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Properties of one-sided ideals of topological rings

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Abstract. A continuous ring isomorphism $\nu : (R,\tau) \to (\widehat{R},\widehat{\tau})$ is said to be semi-topological from the left (right) in the class $\Re$ provided (R,τ) is a left ideal (right ideal, ideal) of a topological ring $(\widetilde{R},\widetilde{\tau}) \in \Re$ and $\nu = \widetilde{\nu}\mid_R$ for a topological homomorphism $\widetilde{\nu} : (\widetilde{R},\widetilde{\tau}) \to (\widehat{R},\widehat{\tau})$. The article contains several criteria for a continuous homomorphism to be semi-topological from the left (right).

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A ring (not necessarily an associative one) is said to be a topological ring provided it is equipped with a topology such that all ring operations are continuous in it.

The following isomorphism theorem is often used in general algebra and, in particular, in ring theory: if A is a subring (subgroup) of a ring (group) R and I is an ideal (normal divisor) in R then there exists a ring (group) isomorphism $\nu : A/(A \cap I) \to (A+I)/I$ of quotient rings (quotient groups); in particular, if $A \cap I = 0$ then the ring (group) A is isomorphic to the ring (group) $(A+I)/I$, i.e. the rings (groups) A and $(A+I)/I$ possess the same algebraic properties.

In the case when the category of topological rings (groups) with continuous ring (group) homomorphisms taken for morphisms is considered, the isomorphisms are precisely those mappings which are isomorphisms of groups (rings) and homeomorphisms of topological spaces.

The analogue of the above isomorphism theorem is known to be not valid for the above mentioned categories. The above mentioned mapping ν is known to be no more than a continuous ring (group) isomorphism (see Theorem 1 in [2]).

Hence the morphism $\nu : A/(A \cap I) \to (A+I)/I$ to be an isomorphism of the category of topological rings (groups) the ring (group) A should possess some additional properties, for instance A should be an ideal (normal divisor) or a one-sided ideal of the topological ring (group) (R,τ_R).

The case when A is an ideal of the topological ring (R,τ) was investigated in [1], and the case when A is a normal divisor of the topological group (R,τ) was investigated in [2].

The present paper is a sequel to [1] and [2]. The case when A is a one-sided ideal of the topological ring (R,τ) is investigated in it .

1 Definition. The homomorphism of topological rings $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is as usually called to be a topological isomorphism iff it is both continuous and open mapping.

2 Remark. Let (R, τ) be an arbitrary topological ring I be its arbitrary (closed) ideal. The canonical homomorphism $\xi : (R, \tau) \to (R, \tau)/I$ (i.e. such that $\xi(r) = r + I$) is known to be a topological homomorphism, and if the mapping $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is a topological ring homomorphism and $I = \ker \varphi$ then the topological rings $(\widehat{R}, \widehat{\tau})$ and $(R, \tau)/I$ are topologically isomorphic.

3 Definition. Let $\Re$ be a class of topological rings and (R, τ) and $(\widehat{R}, \widehat{\tau})$ be elements of $\Re$. Similarly to the definition which is given in [1] we say the continuous isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is semi-topological from the left (right) in the class $\Re$ provided there exists such a topological ring $(\widetilde{R}, \widetilde{\tau}) \in \Re$ that:

– the topological ring (R, τ) is a left (right) ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$;

– the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$.

4 Theorem. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) topological rings;

2. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of subgroups of the additive group of the ring;

3. The class of all (separated) topological rings which are bounded from the right (i.e. for every neighbourhood of zero U there exists such a neighbourhood of zero V that $R \cdot V \subseteq U$);

4. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of left ideals of the ring.

Then if (R, τ) and $(\widehat{R}, \widehat{\tau}) \in \Re$ and $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is a continuous homomorphism then the following assertions are equivalent:

1. The isomorphism φ is semi-topological from the left in $\Re$;

2. For every element $b \in R$ and an arbitrary neighbourhood of zero U in (R, τ) there exist such neighbourhoods of zero $\widehat{V}$ and V in $(\widehat{R}, \widehat{\tau})$ and (R, τ), respectively, that

$$\varphi^{-1}(\widehat{V}) \cdot b \subseteq U \text{ and } \varphi^{-1}(\widehat{V}) \cdot V \subseteq U.$$

3. There exists a topological ring $(\widetilde{R}, \widetilde{\tau}) \in \Re$ such that the topological ring (R, τ) is a left ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$ and the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$ where $(\ker \widetilde{\varphi})^2 = \{0\}$.

Proof. $1 \Rightarrow 2$. Let the isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ be semi-topological from the left in $\Re$. Hence there exists a topological ring $(\widetilde{R}, \widetilde{\tau}) \in \Re$ such that the topological ring (R, τ) is a left ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$ and the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$.

If $b \in R$ and U is an arbitrary neighbourhood of zero in (R, τ) then there exist neighbourhoods of zero $\widetilde{U}$ and $\widetilde{V}$ in $(\widetilde{R}, \widetilde{\tau})$ such that $R \cap \widetilde{U} = U$, $\widetilde{V} \cdot b \subseteq \widetilde{U}$ and $\widetilde{V} \cdot \widetilde{V} \subseteq \widetilde{U}$.

Since (R, τ) is a subring of the topological ring $(\widetilde{R}, \widetilde{\tau})$ then $V = R \cap \widetilde{V}$ is a neighbourhood of zero in (R, τ) and since the homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$ is open then $\widehat{V} = \widetilde{\varphi}(\widetilde{V})$ is a neighbourhood of zero in $(\widehat{R}, \widehat{\tau})$.

Since φ is an isomorphism and $\widetilde{\varphi}$ is its extension then $R \cap \ker \widetilde{\varphi} = \ker \varphi = \{0\}$. Since R is a left ideal $\widetilde{R}$ and $\ker \widetilde{\varphi}$ is an ideal $\widetilde{R}$ then $(\ker \widetilde{\varphi}) \cdot R \subseteq \ker \widetilde{\varphi} \cap R = \{0\}$. Hence

$$\varphi^{-1}(\widehat{V}) \cdot b \subseteq \varphi^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big) \cdot b \subseteq \Big(\widetilde{\varphi}^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big) \cdot b\Big) \cap R = \big((\widetilde{V} + \ker \widetilde{\varphi}) \cdot b\big) \cap R =$$
$$\big(\widetilde{V} \cdot b + \ker \widetilde{\varphi} \cdot b\big) \cap R \subseteq \big(\widetilde{V} \cdot b + \ker \widetilde{\varphi} \cdot R\big) \cap R =$$
$$\big(\widetilde{V} \cdot b\big) \cap R \subseteq \big(\widetilde{V} \cdot b\big) \cap R \subseteq \widetilde{U} \cap R = U \text{ and}$$
$$\varphi^{-1}(\widehat{V}) \cdot V \subseteq \varphi^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big) \cdot V \subseteq \Big(\widetilde{\varphi}^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big) \cdot V\Big) \cap R = \big((\widetilde{V} + \ker \widetilde{\varphi}) \cdot V\big) \cap R =$$
$$\big(\widetilde{V} \cdot V + \ker \widetilde{\varphi} \cdot V\big) \cap R \subseteq \big(\widetilde{V} \cdot V + \ker \widetilde{\varphi} \cdot R\big) \cap R =$$
$$\big(\widetilde{V} \cdot V\big) \cap R \subseteq \big(\widetilde{V} \cdot \widetilde{V}\big) \cap R \subseteq \widetilde{U} \cap R = U,$$

that completes the proof of the implication $1 \Rightarrow 2$.

$2 \Rightarrow 3$. Let (R, τ) and $(\widehat{R}, \widehat{\tau}) \in \Re$ and $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ be such a continuous isomorphism that for every element $b \in R$ and every neighbourhood of zero U in (R, τ) there exist neighbourhoods of zero $\widehat{V}$ and V in $(\widehat{R}, \widehat{\tau})$ and (R, τ), respectively, that

$$\varphi^{-1}(\widehat{V}) \cdot b \subseteq U \text{ and } \varphi^{-1}(\widehat{V}) \cdot V \subseteq U.$$

Consider a discrete ring $\widetilde{R}$ such that its additive group is a direct sum of additive groups of rings R and $\widehat{R}$ and the multiplication is defined as follows:
$(r_1, \widehat{r}_1) \cdot (r_2, \widehat{r}_2) = (r_1 \cdot r_2, \varphi(r_1) \cdot \widehat{r}_2)$.

One can easily check that $\widetilde{R}$ equipped with these operation is a ring and, if the rings R and $\widehat{R}$ are associative then so is the ring $\widetilde{R}$.

We write $\mathbf{B}$ and $\widehat{\mathbf{B}}$ for the set of all neigbourhoods of zero of the topological ring (R, τ) and $(\widehat{R}, \widehat{\tau})$, respectively. For every $V \in \mathbf{B}$ and $\widehat{V} \in \widehat{\mathbf{B}}$ consider the set

$$\widetilde{W}(V, \widehat{V}) = \big\{(r - \varphi^{-1}(\widehat{r}), \widehat{r}) \mid r \in V, \widehat{r} \in \widehat{V}\big\}.$$

Let us prove that assertions BN1 – BN6 from Theorem 1.2.5 in [3] are valid for the set

$$\widetilde{\mathbf{B}} = \{\widetilde{W}(V, \widehat{V}) \mid V \in \mathbf{B}, \widehat{V} \in \widehat{\mathbf{B}}\},$$

i.e. it is a fundamental system of neighbourhoods of zero in a certain ring topology (which need not be separated) $\widetilde{\tau}$ on the ring $\widehat{R}$.

Since $0 \in V$ and $0 \in \widehat{V}$ for every $V \in \mathbf{B}$ and $\widehat{V} \in \widehat{\mathbf{B}}$ then $(0,0) \in \widetilde{W}(V, \widehat{V})$ for every $\widetilde{W}(V, \widehat{V}) \in \widetilde{\mathbf{B}}$ and since $(0,0)$ is a zero in the ring $\widetilde{R}$ then the assertion BN1 is valid.

Let $\widetilde{W}(V_1, \widehat{V}_1)$ and $\widetilde{W}(V_2, \widehat{V}_2) \in \widetilde{\mathbf{B}}$. There exist $V_3 \in \mathbf{B}$ and $\widehat{V_3} \in \widehat{\mathbf{B}}$ such that $V_3 \subseteq V_1 \cap V_2$ and $\widehat{V}_3 \subseteq \widehat{V}_1 \cap \widehat{V}_2$. One can easily see that $\widetilde{W}(V_3, \widehat{V}_3) \subseteq \widetilde{W}(V_1, \widehat{V}_1) \cap \widetilde{W}(V_2, \widehat{V}_2)$, i.e. the assertion BN2 is valid.

Let now $\widetilde{W}(V_1, \widehat{V_1}) \in \widetilde{\mathbf{B}}$. There exist $V_2 \in \mathbf{B}$ and $\widehat{V_2} \in \widehat{\mathbf{B}}$ such that $V_2 + V_2 \subseteq V_1$, $-V_2 \subseteq V_1$ and $\widehat{V}_2 + \widehat{V}_2 \subseteq \widehat{V}_1$, $-\widehat{V}_2 \subseteq \widehat{V}_1$. Then $\widetilde{W}(V_2, \widehat{V}_2) + \widetilde{W}(V_2, \widehat{V}_2) =$

$$\{(r_1 - \varphi^{-1}(\widehat{r}_1), \widehat{r}_1) \mid r_1 \in V_2, \widehat{r}_1 \in \widehat{V}_2\} + \{(r_2 - \varphi^{-1}(\widehat{r}_2), \widehat{r}_2) \mid r_2 \in V_2, \widehat{r}_2 \in \widehat{V}_2\} =$$
$$\{(r_1 - \varphi^{-1}(\widehat{r}_1) + r_2 - \varphi^{-1}(\widehat{r}_2), \widehat{r}_1 + \widehat{r}_2) \mid$$
$$r_1 + r_2 \in V_2 + V_2 \subseteq V_1, \widehat{r}_1 + \widehat{r}_2 \in \widehat{V}_2 + \widehat{V}_2 \subseteq \widehat{V}_1\} \subseteq$$
$$\{(r_3 - \varphi^{-1}(\widehat{r}_3), \widehat{r}_3) \mid r_3 \in V_1, \widehat{r}_3 \in \widehat{V}_1\} = \widetilde{W}(V_2, \widehat{V}_2) \text{ and}$$
$$-\widetilde{W}(V_2, \widehat{V}_2) = -\{(r_1 - \varphi^{-1}(\widehat{r}_1), \widehat{r}_1) \mid r_1 \in V_2, \widehat{r}_1 \in \widehat{V}_2\} =$$
$$\{(-r_1 - \varphi^{-1}(-\widehat{r}_1), -\widehat{r}_1) \mid -r_1 \in -V_2, -\widehat{r}_1 \in -\widehat{V}_2\} \subseteq$$
$$\{(r - \varphi^{-1}(\widehat{r}), \widehat{r}) \mid r \in -V_2 \subseteq V_1, \widehat{r} \in -\widehat{V}_2 \subseteq V_1\} = \widetilde{W}(V_1, \widehat{V}_1),$$

i.e. the assertions BN3 and BN4 are valid.

Let $\widetilde{W}(V_1, \widehat{V}_1) \in \widetilde{\mathbf{B}}$. There exist $V_2 \in \mathbf{B}$ and $\widehat{V_2} \in \widehat{\mathbf{B}}$ such that $V_2 - V_2 \subseteq V_1$ and $\widehat{V}_2 \cdot \widehat{V}_2 \subseteq \widehat{V}_1$. Since the assertion 2 is supposed to be valid then there exist $V_3 \in \mathbf{B}$ and $\widehat{V_3} \in \widehat{\mathbf{B}}$ such that $V_3 \cdot V_3 \subseteq V_2$ and $\varphi^{-1}(\widehat{V_3}) \cdot V_3 \subseteq V_2$. Since the isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is continuous then without loss of generality we can claim that $V_3 \subseteq \widehat{V_3}$.

Than taking into account the definition of the multiplication in the ring $\widetilde{R}$ we get $\widetilde{W}(V_3, \widehat{V}_3) \cdot \widetilde{W}(V_3, \widehat{V}_3) =$

$$\{(r_1 - \varphi^{-1}(\widehat{r}_1), \widehat{r}_1) \mid r_1 \in V_3, \widehat{r}_1 \in \widehat{V}_3\} \cdot \{(r_2 - \varphi^{-1}(\widehat{r}_2), \widehat{r}_2) \mid r_2 \in V_3, \widehat{r}_2 \in \widehat{V}_3\} =$$
$$\{((r_1 - \varphi^{-1}(\widehat{r}_1)) \cdot (r_2 - \varphi^{-1}(\widehat{r}_2)), \varphi(r_1 - \varphi^{-1}(\widehat{r}_1)) \cdot \widehat{r}_2) \mid r_1, r_2 \in V_3, \widehat{r}_1, \widehat{r}_2 \in \widehat{V}_3\} =$$
$$\{(r_1 \cdot (r_2 - \varphi^{-1}(\widehat{r}_2)) - \varphi^{-1}(\widehat{r}_1) \cdot (r_2 - \varphi^{-1}(\widehat{r}_2)),$$
$$\varphi(r_1) \cdot \widehat{r}_2 - \widehat{r}_1 \cdot \widehat{r}_2) \mid r_1, r_2 \in V_3, \widehat{r}_1, \widehat{r}_2 \in \widehat{V}_3\} =$$
$$\{(r_1 \cdot r_2 - \varphi^{-1}(\widehat{r}_1) \cdot r_2 - r_1 \cdot \varphi^{-1}(\widehat{r}_2) + \varphi^{-1}(\widehat{r}_1 \cdot \widehat{r}_2),$$
$$\varphi(r_1) \cdot \widehat{r}_2 - \widehat{r}_1 \cdot \widehat{r}_2) \mid r_1, r_2 \in V_3, \widehat{r}_1, \widehat{r}_2 \in \widehat{V}_3\} =$$
$$\{(r_1 \cdot r_2 - \varphi^{-1}(\widehat{r}_1) \cdot r_2 - (r_1 \cdot \varphi^{-1}(\widehat{r}_2) - \varphi^{-1}(\widehat{r}_1 \cdot \widehat{r}_2)), \varphi(r_1) \cdot \widehat{r}_2 - \widehat{r}_1 \cdot \widehat{r}_2) \mid$$
$$r_1, r_2 \in V_3, \widehat{r}_1, \widehat{r}_2 \in \widehat{V}_3\}.$$

Taking into account the choice of neighbourhoods V_1, V_2, V_3, $\widehat{V}_1$, $\widehat{V}_2$, $\widehat{V}_3$ and elements r_1, r_2, $\widehat{r}_1$, $\widehat{r}_2$ we obtain

$$r_3 = r_1 \cdot r_2 - \varphi^{-1}(\widehat{r}_1) \cdot r_2 \in V_3 \cdot V_3 - \varphi^{-1}(\widehat{V}_3) \cdot V_3 \subseteq V_2 - V_2 \subseteq V_1 \text{ and}$$
$$\widehat{r}_1 = \varphi(r_1) \cdot \widehat{r}_2 - \widehat{r}_1 \cdot \widehat{r}_2 \in \varphi(V_3) \cdot \widehat{V}_3 - \widehat{V}_3 \cdot \widehat{V}_3 \subseteq \widehat{V}_3 \cdot \widehat{V}_3 - \widehat{V}_3 \cdot \widehat{V}_3 \subseteq \widehat{V}_2 - \widehat{V}_2 \subseteq \widehat{V}_1,$$

and hence $\widetilde{W}(V_3, \widehat{V}_3) \cdot \widetilde{W}(V_3, \widehat{V}_3) =$

$$\{(r_1 \cdot r_2 - \varphi^{-1}(\widehat{r}_1) \cdot r_2 - (r_1 \cdot \varphi^{-1}(\widehat{r}_2) - \varphi^{-1}(\widehat{r}_1 \cdot \widehat{r}_2)), \varphi(r_1) \cdot \widehat{r}_2 - \widehat{r}_1 \cdot \widehat{r}_2) \mid$$
$$r_1, r_2 \in V_3, \widehat{r}_1, \widehat{r}_2 \in \widehat{V}_3\} = \{(r_3 - \varphi^{-1}(\widehat{r}_3), \widehat{r}_3) \mid r_3 \in V_1, \widehat{r}_3 \in \widehat{V}_3\} = \widetilde{W}(V, \widehat{V}),$$

i.e. the assertion BN5 is valid.

Let $\widetilde{r} = (r, \widehat{r}) \in \widetilde{R}$ and $\widetilde{W}(V_1, \widehat{V}_1) \in \widetilde{\mathbf{B}}$. There exist $V_2 \in \mathbf{B}$ and $\widehat{V_2} \in \widehat{\mathbf{B}}$ such that $r \cdot V_2 \subseteq V_1$ and $\varphi(r) \cdot \widehat{V}_2 \subseteq \widehat{V}_1$. Hence

$$\widetilde{r} \cdot \widetilde{W}(V_2, \widehat{V}_2) = (r, \widehat{r}) \cdot \{\big(a - \varphi^{-1}(\widehat{a}), \widehat{a}\big) \mid a \in V_2,\ \widehat{a} \in \widehat{V}_2\} =$$
$$\{\big(r \cdot (a - \varphi^{-1}(\widehat{a})), \varphi(r) \cdot \widehat{a}\big) \mid a \in V_2,\ \widehat{a} \in \widehat{V}_2\} =$$
$$\{\big(r \cdot a - r \cdot \varphi^{-1}(\widehat{a})), \varphi(r) \cdot \widehat{a}\big) \mid a \in V_2,\ \widehat{a} \in \widehat{V}_2\} =$$
$$\{\big(r \cdot a - \varphi^{-1}(\varphi(r) \cdot \widehat{a})), \varphi(r) \cdot \widehat{a}\big) \mid a \in V_2,\ \widehat{a} \in \widehat{V}_2\} \subseteq$$
$$\{\big(v - \varphi^{-1}(\widehat{v})), \widehat{v}\big) \mid v \in V_1,\ \widehat{v} \in \widehat{V_1}\} = \widetilde{W}(V_1, \widehat{V}_1),$$

since $r \cdot a \in r \cdot V_2 \subseteq V_1$ and $\varphi(r) \cdot \widehat{a} \in \varphi(r) \cdot \widehat{V} \subseteq V_1$. Except that if $\widetilde{r} = (r, \widehat{r}) \in \widetilde{R}$ and $\widetilde{W}(V_1, \widehat{V}_1) \in \widetilde{\mathbf{B}}$ then there exist V_2, $V_3 \in \mathbf{B}$ and $\widehat{V}_2$, $\widehat{V}_3 \in \widehat{\mathbf{B}}$ such that $V_2 - V_2 + V_2 - V_2 \subseteq V_1$ and $\widehat{V}_2 - \widehat{V}_2 \subseteq \widehat{V}_1$, $V_3 \cdot r \subseteq V_2$, $V_3 \cdot \varphi^{-1}(\widehat{r}) \subseteq V_2$, $\widehat{V}_3 \cdot \widehat{r} \subseteq \widehat{V}_2$. Since the inclusions mentioned in the assertion 2 are valid then we may assert without loss of generality that $\varphi^{-1}(\widehat{V}_3) \cdot r \subseteq V_2$ and $\varphi^{-1}(\widehat{V}_3) \cdot \varphi^{-1}(\widehat{r}) \subseteq V_2$ and, since the isomorphism φ is continuous we may claim that $\varphi(V_3) \subseteq \widehat{V}_3$.

Then

$$\widetilde{W}(V_3, \widehat{V}_3) \cdot \widetilde{r} = \{\big(a - \varphi^{-1}(\widehat{a}), \widehat{a}\big) \mid a \in V_3,\ \widehat{a} \in \widehat{V}_3\} \cdot (r, \widehat{r}) =$$
$$\{\big((a - \varphi^{-1}(\widehat{a})) \cdot r, (\varphi(a - \varphi^{-1}(\widehat{a}))) \cdot \widehat{r}\big) \mid a \in V_3,\ \widehat{a} \in \widehat{V}_3\} =$$
$$\{\big(a \cdot r - \varphi^{-1}(\widehat{a}) \cdot r, \varphi(a) \cdot \widehat{r} - \varphi(\varphi^{-1}(\widehat{a})) \cdot \widehat{r}\big) \mid a \in V_3,\ \widehat{a} \in \widehat{V}_3\} =$$
$$\{\big(a \cdot r - \varphi^{-1}(\widehat{a}) \cdot r + \varphi^{-1}(\varphi(a) \cdot \widehat{r}) - \varphi^{-1}(\widehat{a} \cdot \widehat{r}) - \varphi^{-1}(\varphi(a) \cdot \widehat{r}) + \varphi^{-1}(\widehat{a} \cdot \widehat{r}),$$
$$\varphi(a) \cdot \widehat{r} - \widehat{a} \cdot \widehat{r}\big) \mid a \in V_3,\ \widehat{a} \in \widehat{V}_3\} =$$
$$\{\big(a \cdot r - \varphi^{-1}(\widehat{a}) \cdot r + a \cdot \varphi^{-1}(\widehat{r}) - \varphi^{-1}(\widehat{a}) \cdot \varphi^{-1}(\widehat{r}) - \varphi^{-1}(\varphi(a) \cdot \widehat{r}) + \varphi^{-1}(\widehat{a} \cdot \widehat{r}),$$
$$\varphi(a) \cdot \widehat{r} - \widehat{a} \cdot \widehat{r}\big) \mid a \in V_3,\ \widehat{a} \in \widehat{V}_3\} \subseteq \{\big(b - \varphi^{-1}(\widehat{b}), \widehat{b}\big) \mid b \in V_1,\ \widehat{b} \in \widehat{V}_1\} = \widetilde{W}(V_1, \widehat{V}_1),$$

since

$$b = a \cdot r - \varphi^{-1}(\widehat{a}) \cdot r + a \cdot \varphi^{-1}(\widehat{r}) - \varphi^{-1}(\widehat{a}) \cdot \varphi^{-1}(\widehat{r}) \in$$
$$V_3 \cdot r - V_3 \cdot \varphi^{-1}(\widetilde{r}) + \varphi^{-1}(\widehat{V}_3) \cdot r - \varphi^{-1}(\widehat{V}_3) \cdot \varphi^{-1}(\widehat{r}) \subseteq V_2 - V_2 + V_2 - V_2 \subseteq V_1,$$
$$\widehat{b} = \varphi(a) \cdot \widehat{r} - \widehat{a} \cdot \widehat{r} \in \varphi(V_3) \cdot \widehat{r} - \widehat{V}_3 \cdot \widehat{r} \subseteq \widehat{V}_3 \cdot \widehat{r} - \widehat{V}_3 \cdot \widehat{r} \subseteq \widehat{V}_2 - \widehat{V}_2 \subseteq \widehat{V}_1 \text{ and}$$
$$-\varphi^{-1}(\widehat{b}) = -\varphi^{-1}(\varphi(a) \cdot \widehat{r}) + \varphi^{-1}(\widehat{a} \cdot \widehat{r}).$$

By that the validity of the assertion BN6 has been checked and by Theorem 1.2.5 in [3] the set $\widetilde{\mathbf{B}}$ is a fundamental system of neighbourhoods of zero in a certain ring topology (which need not be separated) $\widetilde{\tau}$ on the ring $\widetilde{R}$.

Now we prove that if the topological rings (R, τ) and $(\widehat{R}, \widehat{\tau})$ are separated then so is $(\widetilde{R}, \widetilde{\tau})$. To do that is sufficient to prove the validity of the assertion BN1' (see Corollary 1.3.7 in [3]) for it.

Let $0 \neq \widetilde{r} = (r, \widehat{r}) \in \widetilde{R}$. If $0 \neq \widehat{r}$ then there exists $\widehat{V} \in \widehat{\mathbf{B}}$ such that $\widehat{r} \notin \widehat{V}$ and hence

$$\widetilde{r} = (r, \widehat{r}) \notin \{(a - \varphi^{-1}(\widehat{a})) \mid a \in R,\ \widehat{a} \in \widehat{V}\} = \{(a, \widehat{a}) \mid a \in R, \widehat{a} \in \widehat{V}\} = \widetilde{W}(R, \widehat{V}).$$

If $0 = \widehat{r}$ then $0 \neq r$ and there exists $V \in \mathbf{B}$ such that $r \notin V$. Hence

$$\widetilde{r} = (r, \widehat{r}) = (r, 0) \notin \{(a - \widehat{a}, \widehat{a}) \mid a \in V,\ \widehat{a} \in \widehat{R}\} = \widetilde{W}(V, \widehat{R}).$$

Hence the assertion BN1' is valid and therefore the topological ring $(\widetilde{R}, \widetilde{\tau})$ is separated.

Let us check that the topological ring (R, τ) is a left ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$.

Indeed, one can easily prove that the set $A = \{(r, 0) \mid r \in R\}$ is a left ideal of the ring $\widetilde{R}$ and since $A \cap \widetilde{W}(V,\ \widehat{V}) =$

$$\{(r, 0) \mid r \in R\} \cap \{(r - \varphi^{-1}(\widehat{r}), \widehat{r}) \mid r \in V,\ \widehat{r} \in \widehat{V}\} = \{(r, 0) \mid r \in V\}$$

for every $\widehat{V} \in \widehat{\mathbf{B}}$ and $V \in \mathbf{B}$, then topological rings (R, τ) and $(A, \widetilde{\tau}|_A)$ are topologically isomorphic. Then we identify an element $r \in R$ with the element $(r, 0) \in A$ and get that the topological ring (R, τ) is a left ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$.

We prove now that the isomorphism $\varphi : R \to \widehat{R}$ can be extended to a homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$.

We set $\widetilde{\varphi}(r, \widehat{r}) = \varphi(r)$ for every element $(r, \widehat{r}) \in \widetilde{R}$. It is obvious that the mapping $\widetilde{\varphi} : \widetilde{R} \to \widehat{R}$ is a homomorphism and, taking into account the identification of the element $r \in R$ with the element $(r, 0) \in A$ get that the mapping $\widetilde{\varphi}$ which is defined on $\widetilde{R}$ is an extension of the isomorphism φ.

Except that one can easily verify that $\ker \widetilde{\varphi} = \{(0, \widehat{r}) \mid \widehat{r} \in \widehat{R}\}$, and $(\ker \widetilde{\varphi})^2 = \{0\}$.

Since

$$\widetilde{\varphi}(\widetilde{W}(V, \widehat{V})) \supseteq \widetilde{\varphi}(\{(0 - \varphi^{-1}(\widehat{r}), \widehat{r})) \mid \widehat{r} \in \widehat{V}\}) =$$
$$\{\varphi(\{(-\varphi^{-1}(\widehat{r}) \mid \widehat{r} \in \widehat{V}\} = \{-\widehat{r} \mid \widehat{r} \in \widehat{V}\} = -\widehat{V}$$

and $-\widehat{V} \in \widehat{\mathbf{B}}$ for every $\widehat{V} \in \widehat{\mathbf{B}}$ and $V \in \mathbf{B}$ then by Proposition 1.5.5 in [3] the homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$ is open.

Let now $\widehat{V} \in \widehat{\mathbf{B}}$. There exists a neighbourhood of zero $\widehat{V}_1 \in \widehat{\mathbf{B}}$ such that $\widehat{V}_1 - \widehat{V}_1 \subseteq \widehat{V}$ and since the isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is continuous then there exists a neighbourhood of zero $V_1 \in \mathbf{B}$ such that $\varphi(V_1) \subseteq \widehat{V}_1$. Then

$$\widetilde{\varphi}(\widetilde{W}(V_1, \widehat{V}_1)) = \widetilde{\varphi}(\{(r - \varphi^{-1}(\widehat{r}), \widehat{r}) \mid r \in V_1, \widehat{r} \in \widehat{V}_1\} =$$
$$\varphi(\{(r - \varphi^{-1}(\widehat{r}) \mid r \in V_1, \widehat{r} \in \widehat{V}_1\}) = \{\varphi(r) - \widehat{r}) \mid r \in V_1, \widehat{r} \in \widehat{V}_1\} \subseteq \widehat{V}_1 - \widehat{V}_1 \subseteq \widehat{V},$$

and, by Proposition 1.5.5 in [3] the homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$ is continuous.

So the implication $2 \Rightarrow 3$ is proved for the case when $\Re$ is a class of all (separated) topological rings.

Let us prove the implication $2 \Rightarrow 3$ for the rest of the classes which are mentioned in the condition of Theorem.

If $\Re$ is the class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of subgroups of the additive group of the ring then to complete the proof of Theorem it is sufficient to check that the above defined sets $\widetilde{W}(V, \widehat{V})$ are subgroups provided so are V and $\widehat{V}$.

Since

$$\widetilde{W}(V, \widehat{V}) - \widetilde{W}(V, \widehat{V}) =$$
$$\{(a - \varphi^{-1}(\widehat{a}), \widehat{a}) \mid a \in V, \widehat{a} \in \widehat{V}\} - \{(b - \varphi^{-1}(\widehat{b}), \widehat{b}) \mid b \in V, \widehat{b} \in \widehat{V}\} =$$
$$\{(a - b - \varphi^{-1}(\widehat{a}) + \varphi^{-1}(\widehat{b}), \widehat{a} - \widehat{b}) \mid a, b \in V, \widehat{a}, \widehat{b} \in \widehat{V}\} =$$
$$\{(a - b - \varphi^{-1}(\widehat{a} - \widehat{b}), \widehat{a} - \widehat{b}) \mid a, b \in V, \widehat{a}, \widehat{b} \in \widehat{V}\} \subseteq$$
$$\{(c - \varphi^{-1}(\widehat{c}), \widehat{c}) \mid c \in V, \widehat{c} \in \widehat{V}\} = \widetilde{W}(V, \widehat{V}),$$

(since $a - b \in V - V = V$ and $\widehat{a} - \widehat{b} \in \widehat{V} - \widehat{V} = \widehat{V}$) then $\widetilde{W}(V, \widehat{V})$ is a subgroup).

Let now $\Re$ be the class the class of all (separated) topological rings which are bounded from the right and (R, τ), $(\widehat{R}, \widehat{\tau}) \in \Re$.

Let us prove that the topological ring $(\widetilde{R}, \widetilde{\tau})$ is also bounded from the right in this case. Indeed, if $\widetilde{W}(V, \widehat{V}) \in \widetilde{\mathbf{B}}$ then there exist $V_1 \in \mathbf{B}$ and $\widehat{V}_1 \in \widehat{\mathbf{B}}$ such that $R \cdot V_1 \subseteq V$ and $\widehat{R} \cdot \widehat{V}_1 \subseteq \widehat{V}$. Then

$$\widetilde{R} \cdot \widetilde{W}(V_1, \widehat{V}_1) = \{(r, \widehat{r}) \mid r \in R, \widehat{r} \in \widehat{R}\} \cdot \{(a - \varphi^{-1}(\widehat{a})) \mid a \in V_1,\ \widehat{a} \in \widehat{V}_1\} =$$
$$\{(r \cdot (a - \varphi^{-1}(\widehat{a})), \varphi(r) \cdot (\widehat{a})) \mid r \in R, a \in V_1, \widehat{a} \in \widehat{V}_1\} =$$
$$\{(r \cdot a - r \cdot \varphi^{-1}(\widehat{a})), \varphi(r) \cdot (\widehat{a})) \mid r \in R, a \in V_1, \widehat{a} \in \widehat{V}_1\} \subseteq$$
$$\{(c - \varphi^{-1}(\widehat{c}), \widehat{c}) \mid c \in V, \widehat{c} \in \widehat{V}\} = \widetilde{W}(V, \widehat{V}),$$

since $r \cdot a \in R \cdot V_1 \subseteq V$, $\varphi(r) \cdot (\widehat{a}) \in \widehat{R} \cdot \widehat{V}_1 \subseteq \widehat{V}$ and $r \cdot \varphi^{-1}(\widehat{a}) = \varphi^{-1}(\varphi(r) \cdot \widehat{a})$.

Let now $\Re$ be the class of all (separated) topological rings, admitting a fundamental system of neighbourhoods of zero consisting of left ideals of the ring. If (R, τ), $(\widehat{R}, \widehat{\tau}) \in \Re$ then these rings are bounded from the right and admit a fundamental system of neighbourhoods of zero consisting of subgroups. Then (see the above two cases) so is $(\widetilde{R}, \widetilde{\tau})$ and, by Proposition 1.6.32 in [3] $(\widetilde{R}, \widetilde{\tau}) \in \Re$.

So the proof of the implication $2 \Rightarrow 3$ is complete for every case mentioned in the condition of Theorem.

To complete the proof of Theorem it is sufficient to prove the implication $3 \Rightarrow 1$. It is obvious since the topological ring $(\widetilde{R}, \widetilde{\tau})$, mentioned in the assertion 3 satisfies the definition 3. □

5 Remark. The following Theorem can be easily obtained from Theorem 4 proceeding to anti-isomorphic rings.

6 Theorem. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) topological rings;

2. The class of all (separated) topological rings, admitting a fundamental system of neighbourhoods of zero consisting of subgroups of the additive group of the ring;

3. The class of all (separated) topological rings which are bounded from the left (i.e. for every neighbourhood of zero U there exists a neighbourhood of zero V such that $V \cdot R \subseteq U$);

4. The class of all (separated) topological rings, admitting a fundamental system of neighbourhoods of zero consisting of right ideals of the ring.

Then if (R,τ) and $(\widehat{R},\widehat{\tau}) \in \Re$ and $\varphi : (R,\tau) \to (\widehat{R},\widehat{\tau})$ is a continuous homomorphism then the following assertions are equivalent:

1. The isomorphism φ is semi-topological from the right in $\Re$;

2. For every element $b \in R$ and arbitrary neighbourhood of zero U in (R,τ) there exist neighbourhoods of zero $\widehat{V}$ and V in $(\widehat{R},\widehat{\tau})$ and (R,τ), respectively, such that

$$b \cdot \varphi^{-1}(\widehat{V}) \subseteq U \text{ and } V \cdot \varphi^{-1}(\widehat{V}) \subseteq U.$$

3. There exists a topological ring $(\widetilde{R},\widetilde{\tau}) \in \Re$, such that the topological ring (R,τ) is a right ideal of the topological ring $(\widetilde{R},\widetilde{\tau})$, and the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R},\widetilde{\tau}) \to (\widehat{R},\widehat{\tau})$ and $(\ker \widetilde{\varphi})^2 = \{0\}$.

7 Remark. The below assertion follows from Theorems 4 and 6 of the present article and from Theorem 1 and Remark 1 from [1].

8 Corollary. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) topological rings;

2. The class of all (separated) topological rings, admitting a fundamental system of zero consisting of subgroups of the additive group of the ring;

Then if (R,τ) and $(\widehat{R},\widehat{\tau}) \in \Re$ and the isomorphism $\varphi : (R,\tau) \to (\widehat{R},\widehat{\tau})$ is semi-topological both from the right and from the left in the class $\Re$ then it is semi-topological in the class $\Re$.[1]

9 Theorem. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) topological rings which are bounded from the left;

2. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of the right ideals of the ring;

3. The class of all (separated) topological rings which are bounded (i.e. are bounded both from the right and from the left);

4. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of the ideals of the ring.

Then if (R,τ) and $(\widehat{R},\widehat{\tau}) \in \Re$ and $\varphi : (R,\tau) \to (\widehat{R},\widehat{\tau})$ is a continuous isomorphism then the following assertions are equivalent:

1. The isomorphism φ is semi-topological from the left in the class $\Re$;

[1] A continuous isomorphism $\varphi : (R,\tau) \to (\widehat{R},\widehat{\tau})$ is said to be semi-topological in the class $\Re$ provided there exists a topological ring $(\widetilde{R},\widetilde{\tau}) \in \Re$ such that the topological ring (R,τ) is an ideal of the topological ring $(\widetilde{R},\widetilde{\tau})$ and the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R},\widetilde{\tau}) \to (\widehat{R},\widehat{\tau})$ (see [1]).

2. For an arbitrary neighbourhood of zero U in (R,τ) there exists a neighbourhood of zero $\widehat{V}$ in $(\widehat{R},\widehat{\tau})$ such that

$$\varphi^{-1}(\widehat{V})\cdot R\subseteq U.$$

3. There exists a topological ring $(\widetilde{R},\widetilde{\tau})\in\Re$ such that the topological ring (R,τ) is a left ideal of the topological ring $(\widetilde{R},\widetilde{\tau})$, the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi}:(\widetilde{R},\widetilde{\tau})\to(\widehat{R},\widehat{\tau})$ and $(\ker\widetilde{\varphi})^2=\{0\}$.

Proof. $1\Rightarrow 2$. Let the isomorphism $\varphi:(R,\tau)\to(\widehat{R},\widehat{\tau})$ be semi-topological in the class $\Re$. Then there exists a topological ring $(\widetilde{R},\widetilde{\tau})\in\Re$ such that the topological ring (R,τ) is a left ideal of the topological ring $(\widetilde{R},\widetilde{\tau})$ and the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi}:(\widetilde{R},\widetilde{\tau})\to(\widehat{R},\widehat{\tau})$.

Let U be an arbitrary neighbourhood of zero in (R,τ). Since the topological ring (R,τ) is a subring of the topological ring $(\widetilde{R},\widetilde{\tau})$ then there exists a neighbourhood of zero $\widetilde{U}$ in $(\widetilde{R},\widetilde{\tau})$ such that $R\cap\widetilde{U}=U$. Since the rings from the class $\Re$ are bounded from the left then there exists a neighbourhood of zero $\widetilde{V}$ in $(\widetilde{R},\widetilde{\tau})$ such that $\widetilde{V}\cdot\widetilde{R}\subseteq\widetilde{U}$. It follows from the openness of the homomorphism $\widetilde{\varphi}:(\widetilde{R},\widetilde{\tau})\to(\widehat{R},\widehat{\tau})$ that $\widehat{V}=\widetilde{\varphi}(\widetilde{V})$ is a neighbourhood of zero in $(\widehat{R},\widehat{\tau})$.

Since φ is an isomorphism and $\widetilde{\varphi}$ is its extension then $R\cap\ker\widetilde{\varphi}=\ker\varphi=\{0\}$. Since R is a left ideal in $\widetilde{R}$ and $\ker\widetilde{\varphi}$ is an ideal in $\widetilde{R}$ then $(\ker\widetilde{\varphi})\cdot R\subseteq\ker\widetilde{\varphi}\cap R=\{0\}$. Hence

$$\varphi^{-1}(\widehat{V})\cdot R=\varphi^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big)\cdot R\subseteq\Big(\widetilde{\varphi}^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big)\Big)\cdot R=(\widetilde{V}+\ker\widetilde{\varphi})\cdot R=$$
$$\widetilde{V}\cdot R+\ker\widetilde{\varphi}\cdot R=\widetilde{V}\cdot R\subseteq\big(\widetilde{V}\cdot\widetilde{R}\big)\cap R\subseteq\widetilde{U}\cap R=U.$$

So the implication $1\Rightarrow 2$ has been proved.

$2\Rightarrow 3$. The assertion 2 of Theorem 4 obviously follows from the assertion 2 of this theorem. Let $(\widetilde{R},\widetilde{\tau})$ be the topological ring which was constructed in the proof of the implication $2\Rightarrow 3$ in Theorem 4. Let us prove first that the topological ring $(\widetilde{R},\widetilde{\tau})$ is bounded from the left in every case mentioned in the condition of Theorem.

Let $\widetilde{W}(V,\widehat{V})\in\widetilde{\mathbf{B}}$ (see the proof of the implication $2\Rightarrow 3$ of Theorem 4). Then there exists a neighbourhood of zero $V_0\in\mathbf{B}$ such that $V_0-V_0+V_0-V_0\subseteq V$ and $\widehat{V}_0\in\widehat{\mathbf{B}}$ such that and $\widehat{V}_0-\widehat{V}_0\subseteq\widehat{V}$. Since the rings from the class $\Re$ are bounded from the left in every cases mentioned in the condition of Theorem then there exist neighbourhoods of zero $V_1\in\mathbf{B}$ and $\widehat{V}_1\in\widehat{\mathbf{B}}$ such that $V_1\cdot R\subseteq V_0$ and $\widehat{V}_1\cdot\widehat{R}\subseteq\widehat{V}_0$.

By the assertion 2 of present Theorem there exists a neighbourhood of zero $\widehat{V}_2\in\widehat{\mathbf{B}}$ such that $\widehat{V}_2\subseteq\widehat{V}_1$ and $\varphi^{-1}(\widehat{V}_2)\cdot R\subseteq V_0$.

Since the isomorphism $\varphi:(R,\tau)\to(\widehat{R},\widehat{\tau})$ is continuous then there exists a neighbourhood of zero $V_2\in\mathbf{B}$ such that $V_2\subseteq V_1$ and $\varphi(V_2)\subseteq\widehat{V}_2$. Then

$$\widetilde{W}(V_2,\widehat{V}_2)\cdot\widetilde{R}=\{(a-\varphi^{-1}(\widehat{a}),\widehat{a})\mid a\in V_2,\widehat{a}\in\widehat{V}_2\}\cdot\{(r,\widehat{r})\mid r\in R,\widehat{r}\in\widehat{R}\}=$$
$$\{\big((a-\varphi^{-1}(\widehat{a}))\cdot r,\varphi(a-\varphi^{-1}(\widehat{a}))\cdot\widehat{r}\big)\mid a\in V_2,\widehat{a}\in\widehat{V}_2,r\in R,\widehat{r}\in\widehat{R}\}=$$
$$\{\big((a-\varphi^{-1}(\widehat{a}))\cdot r+\varphi^{-1}(\varphi(a-\varphi^{-1}(\widehat{a}))\cdot\widehat{r})-\varphi^{-1}(\varphi(a-\varphi^{-1}(\widehat{a}))\cdot\widehat{r}),$$
$$\varphi(a-\varphi^{-1}(\widehat{a}))\cdot\widehat{r}\big)\mid a\in V_2,\widehat{a}\in\widehat{V}_2,r\in R,\widehat{r}\in\widehat{R}\}=$$

$$\{\big(a \cdot r - \varphi^{-1}(\widehat{a}) \cdot r + a \cdot \varphi^{-1}(\widehat{r}) - \varphi^{-1}(\widehat{a}) \cdot \varphi^{-1}(\widehat{r}) - \varphi^{-1}\big(\varphi(a) \cdot \widehat{r} - \varphi^{-1}(\widehat{a} \cdot \widehat{r})\big), \varphi(a) \cdot \widehat{r} - \widehat{a} \cdot \widehat{r}\big) \mid a \in V_2, \widehat{a} \in \widehat{V}_2, r \in R, \widehat{r} \in \widehat{R}\} \subseteq \{(b - \varphi^{-1}(\widehat{b}), \widehat{b}) \mid b \in V, \widehat{b} \in \widehat{V}\} = \widetilde{W}(V, \widehat{V}),$$

since

$$b = a \cdot r - \varphi^{-1}(\widehat{a}) \cdot r + a \cdot \varphi^{-1}(\widehat{r}) - \varphi^{-1}(\varphi(a)) \cdot \varphi^{-1}(\widehat{r}) = a \cdot r - \varphi^{-1}(\widehat{a}) \cdot r + a \cdot \varphi^{-1}(\widehat{r}) - \varphi^{-1}(\widehat{a}) \cdot \varphi^{-1}(\widehat{r}) \in V_2 \cdot R - \varphi^{-1}(\widehat{V}_2) \cdot R + V_2 \cdot R - \varphi^{-1}(\widehat{V}_2) \cdot R \subseteq V_1 \cdot R - \varphi^{-1}(\widehat{V}_2) \cdot R + V_1 \cdot R - \varphi^{-1}(\widehat{V}_2) \cdot R \subseteq V_0 - V_0 + V_0 - V_0 \subseteq V \text{ and}$$
$$\widehat{b} = \varphi(a) \cdot \widehat{r} - \widehat{a} \cdot \widehat{r} \in \varphi(V_2) \cdot \widehat{R} - \widehat{V}_1 \cdot \widehat{R} \subseteq \widehat{V}_1 \cdot \widehat{R} - \widehat{V}_1 \cdot \widehat{R} \subseteq \widehat{V}_0 - \widehat{V}_0 \subseteq \widehat{V}.$$

The boundedness of the topological ring $(\widetilde{R}, \widetilde{\tau})$ follows from that arbitrariness of $\widetilde{W}(V, \widehat{V})$.

To complete the proof of the implication $2 \Rightarrow 3$ it is sufficient to check that $(\widetilde{R}, \widetilde{\tau}) \in \Re$ for every case mentioned in the condition of Theorem provided (R, τ) and $(\widehat{R}, \widehat{\tau}) \in \Re$.

Indeed, if $\Re$ is the class of all (separated) topological rings which are bounded from the left then it is so.

If $\Re$ is the class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of right ideals then topological rings (R, τ) and $(\widehat{R}, \widehat{\tau})$ admit a fundamental system of neighbourhoods of zero consisting of subgroups. Hence by Theorem 4 the topological ring $(\widetilde{R}, \widetilde{\tau})$ also admits a fundamental system of neighbourhoods of zero consisting of subgroups and it has been proved above that it is bounded from the left. Hence, by Theorem 1.6.32 in [3] the topological ring $(\widetilde{R}, \widetilde{\tau})$ admits a fundamental system of neighbourhoods of zero consisting of right ideals, i.e. $(\widetilde{R}, \widetilde{\tau}) \in \Re$.

Let now $\Re$ be the class of all (separated) topological rings which are bounded. Then the topological rings (R, τ) and $(\widehat{R}, \widehat{\tau})$ are bounded from the right and by Theorem 4 the topological ring $(\widetilde{R}, \widetilde{\tau})$ is bounded from the right. It has been proved above that it is bounded from the left and hence is bounded, i.e. $(\widetilde{R}, \widetilde{\tau}) \in \Re$.

If $\Re$ is the class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of ideals then topological rings (R, τ) and $(\widehat{R}, \widehat{\tau})$ admit a fundamental system of neighbourhoods of zero consisting of left ideals. Then by Theorem 4 the topological ring $(\widetilde{R}, \widetilde{\tau})$ also admits a fundamental system of neighbourhoods of zero consisting of left ideals and it has been proved above that it is bounded from the left. Hence (R, τ) and $(\widehat{R}, \widehat{\tau})$ are bounded and admit a fundamental system of neighbourhoods of zero consisting of subgroups. Then by Theorem 1.6.32 in [3] it admits a fundamental system of neighbourhoods of zero consisting of ideals, i.e. $(\widetilde{R}, \widetilde{\tau}) \in \Re$.

So we have proved that $(\widehat{R}, \widehat{\tau}) \in \Re$ in every case mentioned in the condition of Theorem. This completes the proof of the implication $2 \Rightarrow 3$.

To complete the proof of Theorem is sufficient to check the implication $3 \Rightarrow 1$. It is obvious since the topological ring $(\widetilde{R}, \widetilde{\tau})$ mentioned in the assertion 3 of the current Theorem satisfies Definition 3. □

10 Remark. The following Theorem can be easily obtained from Theorem 9 proceeding to anti-isomorphic rings.

11 Theorem. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) topological rings which are bounded from the right;

2. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of left ideals of the ring;

3. The class of all (separated) topological rings which are bounded;

4. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of ideals of the ring.

Hence if (R, τ) and $(\widehat{R}, \widehat{\tau}) \in \Re$ and $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is a continuous isomorphism then the following assertions are equivalent:

1. The isomorphism φ is semi-topological in the class $\Re$;

2. For every neighbourhood of zero U in (R, τ) there exists a neighbourhood of zero $\widehat{V}$ in $(\widehat{R}, \widehat{\tau})$ such that

$$R \cdot \varphi^{-1}(\widehat{V}) \subseteq U.$$

3. There exists a topological ring $(\widetilde{R}, \widetilde{\tau}) \in \Re$ such that the topological ring (R, τ) is a right ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$, the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$ and $(\ker \widetilde{\varphi})^2 = \{0\}$.

12 Remark. The below Theorem is proved similarly to Theorem 3 in [1] and is its two-sided analogue.

13 Theorem. *If $\mathcal{K}$ is the class of all bounded topological rings or the class of all topological rings admitting a fundamental system of neighbourhoods of zero consisting of ideals then the continuous isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is semi-topological iff for every neighbourhood of zero U in (R, τ) there exists a neighbourhood of zero $\widehat{V}$ in $(\widehat{R}, \widehat{\tau})$ such that $\varphi^{-1}(\widehat{V}) \cdot R \subseteq U$ and $R \cdot \varphi^{-1}(\widehat{V}) \subseteq U$.*

14 Remark. The below assertion follows from Theorems 9 and 13 and Remark 12 of the present article.

15 Corollary. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) bounded topological rings;

2. The class of all (separated) topological rings, admitting a fundamental system of neighbourhoods of zero consisting of ideals of the ring.

Then if (R, τ) and $(\widehat{R}, \widehat{\tau}) \in \Re$ and the isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is semi-topological from the right in the class $\Re$ and is semi-topological from the left in the class $\Re$ then it is semi-topological.

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