - \sum_{v\in V} x_{([v,u],u)}.$$

Using (7) we reduce the first part of this equality to q_u and hence condition (1) is true.

It is evident that costs of feasible flows in these two problems are equal, so in such a way we reduced the problem with restrictions on edge capacities to the problem without restrictions on edge capacities.

We would like to note that the same argumentation can be held to solve the considered network problem in the case when there are two-side restrictions on edge capacity:

$$r_e \le x_e \le c_e, \quad \forall\, e \in E,$$

where r_e and c_e are lower and upper boundaries of the capacity of the edge e at time t correspondingly. This case can easily be reduced to the one with only one-side restrictions [1]. We introduce one additional artificial source b_1 and one additional artificial sink b_2. For every arc $e = (u, v)$, where $r_e \neq 0$ we introduce arcs (b_1, v) and (u, b_2) with r and 0 as the upper and lower boundaries of the capacity of the edges. We reduce c to $c - r$, but r to 0. We also introduce the arc (b_2, b_1) with $c_{(b_2,b_1)} = \infty$ and $r_{(b_2,b_1)} = 0$.

4 The Minimum Cost Flow Problem with Cost Functions that Do Not Depend on Flow

Further we will study the minimum cost flow problem without restrictions on edge capacities and with cost functions that do not depend on flow, i.e. when the cost functions are constant on the constructed time-expanded network. Obviously that in the case of constant cost functions the structure of optimal solution does not depend on flow distribution on network. When there is only one source and one sink the considered problem becomes a problem of finding the shortest path from a source to a sink. For solving this problem there is a plenty of algorithms [1, 3].

The formulated minimum cost flow problem with constant cost functions is related to network synthesis problems and Steiner trees. The network synthesis problem is formulated as follows. Let the graph $G = (V, E)$, $|V| = n$, with source $\tilde{v} \in V$ be given. Moreover with every arc $e \in E$ a length φ_e is associated. The problem consists in finding the graph $G^* = (V, E^*)$, $E^* \subset E$, in which there is a path from the vertex $\tilde{v}$ to every other vertex $u \in V \setminus \{\tilde{v}\}$ and the total length of its arcs is minimal.

It is easy to show that the optimal graph G^* is a tree rooted at the source. One of the methods for solving this problem is to generate all trees with the root $\tilde{v}$ that allow flows, to calculate the cost of the flow and to select the tree with the minimal cost. Evidently this method can be applied only in the case when the number of vertices $u \in V \setminus \{\tilde{v}\}$, for which $q_u > 0$ is not too big. Nevertheless this approach can be used for some practical problems.

The more general network synthesis problem is formulated as follows. Let the directed graph $G = (V, E)$, $|V| = n$ be given. With every arc $e \in E$ of this graph a length φ_e is associated. Moreover a subset of vertices $\widetilde{V}$, $|\widetilde{V}| = p$ $(p < n)$, is given, for which for every $u \in V \setminus \widetilde{V}$ there is a path $P(v, u)$ from the vertex $v \in \widetilde{V}$ to u. The problem consists in finding the graph $G^* = (V, E^*)$, $E^* \subset E$, which satisfies this condition and the total length of which is minimal. Evidently the optimal graph G^* is a tree with the base $\widetilde{V} \subset V$. An algorithm for finding the minimal tree with the given base is proposed in [6].

The particular case of the network synthesis problem is the Steiner problem, which is formulated as follows. Let the directed graph $G = (V, E)$ with the root vertex $\tilde{v}$ and the subset of vertices $U \subset V$, where a nonnegative length φ_e is associated with each arc $e \in E$, be given. It is necessary to find a tree $T^* = (V^*, E^*)$, $V^* \subset V$, that contains subset of vertices U, i. e. $U \subset V^*$, and for which the sum of lengths of its edges is minimal. In our problem subset U represents a set of stocks on the network. Though the problem of constructing Steiner tree is NP-complete, many heuristic algorithms have been designated to approximate the result within polynomial time [4, 7].

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