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Measure of stability and quasistability to a vector integer programming problem in the l_1 metric*

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Abstract. This paper is devoted to a multicriterion vector integer programming problem with Pareto principle of optimality. Quantitative characteristics of two types of stability under perturbations of the vector criterion parameters with l_1 metric are obtained.

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1 Introduction

Multiobjective combinatorial models of decision making are widespread in design, control, economics and many other fields of applied research. Therefore interest of mathematicians in multiobjective problems of discrete optimization keeps very high, as confirmed by the intensive publishing activity (see, for example, bibliography in [1]). One of the areas of investigations in such problems is stability of the problem solution to perturbations of initial data (of the problem parameters). Various settings of stability problem give rise to numerous directions of research. Not touching upon wide spectrum of questions appeared in this area we only refer to the extensive bibliography [2] and to the monographs [3–5].

Present work is concerned with investigations of quantitative characteristics of stability. Such a characteristic, usually called stability radius, is defined as the limit level of perturbations of the problem parameters, which save a given property of a solution set (or of a certain solution). The perturbed parameters are usually coefficients of the scalar or vector criterion. As a rule, the results of investigation of a stability radius are its formal expressions, estimations and algorithms of its calculation. In the case of a single objective, formulae of stability radius are obtained for problems of Boolean programming, problems on systems of subsets and on graphs [6], for some scheduling problems [5, 7]. Such formulas are the basis of investigations for algorithmic aspects of the stability analysis of discrete optimization problems (see, for example,[8–10]).

Our research continues the cycle of works, devoted to the stability of the vector (multicriterion) integer programming problems [11–17]. In this paper we analyse the

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discrete analogues of the Hausdorff lower and upper semicontinuity of the Pareto optimal mapping to estimate limit levels of perturbations of the partial criteria coefficients mentioned above of the vector integer programming problem in the case of l_1 metric. Note that analogous results were obtained earlier in [15] for vector integer programming problem in the case of l_∞ metric.

2 Basic definitions

Consider n-criterion problem of the vector integer programming with m variables:

$$Cx = (C_1x, C_2x, \ldots, C_nx)^T \to \min_{x \in X},$$

where $C = [c_{ij}]_{n\times m} \in \mathbf{R}^{n\times m}$, $n, m \in \mathbf{N}$, C_i is i-th row of the matrix C, i.e. $C_i = (c_{i1}, c_{i2}, \ldots, c_{im})$, $i \in N_n = \{1, 2, \ldots, n\}$, X is a finite set of (feasible) solutions in $\mathbf{Z}^m$, $|X| > 1$.

Under a vector integer programming problem we understand the problem of finding the Pareto set, i.e. the set of efficient (Pareto optimal) solutions

$$P^n(C) = \{x \in X : \ \pi(x, C) = \emptyset\},$$

where $\pi(x, C) = \{x' \in X : \ Cx \geq Cx', \ Cx \neq Cx'\}$.

We denote this problem by $Z^n(C)$.

We also define the set of weakly efficient solutions (the Slater set [18])

$$Sl^n(C) = \{x \in X : \ \sigma(x, C) = \emptyset\}$$

and the set of strictly efficient solutions (the Smale set [19])

$$Sm^n(C) = \{x \in X : \ \eta(x, C) = \emptyset\},$$

where

$$\sigma(x, C) = \{x' \in X : \ C_ix > C_ix', \quad i \in N_n\},$$
$$\eta(x, C) = \{x' \in X \setminus \{x\} : \ Cx \geq Cx'\}.$$

For any matrix $C \in \mathbf{R}^{n\times m}$ the following inclusions are evident

$$Sm^n(C) \subseteq P^n(C) \subseteq Sl^n(C).$$

It is obvious that in the case, where $n = 1$, the considered problem turns into ordinary scalar integer programming problem $Z^1(C)$, $C \in \mathbf{R}^m$, on the bounded set of feasible solutions. The Pareto set coincides with the Slater set ($P^1(C) = Sl^1(C)$) and they turn into the set of optimal solutions.

Adding a perturbing matrix $C' \in \mathbf{R}^{n\times m}$ to the matrix C, we model perturbations of parameters of the problem. Thus, perturbed problem $Z^n(C + C')$ has the form

$$(C + C')x \to \min_{x \in X}.$$

The Pareto set of this problem is $P^n(C+C')$. For any number $k \in \mathbf{N}$ we define two metrics l_1 and l_∞ in space $\mathbf{R}^k$, i.e. under norms of the vector $z = (z_1, z_2, \dots, z_k) \in \mathbf{R}^k$ we understand correspondingly the numbers

$$||z||_1 = \sum_{i \in N_k} | z_i |, \qquad ||z||_\infty = \max\{| z_i |: \ i \in N_k\}.$$

This allows us formulate the question about quantitative characteristics of the stability. Later in p. 4 and 5 we deduce the corresponding formulas of the limit levels of perturbations. Under the norm of a matrix $C' = [c'_{ij}]_{n \times m}$ we understand the norm of the vector which consists of all elements of the matrix, i.e. the norm of the vector $(c'_{11}, c'_{12}, \dots, c'_{n,m-1}, c'_{nm})$.

We define the set of perturbing matrices in the space with l_1 metric for an arbitrary number $\varepsilon > 0$:

$$\Omega(\varepsilon) = \{C' \in \mathbf{R}^{n \times m} : \ ||C'||_1 < \varepsilon\}.$$

Definitions 1–4 given below are well known (see, for example,[11, 13, 15], in that case the metric l_∞ is defined in the space of perturbing parameters of a vector integer programming problem).

Definition 1. *The vector integer programming problem $Z^n(C)$, $n \geq 1$, is called stable to perturbations of elements of matrix C if there exists a number $\varepsilon > 0$ such that for any perturbing matrix $C' \in \Omega(\varepsilon)$ the following inclusion holds:*

$$P^n(C + C') \subseteq P^n(C).$$

It is evident that the stability of the problem is equivalent to the Hausdorff upper semicontinuity [3, 4, 20] at the point $C \in \mathbf{R}^{n \times m}$ of the optimal mapping

$$P^n : \ \mathbf{R}^{n \times m} \to 2^X, \tag{1}$$

i.e. the point-to-set (set-valued) mapping that assigns the Pareto set $P^n(C)$ to each collection of the problem parameters from metric space $\mathbf{R}^{n \times m}$.

Let us consider a quantitative evaluation of stability .

Definition 2. *Under stability radius of the vector integer programming problem $Z^n(C)$, $n \geq 1$, we understand the number*

$$\rho_1^n(C) = \sup\{\varepsilon > 0 : \ \forall C' \in \Omega(\varepsilon) \ \ (P^n(C + C') \subseteq P^n(C))\}$$

if the problem $Z^n(C)$ is stable, and $\rho_1^n(C) = 0$ otherwise.

In other words, the stability radius of the problem $Z^n(C)$ is the limit level of perturbations of elements of matrix C in the space $\mathbf{R}^{n \times m}$ with metric l_1, which does not lead to appearance of new efficient solutions.

It is clear that the problem $Z^n(C)$ is always stable and its stability radius is equal to infinity if the equation $P^n(C) = X$ holds. The problem $Z^n(C)$, for which the set $\bar{P}^n(C) = X \setminus P^n(C)$ is non-empty, is called non-trivial.

Now consider the case where the stability of problem $Z^n(C)$ is defined as the discrete analogue of the Hausdorff lower semicontinuity at the point C of optimal mapping (1). For the vector integer programming problem, the lower semicontinuity means that there exists a neighborhood of the point C in space $\mathbf{R}^{n\times m}$ where the Pareto set can only expand.

Definition 3. *The vector integer programming problem $Z^n(C)$, $n \geq 1$, is called quasistable to perturbations of the elements of matrix C), if there exists a number $\varepsilon > 0$ such that for any perturbing matrix $C' \in \Omega(\varepsilon)$ the following inclusion holds*

$$P^n(C) \subseteq P^n(C + C').$$

Definition 4. *Under the quasistability radius of the vector integer programming problem $Z^n(C)$, $n \geq 1$, we understand the number*

$$\rho_2^n(C) = \sup\{\varepsilon > 0 : \ \forall C' \in \Omega(\varepsilon) \ \ (P^n(C) \subseteq P^n(C + C'))\},$$

if the problem $Z^n(C)$ is quasistable, and $\rho_2^n(C) = 0$ otherwise.

In that way, the quasistability radius determines the limit level of perturbations preserving all efficient solutions of the initial problem.

3 Auxiliary statements

For any solution $x \in \bar{P}^n(C)$ we define the set

$$P_x(C) = P^n(C) \cap \sigma(x, C).$$

The following properties are obvious.

Property 1. *If $P^n(C) = Sl^n(C)$, then $P_x(C) \neq \emptyset$ for any solution $x \in \bar{P}^n(C)$.*

By definition, put $[z]^+ = \max\{0, z\}$, where $z \in \mathbf{R}$.

Property 2. *If the inequality*

$$(C_i + C_i')(x - x') \leq 0 \tag{2}$$

holds for any index $i \in N_n$, then

$$[C_i(x - x')]^+ \leq ||C_i'||_1 ||x - x'||_\infty. \tag{3}$$

Clearly, inequality (3) holds for $C_i(x - x') \leq 0$. If $C_i(x - x') > 0$, then it follows from (2) and linearity of function $C_i(x - x')$ that

$$[C_i(x - x')]^+ = C_i(x - x') = (C_i + C_i')(x - x') - C_i'(x - x') \leq$$

$$\leq -C_i'(x - x') \leq ||C_i'||_1 ||x - x'||_\infty.$$

Property 3. *If $x \in \bar{P}^n(C)$ and*

$$P^n(C) \cap \sigma(x, C + C') = \emptyset, \tag{4}$$

then there exists a solution $x^* \in \bar{P}^n(C)$ such that $x^* \in Sl^n(C + C')$.

Since $\sigma(x, C + C') = \emptyset$, we can put $x^* = x$. If $\sigma(x, C + C') \neq \emptyset$, then taking into account external stability of the Slater set (see, for example,[18]) there exists a solution $x^* \in \sigma(x, C + C')$ such that $x^* \in Sl^n(C + C')$. It follows from (4) that $x^* \in \bar{P}^n(C)$.

Denote

$$\frac{C_p(x - x')}{C_q(x - x')}$$

by $\gamma(x, x', p, q)$ for any $p, q \in N_n$, $x \in \bar{P}^n(C)$, $x' \in P_x(C)$. It is clear that the values $\gamma(x, x', p, q)$ and $||C_p||_1$ are positive for any parameters x, x', p, q under the assumption $P^n(C) = Sl^n(C)$.

Lemma 1. *Let $P^n(C) = Sl^n(C)$, $x \in \bar{P}^n(C)$, $p, q \in N_n$ and number ψ be positive and such that*

$$||C_q||_1 \max\{\gamma(x, x', p, q) : \ x' \in P_x(C)\} \leq \psi. \tag{5}$$

Then for any number $\varepsilon > \psi$ there exist $C' \in \Omega(\varepsilon)$ and $x^* \in \bar{P}^n(C)$ such that

$$x^* \in Sl^n(C + C'). \tag{6}$$

Proof. It follows directly from Lemma that the inequalities

$$\varepsilon > \psi \geq ||C_q||_1 \zeta(x),$$

where $\zeta(x) = \max\{\gamma(x, x', p, q) : \ x' \in P_x(C)\}$ hold. According to Corollary 1, the set $P_x(C)$ is not empty. It is obvious that there exists number $\delta > 0$ such that

$$\varepsilon > (1 + \delta)||C_q||_1 \zeta(x) > \psi. \tag{7}$$

We define the perturbing matrix $C' = [c'_{ij}]_{n \times m}$ by

$$c'_{ij} = \begin{cases} -\alpha_j, & \text{if} \quad i = p, \ j \in N_m, \\ 0, & \text{if} \quad i \in N_n \setminus \{p\}, \ j \in N_m, \end{cases}$$

where $\alpha_j = (1 + \delta) \ c_{qj} \ \zeta(x)$. Hence, taking into account (7), we have

$$C' \in \Omega(\varepsilon),$$

$$C'_p = -(1 + \delta) C_q \zeta(x), \tag{8}$$

$$C'_i = (0, 0, \ldots, 0) \in \mathbf{R}^m, \qquad i \in N_n \backslash \{p\}. \tag{9}$$

Let us show that equality (4) holds, i.e. there are no solutions from $P^n(C)$ belonging to $\sigma(x, C + C')$. Let $x^0 \in P^n(C)$.

Case 1: $x^0 \in P_x(C)$. Combining equality (8) and the definition of $\zeta(x)$, we obtain

$$(C_p + C'_p)(x - x^0) = C_p(x - x^0) - (1 + \delta)\zeta(x)C_q(x - x^0) \leq$$

$$\leq C_p(x - x^0) - (1 + \delta)\gamma(x, x^0, p, q)C_q(x - x^0) = -\delta C_p(x - x^0) < 0.$$

Hence, $x^0 \notin \sigma(x, C + C')$.

Case 2: $x^0 \in P^n(C) \setminus P_x(C)$. If there exists an index $s \in N_n \setminus \{p\}$ such that $C_s(x - x^0) < 0$, then it follows from (9) that the inequality $(C_s + C'_s)(x - x^0) < 0$ holds.

If for any index $i \in N_n \setminus \{p\}$ the inequality $C_i(x - x^0) \geq 0$ holds, then it follows from $x \in \bar{P}^n(C)$ and $x^0 \in P^n(C) \setminus P_x(C))$ that the inequality $C_p(x - x^0) < 0$ is true. Hence, we have from equality (8):

$$(C_p + C'_p)(x - x^0) \leq C_p(x - x^0) < 0.$$

Consequently we obtain $x^0 \notin \sigma(x, C + C')$ in this case.

Thus, equality (4) holds. Hence it follows from Corollary 3 that there exists a solution $x^* \in \bar{P}^n(C)$ such that inclusion (6) holds.

Lemma 1 is proved.

From Lemma 4.3 [4] (see also Theorem 3.2 [15]) we obtain

Lemma 2. *For any solution $x \in Sl^n(C) \setminus P^n(C)$ and for any number $\varepsilon > 0$ there exists a matrix $C^* \in \Omega(\varepsilon)$ such that $x \in P^n(C + C^*)$.*

Lemma 3. *Let for number ξ and for solutions x and x' the inequalities*

$$0 < \xi||x - x'||_\infty \leq \sum_{i \in N_n} [C_i(x - x')]^+ \tag{10}$$

hold. Then for any perturbing matrix $C' \in \Omega(\xi)$ we have $x \notin \pi(x', C + C')$.

Proof. Suppose, to the contrary, that there exists a perturbing pair $C' \in \Omega(\xi)$ such that $x \in \pi(x', C + C')$. Then for any index $i \in N_n$ inequality (2) is valid. Hence, it follows from Corollary 2 that inequality (3) holds. Since $C' \in \Omega(\xi)$, we have

$$\sum_{i \in N_n} [C_i(x - x')]^+ \leq \sum_{i \in N_n} ||C'_i||_1 ||x - x'||_\infty =$$

$$= ||C'||_1 ||x - x'||_\infty < \xi ||x - x'||_\infty,$$

which gives a contradiction with condition (10).

Lemma 3 is proved.

Lemma 4. *Let $x, x' \in X$, $x \neq x'$. Let vector $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ consist of positive elements such that*

$$\eta_i ||x - x'||_\infty > [C_i(x - x')]^+, \qquad i \in N_n. \tag{11}$$

Then for any number $\varepsilon > \|\eta\|_1$ there exists a perturbing matrix $C' \in \Omega(\varepsilon)$ such that $x \in \pi(x', C + C')$.

Proof. It is enough to build a perturbing matrix $C' \in \Omega(\varepsilon)$ such that

$$(C_i + C'_i)(x - x') < 0, \qquad i \in N_n. \tag{12}$$

Let $q = \arg\max\{|x_j - x'_j| : \ j \in N_n\}$. Define the elements of perturbing matrix $C' = [c'_{ij}]_{n\times m}$ by the formula

$$c'_{ij} = \begin{cases} \beta_i \operatorname{sign}(x'_q - x_q), & \text{if} \quad i \in N_m,\ j = q, \\ 0, & \text{if} \quad i \in N_n,\ j \neq q. \end{cases}$$

It is clear that $C' \in \Omega(\varepsilon)$. It follows from the above formula that

$$C'_i(x-x') = \sum_{j\in N_m} c'_{ij}(x_j-x'_j) = c'_{iq}(x_q-x'_q) = -\beta_i|x_q-x'_q| = -\beta_i||x-x'||_\infty, \quad i \in N_n$$

holds. Hence, combining the linearity of the function $C_i(x - x')$ and ratio (11), we prove inequalities (12):

$$(C_i + C'_i)(x - x') = C_i(x - x') + C'_i(x - x') =$$

$$= C_i(x - x') - \beta_i||x - x'||_\infty \le [C_i(x - x')]^+ - \beta_i||x - x'||_\infty < 0, \qquad i \in N_n.$$

Lemma 4 is proved.

4 Stability radius

It is well known [21] (see also [3,4,13,15]) that necessary and sufficient condition for non-stability of the problem $Z^n(C)$ is that the Pareto set $P^n(C)$ does not coincide with the Slater set $Sl^n(C)$. In this case the stability radius of the problem $Z^n(C)$ is equal to zero.

It remains to consider the case where $P^n(C) = Sl^n(C)$.

Theorem 1. *Let*

$$P^n(C) = Sl^n(C),$$

$$\varphi = \min_{x\in\bar{P}^n(C)} \ \max_{x'\in P_x(C)} \ \min_{i\in N_n} \frac{C_i(x - x')}{||x - x'||_\infty},$$

$$\psi = \min_{x\in\bar{P}^n(C)} \ \min_{(i,k)\in N_n\times N_n} \ \max_{x'\in P_x(C)} \frac{C_i(x - x')}{C_k(x - x')}||C_k||_1.$$

Then the stability radius $\rho_1^n(C)$ of any non-trivial vector integer programming problem $Z^n(C)$, $n \ge 1$, has the following bounds: $0 < \varphi \le \rho_1^n(C) \le \psi$.

Proof. It follows from Corollary 1 that for any solution $x \in \bar{P}^n(C)$ the set $P_x(C)$ is nonempty. Hence, $\varphi > 0$.

We now prove that $\rho_1(C) \ge \varphi$. Let $C' \in \Omega(\varphi)$. Then it follows directly from the definition of φ that for any $x \in \bar{P}^n(C)$ there exists a solution $x^0 \in P_x(C)$ such that:

$$\|C'_i\|_1 \le \|C'\|_1 < \varphi \le \frac{C_i(x - x^0)}{\|x - x^0\|_\infty}, \quad i \in N_n$$

holds. Therefore, we have

$$(C_i + C'_i)(x - x^0) \geq C_i(x - x^0) - \|C'_i\|_1 \|x - x^0\|_\infty > 0, \quad i \in N_n,$$

i.e. $x^0 \in \pi(x, C + C')$. Hence, $x \in \bar{P}^n(C + C')$. Thus, for any matrix $C' \in \Omega(\varphi)$ the inclusion $P^n(C + C') \subseteq P^n(C)$ holds. Consequently, $\rho_1^n(C) \geq \varphi$.

In particular, let us prove that $\rho_1^n(C) \leq \psi$. Suppose that $\varepsilon > \psi$. It follows from the definition of ψ that there exist indices $p, q \in N_n$ and solution $x \in \bar{P}^n(C)$ such that the inequality (5) is fulfilled. It follows from lemma 1 that for any number ε_1, where $\varepsilon > \varepsilon_1 > \psi > 0$, there exist $C' \in \Omega(\varepsilon_1)$ and $x^* \in \bar{P}^n(C)$ such that $x^* \in Sl^n(C + C')$.

There are only two cases.

Case 1: $x^* \in P^n(C + C')$. Since $x^* \in \bar{P}^n(C))$, it follows that $P^n(C + C') \not\subseteq P^n(C)$, $C' \in \Omega(\varepsilon)$.

Case 2: $x^* \in Sl^n(C + C') \setminus P^n(C + C')$. It follows from Lemma 2 that for $\varepsilon_2 := \varepsilon - \varepsilon_1 > 0$ there exists a matrix $C'' \in \Omega(\varepsilon_2)$ such that $x^* \in P^n(C + C' + C'')$. In other words, for any number $\varepsilon = \varepsilon_1 + \varepsilon_2 > \psi$ there exists matrix $C^0 = C' + C''$ such that $P^n(C + C^0) \not\subseteq P^n(C)$, $C^0 \in \Omega(\varepsilon)$.

Combining the results of considered above cases, we see that the inequality $\rho_1^n(C) < \varepsilon$ holds for any $\varepsilon > \psi$. Consequently, $\rho_1^n(C) \leq \psi$.

Theorem 1 is proved.

As corollaries of Theorem 1 and of the mentioned above criterion of stability of the non-trivial problem $Z^n(C)$, we obtain the following results.

Corollary 1. *For the stability radius of any non-trivial vector integer programming problem $Z^n(C)$, $n \geq 1$, we have*

$$\min_{x \in \bar{P}^n(C)} \max_{x' \in P^n(C) \cap \pi(x, C)} \min_{i \in N_n} \frac{C_i(x - x')}{||x - x'||_\infty} \leq \rho_1^n(C) \leq \min_{i \in N_n} \|C_i\|_1 \leq \|C\|_1. \tag{13}$$

If the lower bound is equal to zero, then $\rho^n(C) = 0$.

Proof. Suppose that $i = k$ in the expression ψ (see Theorem 1). Then

$$\rho_1^n(C) \leq \|C_i\|_1, \qquad i \in N_n.$$

Hence, the upper bound is valid in (13).

Now we show that

$$\rho_1^n(C) \geq \varphi', \tag{14}$$

where φ' is the left-hand side of (13).

At first we consider the case $P^n(C) \neq Sl^n(C)$. Then $\rho_1^n(C) = 0$. Let us show that $\varphi' = 0$. It is obvious that there exists solution $x^0 \in \bar{P}^n(C) \cap Sl^n(C)$. Therefore for any solution $x' \in P^n(C) \cap \pi(x, C)$ there exists an index $s \in N_n$ for which $C_s(x^0 - x') = 0$. Hence, $\varphi' = 0$.

Now we will consider the case $P^n(C) = Sl^n(C)$ and prove that $\varphi' = \varphi$ (see Theorem 1). By definition, put

$$\tau(x, x') = \min\Big\{\frac{C_i(x - x')}{||x - x'||_\infty} : \ i \in N_n\Big\}.$$

According to the evident inclusion

$$P_x(C) \subseteq P^n(C) \cap \pi(x, C),$$

we define the set

$$Q(x) = (P^n(C) \cap \pi(x, C)) \setminus P_x(C).$$

From the above notations it is clear that

$$\tau(x, x') \begin{cases} = 0, & \text{if} \quad x' \in Q(x), \\ > 0 & \text{if} \quad x' \in P_x(C). \end{cases}$$

Since the set $P_x(C)$ is nonempty (in view of Corollary 1), it is clear that $\varphi' = \varphi$.

We will prove that $\rho_1^n(C) = 0$ for $\varphi' = 0$. In this case it follows directly from the definition of φ' that there exists a solution $x^0 \in \bar{P}^n(C)$ such that for any solution $x' \in P^n(C) \cap \pi(x^0, C)$ there exists an index $s \in N_n$ for which $C_s(x^0 - x') = 0$. In other words, there is no solution $x' \in P^n(C) \cap \pi(x^0, C)$ belonging to $\sigma(x^0, C)$. Hence, $P^n(C) \cap \sigma(x^0, C) = \emptyset$. Thus, it follows from corollary 3 that there exists a solution $x^* \in \bar{P}^n(C)$ such that $x^* \in Sl^n(C)$, i.e. $P^n(C) \neq Sl^n(C)$. According to the stability criterion of the problem $Z^n(C)$, we have that the problem $Z^n(C)$ is non-stable. Consequently, $\rho_1^n(C) = 0$.

Corollary 1 is proved.

From Corollary 1, we have the following statement.

Corollary 2. *Let the vector integer programming problem $Z^n(C)$, $n \geq 1$, be non trivial. Let matrix $C \in \mathbf{R}^{n\times m}$ contain at least one null row. Then the problem $Z^n(C)$ is nonstable.*

Corollary 3. *If the vector integer programming problem $Z^n(C)$, $n \geq 1$, has a unique efficient solution x^0, then*

$$\rho_1^n(C) = \min_{x \in \bar{P}^n(C)} \ \min_{i \in N_n} \frac{C_i(x - x^0)}{||x - x^0||_\infty}. \tag{15}$$

Proof. We denote the right-hand side of formula (15) by θ. It follows from Corollary 1 that the inequality $\rho_1^n(C) \geq \theta$ holds. Hence, to prove corollary 3 it is enough to show that $\rho_1^n(C) \leq \theta$.

Let $\varepsilon > \theta$. It follows from the definition of θ that there exist a solution $x^* \in \bar{P}^n(C)$ and an index $s \in N_n$ such that

$$C_s(x^* - x^0) = \theta||x^* - x^0||_\infty. \tag{16}$$

Let $q = \arg\max\{|x_j^* - x_j^0| : \ j \in N_n\}$ and the elements of perturbing matrix $C' = [c'_{ij}]_{n\times m}$ be defined by the formula:

$$c'_{ij} = \begin{cases} \gamma \ \mathrm{sign}(x_q^0 - x_q^*), & \text{if} \quad i = s, \ j = q, \\ 0, & \text{if} \quad i \in N_n \setminus \{s\}, \ j \in N_m \setminus \{q\}, \end{cases}$$

where $\theta < \gamma < \varepsilon$.

It is obvious that $C' \in \Omega(\varepsilon)$. From the construction of matrix C' we have

$$C'_s(x^* - x^0) = \sum_{j\in N_m} c'_{sj}(x^* - x^0) = c'_{sq}(x^* - x^0) = -\gamma|x_q^* - x_q^0| = -\gamma||x^* - x^0||_\infty.$$

From the above qualities and equality (16), we have

$$(C_s + C'_s)(x^* - x^0) = C_s(x^* - x^0) - \gamma||x^* - x^0||_\infty = (\theta - \gamma)||x^* - x^0||_\infty < 0,$$

i.e. $x^0 \notin \pi(x^*, C+C')$. If $\pi(x^*, C+C') = \emptyset$, then $x^* \in P^n(C+C')$. If $\pi(x^*, C+C') \neq \emptyset$, then due to external stability of the Pareto set $P^n(C+C')$ (see, for example,[18]) there exists a solution $\hat{x} \in \pi(x^*, C + C')$ such that $\hat{x} \in P^n(C + C')$.

Thus in the case, $P^n(C) = \{x^0\}$, for any number $\varepsilon > \theta$ there exist matrix $C' \in \Omega(\varepsilon)$ and solution $x' \neq x$ such that $x' \in P^n(C+C')$, i.e. $P^n(C+C') \not\subseteq P^n(C)$. Thus, for any number $\varepsilon > \theta$ the inequality $\rho_1^n(C) < \varepsilon$ holds. Hence, $\rho_1^n(C) \leq \theta$.

Corollary 3 is proved.

It follows from Corollary 3 that the lower bound φ in Theorem 1 is attainable for $|P^n(C)| = 1$.

Since $P^1(C) = Sl^1(C)$, as a corollary of Theorem 1, we have

Corollary 4. *Singlecriterion (scalar) integer programming problem $Z^1(C)$ ($C \in \mathbf{R}^m$) is always stable.*

5 Quasistability radius

Theorem 2. *The quasistability radius $\rho_2^n(C)$ of the vector integer programming problem $Z^n(C)$, $n \geq 1$, is expressed by the formula*

$$\rho_2^n(C) = \min_{x'\in P^n(C)} \ \min_{x\in X\setminus\{x'\}} \sum_{i\in N_n} \frac{[C_i(x - x')]^+}{||x - x'||_\infty}. \tag{17}$$

Proof. It is evident that the right-hand side of formula (17) is nonnegative for any matrix C. We denote it by ξ.

First let us prove the inequality $\rho_2^n(C) \geq \xi$. If $\xi = 0$, then the inequality is evident.

Let $\xi > 0$ and $C' \in \Omega(\xi)$. It follows from the definition of the value ξ that for any vectors $x' \in P^n(C)$ and $x \in X \setminus \{x'\}$ the inequality

$$\xi||x - x'||_\infty \leq \sum_{i\in N_n} [C_i(x - x')]^+$$

holds. Hence, it follows from Lemma 3 that $x \notin \pi(x', C + C')$, i.e. in view of $x' \notin \pi(x', C + C')$ the set $\pi(x', C + C')$ is non-empty. Therefore $x' \in P^n(C)$ belongs to the set $P^n(C+C')$ for any perturbing matrix $C' \in \Omega(\xi)$, i.e. $P^n(C) \subseteq P^n(C+C')$. Consequently, $\rho_2^n(C) \geq \xi$.

Now we show that $\rho_2^n(C) \leq \xi$. Let $\varepsilon > \xi$. It follows directly from the definition of the number ξ that there exist solutions $x' \in P^n(C)$ and $x \neq x'$ such that

$$\xi ||x - x'||_\infty = \sum_{i \in N_n} [C_i(x - x')]^+.$$

Hence, it is obvious that there exist positive numbers $\eta_i, i \in N_n$, such that

$$\eta_i ||x - x'||_\infty > [C_i(x - x')]^+, \quad i \in N_n, \qquad \varepsilon > \sum_{i \in N_n} \eta_i > \xi.$$

Thus, it follows from Lemma 4 that there exists a matrix $C' \in \Omega(\varepsilon)$ for which $x \in \pi(x', C + C')$, i.e. $x' \notin P^n(C + C')$. This means that the inequality $\rho_2^n(C) < \varepsilon$ holds for any number $\varepsilon > \xi$. Consequently, $\rho_2^n(C) \leq \xi$.

Theorem 2 is proved.

Any problem on a system of subsets of a finite set is equivalent to a boolean programming problem. Thus formula (17) easily moves to the well-known [17, 22] formula of the quasistability radius of the vector integer programming problem with linear criteria.

Corollary 5. *A necessary and sufficient condition for the quasistability of the vector integer programming problem $Z^n(C)$, $n \geq 1$, is the equality $P^n(C) = Sm^n(C)$ [3].*

Corollary 6. *Singlecriterion (scalar) vector integer programming problem $Z^1(C)$ ($C \in \mathbf{R}^m$) is quasistable if and only if it has a unique optimal solution.*

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