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A test for completeness with respect to implicit reducibility in the chain super-intutionistic logics

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Abstract. We examine chain logics $C_2, C_3, \dots$, which are intermediary between classical and intuitionistic logics. They are also the logics of pseudo-Boolean algebras of type $< E_m, \&, \vee, \supset, \neg >$, where E_m is the chain $0 < \tau_1 < \tau_2 < \dots < \tau_{m-2} < 1$ $(m = 2, 3, \dots)$. The formula F is called to be implicitly expressible in logic L by the system Σ of formulas if the relation

$$L \vdash (F \sim q) \sim ((G_1 \sim H_1) \& \dots \& (G_k \sim H_k))$$

is true, where q do not appear in F, and formulas G_i and H_i, for $i = 1, \dots, k$, are explicitly expressible in L via Σ. The formula F is said to be implicitly reducible in logic L to formulas of Σ if there exists a finite sequence of formulas $G_1, G_2, \dots, G_l$ where G_l coincides with F and for $j = 1, \dots, l$ the formula G_j is implicitly expressible in L by $\Sigma \cup \{G_1, \dots, G_{j-1}\}$. The system Σ is called complete relative to implicit reducibility in logic L if any formula is implicitly reducible in L to Σ.
The paper contains the criterion for recognition of completeness with respect to implicit reducibility in the logic C_m, for any $m = 2, 3, \dots$. The criterion is based on 13 closed pre-complete classes of formulas.

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The criterion of functional completeness in classical logic [1, 2] gives an algorithm which permits, for each finite system of Boolean functions given by formulas or tables, to recognise if it is possible to obtain any Boolean function via this system using superpositions or not. Analogous criteria of completeness have been obtained in general k-valued logic, $k > 2$ [2, 3], in propositional intuitionistic logic [4], etc. Each of these criteria is based on a finite number of closed (relative to expressibility in corresponding logic [5]) classes of functions or formulas that are pre-complete (i. e. maximal and non-complete).

In connection with the fact that in general 3-valued logic and even in its fragment – in logic of First Iaśkowski's Matrix [6] there is continuum of closed classes [4, 7]. A.V. Kuznethov [9] introduced the concepts of implicit expressibility, implicit reducibility and parametrical expressibility, which are natural generalizations of usual expressibility. He found a criterion for parametrical expressibility in any general k-valued logic for $k \geq 2$.

The research of the mentioned generalizations of expressibility in nonclassical logics is an actual problem. In the present paper the conditions of implicit reducibility of the set of all formulas in the chain super-intuitionistic logic, which is intermediate between classical and intuitionistic ones, are found. The criterion of completeness with respect to impicit reducibility in these logics is given. This criterion is based on 13-classes of formulas.

Formulas (of propositional logic) are constructed from variables p, q, r (possibly with indexes) by means of logical operations: & (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\neg$ (negation). In the work the formulas are designated with capital letters of the Latin alphabet. Using the mark $\rightleftharpoons$, and reading it as "means" we introduce designations for seven formulas: $1 \rightleftharpoons (p \supset p)$, $0 \rightleftharpoons (p \,\&\, \neg p)$, $\perp F \rightleftharpoons (F \vee \neg F)$ (ternondation), $(F \sim G) \rightleftharpoons ((F \supset G) \,\&\, (G \supset F))$ (equivalence), $(F \cdot G) \rightleftharpoons ((F \sim G) \,\&\, \neg\neg G)$, $(F \,\&'\, G) \rightleftharpoons ((F \,\&\, G) \sim \perp (F \sim G)$ and $(F, G, H) \rightleftharpoons ((F \,\&\, G) \vee (F \,\&\, H) \vee (G \,\&\, H))$ (median). In the interpretation of formulas, the symbol $F[\alpha_1, \ldots, \alpha_n]$ designates the result of substitution in the formula F of the values $\alpha_1, \ldots, \alpha_n$ for variables $p_1, \ldots, p_n$, respectively.

Intuitionistical and classical propositional calculuses are based on the mentioned concept of formula. By these calculuses intuitionistical and classical logics are defined. Thus we determine the logic of that calculus as the set of all formulas deducible in the given calculus. The classical logic in this sense coincides, as it is known, with the set of formulas valid on the classical matrix.

In this paper we examine logics that are intermediary between classical logic and intuitionistic one [10, 11]. They are constructed on finite or infinite chains (i.e. linear ordered sets) of true values. It is known that the logic is called a chain [5] if the formula $((p \supset q) \vee (q \supset p))$ is true in it. In the considered m-valued logic $(m = 2, 3, \ldots)$ the variables will take values from the set E_m, where $E_m = \{0, 1, \tau_1, \tau_2, \ldots, \tau_{m-2}\}$ if m is finite and $E_m = \{0, 1, \tau_1, \tau_2, \ldots\}$ if m is infinite. Instead of τ_1 and τ_2 we will write τ and ω, respectively. We remind that the set of all functions as mappings from E_m into E_m is usually called general m-valued logic P_m. Further we consider the linear ordering on the set E_m by the relation $0 < \tau_1 < \tau_2 < \ldots$ $\ldots < \tau_{m-2} < 1$. We define the operations $\&, \vee, \supset$, and $\neg$ on E_m as follows:

$$\begin{array}{l} p \,\&\, q = \min(p, q), \\ p \vee q = \max(p, q), \end{array} \qquad p \supset q = \begin{cases} 1 & \text{if } p \leq q, \\ q & \text{if } p > q, \end{cases} \qquad \neg p = p \supset 0.$$

In the considered interpretation of symbols $\&, \vee, \supset$ and $\neg$ each formula expresses some function of general m-valued logic. Let us observe that the function $\lrcorner\, p$ of P_3 defined by the equalities $\lrcorner\, 0 = \lrcorner\, \tau_1 = 1$ and $\lrcorner\, 1 = 0$ is not expressed by any formula. We remind [8, 12] that the pseudo-Boolean algebra is the system $\mathfrak{A} =< M; \&, \vee, \supset, \neg >$ that is a lattice by & and $\vee$, where $\supset$ is relative pseudo-complement and $\neg$ is pseudo-complement. The logic of this algebra is defined as the set of all formulas that are true on $\mathfrak{A}$, i.e. formulas identically equal to the greatest element 1 of this algebra. We will denote the algebra $< E_m; \&, \vee, \supset, \neg >$ $(m = 2, 3, \ldots)$ by Z_m. The logic of this algebra LZ_m is denoted by C_m. It is also possible to define the logic C_1 of one-element algebra which includes the set of all formulas and is contradictory. The smallest chain logic, called Dummett logic [10], coincides with the intersection of all m-valued chain logics with m positive integer number.

Two formulas F and G are called equivalent in logic L (write $L \vdash (F \sim G)$) if the equivalence $F \sim G$ in L is true. Two formulas are equivalent in logic C_m $(m = 1, 2, \ldots)$ if and only if the operators of algebra Z_m, expressed by them, are equal.

Therefore instead of the relation $C_m \vdash (F \sim G)$ we sometimes will use the equality $F = G$ on Z_m. If the formula $F \sim G$ contains only the variables $p_1, p_2, \ldots, p_n$ and the inequality $(F \sim G)[p1/\alpha_1, \ldots, p_n/\alpha_n] \neq 1$ is true on Z_m, then we will use the notation $(F \neq G)[p1/\alpha_1, \ldots, p_n/\alpha_n]$. The formula F is called explicitly expressible in logic L by the system of formulas of Σ [9] if it is possible to obtain the formula F from variables and formulas of Σ using a finite number of times the weak substitution rule, and the rule of replacement by equivalents in L. The relation of explicit expressibility is transitive. The formula F is called directly expressible via the system of formulas of Σ if it is possible to obtain F from variables and formulas of Σ by using a finite number of times the weak substitution rule. The relation of direct expressibility is transitive.

The formula F is called implicitly expressible in logic L [9] via the system of formulas Σ if there exist the formulas G_i and H_i $(i = 1, \ldots, k)$ explicitly expressible in L by Σ such that the predicate $L \vdash (F \sim q)$, where q is a variable not contained in F, is equivalent to the predicate $L \vdash ((G_1 \sim H_1)\,\&\, \ldots \,\&\, (G_k \sim H_k))$.

Because the relation of implicit expressibility, generally speaking, is not transitive, we are going to introduce a new concept. The formula F is called implicitly reducible in logic L via formulas of Σ if there exists a finite sequence of formulas $G_1, G_2, \ldots, G_l$, where G_l coincides with F and each term of this sequence can be implicitly expressible in L by Σ and terms of the sequence placed before it. We will say that the system Σ' of formulas is implicitly reducible in L to the system Σ if each formula of Σ' is implicitly reducible in L to Σ. It is clear that the relation of implicit reducibility is transitive. The system Σ of formulas is called complete with respect to implicit reducibility in logic L if each formula (in language of this logic) is implicitly reducible in L to Σ. The system Σ of formulas is said to be pre-complete with respect to implicit reducibility in L if Σ is not complete by this reducibility in L, but the system $\Sigma \cup \{F\}$ is complete relative to implicit reducibility in L, for any formula F.

Two functions $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_k)$ of P_m are called permutable [13] if the identity $f(g(x_{11}, \ldots, x_{1k}), \ldots, g(x_{n1}, \ldots, x_{nk})) = g(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1k}, \ldots, x_{nk}))$ is true. The set of all functions of P_m, permutable with the given function f, is called the centralizer of function f (denoted $\prec f \succ$)[13]. The set of all formulas which in the interpretation on Z_m are permutable with the function f (from P_m) is called the formula centralizer on the algebra Z_m of function f. We say the function $f(x_1, \ldots, x_n)$ of P_m preserves the predicate (relation) $R(x_1, \ldots, x_w)$ if for any possible values of variables $x_{ij} \in E_m$ $(i = 1, \ldots, w;\ j = 1, ..n)$, from the truth of $R(x_{11}, x_{21}, \ldots, x_{w1}), \ldots, R(x_{12}, x_{22}, \ldots, x_{w2}), \ldots, R(x_{1n}, x_{2n}, \ldots, x_{wn})$ follows the truth of $R(f(x_{11}, x_{12}, \ldots, x_{1n}), \ldots, f(x_{21}, x_{22}, \ldots, x_{2n}), \ldots, f(x_{w1}, x_{w2}, \ldots, x_{wn}))$. The centralizer $\prec f(x_1, \ldots, x_n) \succ$ coincides with the set of all functions of P_m which preserve the predicate $f(x_1, \ldots, x_n) = x_{n+1}$, where the variable x_{n+1} differs from $x_1, \ldots, x_n$ [9]. We say that the formula F preserves, on the algebra Z_m, the predicate R if the function of logic C_m, expressed by formula F, preserves R. The predicate could be replaced by the corresponding to it matrix (α_{ij}) $(i = 1, \ldots, w;\ j = 1, \ldots, t)$ of elements of algebra Z_m [14] such that the predicate R is true on all those and only those sets of elements that are columns in this matrix. Let us observe that each formula of the system $\{p \,\&\, q, p \vee q, p \supset q, \neg p\}$ preserves on the algebra Z_m $(m = 3, 4, \ldots)$ the below predicates and matrices, therefore any formula preserves them too:

$$\neg x = \neg y, \quad x \neq \tau_j \quad (j = 1, 2, \ldots, m-2), \tag{1}$$

$$\begin{pmatrix} 0 & \tau & 1 \\ 0 & \tau_j & 1 \end{pmatrix} \quad (j = 1, 2, \ldots, m-2), \tag{2}$$

$$\begin{pmatrix} 0 & \tau & \omega & 1 \\ 0 & \tau_v & \tau_w & 1 \end{pmatrix} \quad (v,\, w = 1, 2, \ldots, m-2;\ v < w), \tag{3}$$

$$\begin{pmatrix} 0 & \tau_j & 1 & 1 \\ 0 & \tau_v & \tau_w & 1 \end{pmatrix} \quad (j,\, v,\, w = 1, 2, \ldots, m-2;\ v < w). \tag{4}$$

We present the next affirmation without any proof.

Affirmation. *If the function f belongs to the class C_m $(m = 2, 3, \ldots)$ then the following identity:*

$$f(\neg\neg x_1, \ldots, \neg\neg x_n) = \neg\neg f(x_1, \ldots, x_n) \tag{5}$$

is true.

Let us observe that the class of all formulas that preserve on Z_m some predicate is closed relative to the explicit expressibility in logic C_m, but it is not necessarily closed relative to the implicit expressibility in this logic [9]. It is easy to see that any class of formulas is closed relative to the implicit reducibility in logic C_m if and only if it is closed relative to the implicit expressibility. We remind that the centralizer of one or another function is closed relative to the implicit expressibility. It is obvious that for each $m = 1, 2, \ldots$, if the class of functions K is closed relative to the implicit expressibility in logic C_m then K is closed relative to the implicit expressibility in any logic C_n where $n \geq m$.

Let us define the functions f_1 and f_2 from P_4 as follows:

$$f_1(0) = 0, \quad f_1(\tau) = 1, \quad f_1(w) = \omega, \quad f_1(1) = 1,$$
$$f_2(0) = 0, \quad f_2(\tau) = \omega, \quad f_2(w) = \omega, \quad f_2(1) = 1.$$

We denote the classes of formulas preserving the predicates $x = 0$, $x = 1$, $\neg x = y$, $x \,\&\, y = z$, $x \vee y = z$, $(x \sim (y \sim z)) = u$ on Z_2, $\lrcorner\lrcorner x = y$, $\perp x = \perp y$, $(x \,\&\, y = z) \,\&\, (\neg x = \neg y)$, $((x \sim y) \,\&\, \neg\neg y = z) \,\&\, (\neg x = \neg y)$, $((x \,\&\, y) \sim ((x \sim y) \vee \neg(x \sim y)) = z) \,\&\, (\neg x = \neg y)$ on Z_3, $f_1(x) = y$, $f_2(x) = y$, respectively, on Z_4 by symbols $\Omega_0, \Omega_1, \ldots, \Omega_{12}$. Let us observe that the class Ω_5 on algebra Z_2 coincides with known class of linear Boolean functions. Remind that the closure relative to the implicit expressibility in C_2 of classes $\Omega_0, \ldots, \Omega_5$ is based on the fact that they are centralizers of some functions. Analogous closure in C_3 of classes $\Omega_6, \ldots, \Omega_{10}$ is shown in [15]. It follows that these classes are closed relative to the implicit expressibility in any other logic C_m, where $m \geq 3$.

Assertion 1 (A.V. Kuznetsov [9]). *In order that the system Σ of formulas could be complete by the implicit reducibility in logic C_2 it is necessary and sufficient that Σ be not included in any of clases $\Omega_0, \ldots, \Omega_5$.*

According to [15] the next criterion of completeness relative to the implicit reducibility in logic of First Iaśkowski's Matrix is true:

Assertion 2. *In order that the system Σ of formulas could be complete with respect to the implicit reducibility in logic C_3 it is necessary and sufficient that for each $i = 0, \ldots, 10$ should exist a formula of Σ which doesn't belong to the class Ω_i.*

The next criteria of completeness with respect to the reducibility in any chain logic included in C_4 are true:

Theorem 1. *For any $m = 4, 5, \ldots$, in order that the system Σ of formulas could be complete by the implicit reducibility in logic C_m it is necessary and sufficient that Σ be complete by implicit reducibility in logic C_3 and be not included in the following two formula centralizers on algebra Z_4:*

$$\prec f_1(p) \succ, \quad \prec f_2(p) \succ. \tag{6}$$

Proof. *Necessity.* Let the system Σ be complete with respect to the implicit reducibility in logic C_m $(m \geq 4)$. Then, since the implicit reducibility in logic C_m $(m \geq 2)$ implies the implicit reducibility in C_{m-1}, it results that Σ is complete by the implicit reducibility in C_3. Because formula centralizers are closed relative to the implicit reducibility in logic C_4, then they are closed relative to the implicit reducibility in C_m where $m > 4$. Moreover, they are not complete in C_m, because they don't contain for example the formula $((x \supset y) \,\&\, \neg\neg y)$. So no one of them could contain Σ.

Sufficiency. Let Σ be complete by the implicit reducibility in logic C_3 and be not included in any of two formula centralizers (6). Then Σ is complete by the implicit reducibility in C_2, since there exist, in accordance with Assertion 2, the formulas $F_0, \ldots, F_{10}$ which don't belong to $\Omega_0, .., \Omega_{10}$, and also there exist F_{11}, F_{12}, which don't belong to Ω_{11}, Ω_{12}, respectively. Let us suppose that these formulas don't contain other variables except $p_1, \ldots, p_n$. It is sufficient to prove that every formula of system $\{p \,\&\, q,\ p \vee q,\ p \supset q,\ \neg p\}$ is implicitly reducible to the system Σ of formulas in C_m $(m = 4, 5, \ldots)$. It is known [10] that in any chain logic C_m the relation

$$C_m \vdash (p \vee q) \sim (((p \supset q) \supset q) \,\&\, ((q \supset p) \supset p))$$

is true. The conjunction is implicitly expressible via the implication in any chain logic C_m, because the relation

$$C_m \vdash ((p \,\&\, q) \sim r) \sim (((p \supset (q \supset r)) \sim 1) \& ((r \supset p) \supset 1) \& ((r \supset q) \sim 1))$$

is true. It remains to prove that the formulas $\neg p$ and $p \supset q$ are implicit reducible to the system Σ in any chain logic included in C_4.

This fact results from the next lemmas.

Lemma 1. *If the formula $\neg p$ is implicitly reducible to the system Σ of formulas in logic C_3 then this formula is implicitly reducible to Σ in logic C_m, for any $m = 3, 4, \ldots$*

Lemma 2. *If the formula 0 is implicitly reducible to the system Σ of formulas in logic C_2 then this formula is implicitly reducible to Σ in logic C_m, for any $m = 3, 4, \ldots$*

Lemma 3. *The formula 1 is explicitly expressible through 0 and $\neg p$ in C_m, for any $m = 3, 4, \ldots$*

Lemma 4. *If the formula $\perp p$ is implicitly reducible to the system Σ of formulas in logic C_3 then this formula is implicitly reducible to Σ in logic C_m, for any $m = 3, 4, \ldots$*

Lemma 5. *If the formula $p \,\&\, q$ is implicitly reducible to the system Σ of formulas in logic C_2 then the formula $\neg\neg(p \,\&\, q)$ is explicitly expressible through 0, 1, $\neg p$ and Σ in logic C_m, for any $m = 3, 4, \ldots$*

Lemma 6. *If the formula $\neg p \,\&\, q$ is implicitly reducible to the system Σ of formulas in logic C_3 then the formulas $\neg p \,\&\, q$ and $\neg p \vee q$ are implicitly expressible through 0, 1, $\neg p$, $\perp p$, $\neg\neg(p \,\&\, q)$ and Σ in the logic C_m, for any $m = 3, 4, \ldots$*

In order to obtain the implication we further present 5 lemmas without proofs.

Lemma 7. *At least one of 4 following formulas:*

$$p \supset q,\ p \sim q,\ \perp p \vee \perp q,\ (p \,\&\, q) \sim ((p \sim q)\ \vee\ \neg\,(p \sim q)) \tag{7}$$

is explicitly expressible in C_m through formulas of the system

$$\{0,\ 1,\ \neg p,\ \perp p,\ \neg p \,\&\, q,\ \neg p \vee q\} \tag{8}$$

and F_8, for any $m = 3, 4, \ldots$

Lemma 8. *At least one of 3 formulas:*

$$p \supset q,\ \perp p\ \vee \perp q,\ (p \,\&\, q) \sim ((p \sim q)\ \vee\ \neg\,(p \sim q)) \tag{9}$$

is explicitly expressible through formulas of the system (8) and F_8, F_9 in C_m, for any $m = 3, 4, \ldots$

Lemma 9. *At least one of following 4 systems:*

$$\{p \supset q\},\quad \{(p \sim q) \vee q\},\quad \{(p \,\&\, q) \sim ((p \sim q)\ \vee\ \neg\,(p \sim q)), T'\}, \{\perp p \vee \perp q, T'\}, \tag{10}$$

is explicitly expressible through formulas of the system (8) and F_8, F_9 and F_{10} in C_m, for any $m = 3, 4, \ldots$, where

$$T'[\tau, \tau, 1] = T'[\tau, 1, \tau] = \tau,\quad T'[\tau, 1, 1] = 1. \tag{11}$$

Lemma 10. *The implication $(p \supset q)$ is implicitly expressible in C_m, for any $m = 4, 5, \ldots$, through formulas of system (8), formula F_{11} and any of two systems $\{(p \sim q) \vee q\}$ or $\{(p \,\&\, q) \sim ((p \sim q)\ \vee\ \neg\,(p \sim q)), T'\}$, where T' is 3-ary formula, which satisfies (11) conditions.*

Lemma 11. *The formula $p \supset q$ is implicitly expressible in C_m, for any $m = 4, 5, \ldots$, through formulas of (8), formulas F_{11}, F_{12} and the system $\{\perp p \vee \perp q, T'\}$, where T' is the 3-ary formula satisfying conditions (11).*

From the formulated above lemmas it results that conditions of theorem are sufficient, namely the formula $\neg p$ is implicitly reducible to the system Σ of formulas in logic C_m, for any $m = 3, 4, \ldots$. Lemmas 1 – 11 allow us to deduce that the implication $p \supset q$ is implicitlty reducible to Σ in any chain logic C_m included in C_4. So, according to lemmas 1 – 6 the formulas of the system (8) are implicitly reducible to Σ. Lemmas 7 – 9 permit to conclude that at least one of 4 systems of formulas (10) is explicitly expressible in logic C_m through formulas (8) and F_8 – F_{10}. Therefore it remains to observe that one of these systems consists of $p \supset q$, but the implication is implicitly expressible in C_m through any other of 3 systems and formulas F_{11}, F_{12} and (8), in accordance with Lemmas 10 and 11.

From Assertion 1, 2 and Theorem 1 the next criterion of completeness with respect to implicit reducibility in an arbitrary chain logic results.

Theorem 2. *In order that the system of formulas Σ could be complete relative to the implicit reducibility in any chain logic L, including Dummett logic, it is necessary and sufficient that the next conditions be satisfied simultaneously:*

1) if $L \subseteq C_2$ then the system Σ is included neither in Ω_0, nor in Ω_1, nor in Ω_2, nor in Ω_3, nor in Ω_4, nor in Ω_5;

2) if $L \subseteq C_3$ then Σ is also included neither in Ω_6, nor in Ω_7, nor in Ω_8, nor in Ω_9, nor in Ω_{10};

3) if $L \subseteq C_4$ then Σ is also included neither in Ω_{11}, nor in Ω_{12}.

Proof. *Necessity* results from the fact that all these classes are closed relative to the implicit reducibility in C_m and are pairwise incomparable to inclusions.

Sufficiency. Let conditions 1)–3) be satisfied. Then the system Σ is complete relative to implicit reducibility in C_2 according to Assertion 1, and it is complete relative to the implicit reducibility in C_3 by Assertion 2 and it is complete relative to implicit reducibility in any chain logic C_m, included in C_4 according to Theorem 1.

From this criterion the next corollaries follow.

Theorem 3. *For any chain logic L (including Dummett logic) there exists an algorithm that allows to recognize for any finite system Σ of formulas if Σ is complete relative to implicit reducibility in logic L or not.*

From Assertion 1 it results that the classes $\Omega_0, \Omega_1, \ldots, \Omega_5$ and only they are pre-complete relative to implicit reducibility in C_2, and the classes $\Omega_0, \Omega_1, \ldots, \Omega_{10}$ and only they are pre-complete by implicit reducibility in C_3.

Theorem 4. *The next 13 classes: $\Omega_0, \Omega_1, \ldots, \Omega_{12}$ of formulas and only they are pre-complete relative to implicit reducibility in logic C_m, for any $m = 4, 5, \ldots$.*

A system Σ of formulas is called weak complete with respect to implicit reducibility in logic L if the system $\Sigma \cup \{p \supset p,\ p \& \neg p\}$ is complete by implicit reducibility in L.

Theorem 5 (criterion of weak completeness with respect to implicit reducibility in an arbitrary chain logic). *In order that the system* Σ *of formulas be weak complete relative to implicit reducibility in chain logic L it is necessary and sufficient that the next conditions be satisfied simultaneously:*

1) if $L \subseteq C_2$ *then system* Σ *is included neither in* Ω_3*, nor in* Ω_4*, nor in* Ω_5*;*

2) if $L \subseteq C_3$ *then system* Σ *is also included neither in* Ω_6*, nor in* Ω_7*, nor in* Ω_8*, nor in* Ω_9*, nor in* Ω_{10}*;*

3) if $L \subseteq C_4$ *then system* Σ *is also included neither in* Ω_{11}*, nor in* Ω_{12}*.*

The logics L_1 and L_2 are called equal relative to completeness by implicit reducibility if any system Σ of formulas is complete by implicit reducibility in L_1 if and only if this system is complete by implicit reducibility in L_2.

Theorem 6. *Any chain logic is equal relative to completeness with respect to implicit reducibility to one and only one of the next 4 logics: the absolute contradictory logic, the classical logic, the logic* C_3 *and* C_4 *logic.*

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