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The Multidimensional Directed Euler Tour of Cubic Manifold

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Abstract. In the paper [3] we tried to generalize the problem of existence of a directed $(n-1)$-dimensional Euler tour for the abstract directed n-dimensional manifold, which is a *complex of multi-ary relations* [5], namely by means of abstract simplexes. In the paper [3] we show the existence of such kind of tour only for manifolds of odd dimension because we have not enough conditions to do more. In the present paper we will show conditions of existence for a directed Euler tour of abstract manifolds with even dimensions. In this purpose, we will introduce some new definitions which permit us to define manifolds by so-called *abstract cubes*.

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For clarity, let's remind some notions and introduce other new ones, although they will be rather numerous.

Definition 1 [2]. *The complex of multi-ary relations, $K^n = \{S^m_\lambda : \lambda \in \Lambda, card\Lambda < \infty, 0 \le m \le n\}$, denoted V^n, is called an* ***abstract manifold of dimension n*** *if it satisfies the following postulates:*

A. $\forall S^{n-1} \in V^n$ is a common face exactly for two abstract n-dimensional simplexes;

B. for $\forall S^n_i, S^n_j \in V^n$, $i \neq j$, exists a sequence of n-dimensional simplexes $S^n_1 = S^n_i, S^n_2, \ldots, S^n_k = S^n_j$, $k \ge 2$, where $S^n_r \cap S^n_{r+1} = S^{n-1}_{r,r+1}$, $r \in \{1, 2, \ldots, k-1\}$;

C. for $\forall S^m \in V^n$ it holds that $\exists S^n \in V^n$ such that S^m is a face of S^n, $m \in \{0, 1, \ldots, n-1\}$;

D. for two disjoint simplexes $\forall S^n_i, S^n_j \in V^n$, where $S^n_i \cap S^n_j = S^m$, it holds that $\exists S^n_1 = S^n_i, S^n_2, \ldots, S^n_k = S^n_j$ such that $\bigcap\limits_{l=1}^{k} S^n_l = S^m$.

From the postulate A it results immediately that $\forall S^n \in V^n$ takes part in the building of V^n just one time.

Definition 2 [4]. *Let K^n be a complex of multi-ary relations and $S^m = [x_{i_0}, x_{i_1}, \ldots, x_{i_m}]$ be a simplex from K^n.*

Let's consider $\overset{\circ}{S}{}^m = (x_{i_0}, x_{i_1}, \ldots, x_{i_m}) = S^m \backslash \{F_\lambda\}$, $\lambda \in \Lambda'$, where $\{F_\lambda\}$, $\lambda \in \Lambda'$ is *the set of all proper faces* [5] *of* S^m. $\overset{\circ}{S^m}$ *is called a* ***m-dimensional vacuum*** *of the simplex* S^m, $m \in \{1, 2, \ldots, n\}$.

There are two ways for generalization of an Euler tour/directed Euler tour for K^n in the graph theory. We consider the case when $K^n \equiv V^n$.

a) Let Z be the group of integer numbers, $F : V^n \to Z$ be a family of single–valued maps that satisfies: for $\forall S^m \in V^n$ and $\forall f \in F$ it holds that $f(-S^m) = -f(S^m)$, $0 \leq m \leq n$. We will denote $f(S^m) = g \in Z$ and will use the notation gS^m for $f(S^m)$.

Definition 3 [3]. *The formal sum*

$$l^m = g_1 S_1^m + g_2 S_2^m + \ldots, + g_{\alpha_m} S_{\alpha_m}^m, \tag{1}$$

where α_m *is the number of all m-dimensional simplexes of manifold* V^n*, is called a* ***m-dimensional*** Δ***-chain*** [5] *of the manifold* V^n*. Moreover, if in* (1) $|g_i| = 1$, $i \in \{1, 2, \ldots, \alpha_m\}$ *and* $\Delta l^m = 0$ [5]*, then the* Δ*-chain is called a* ***m-dimensional Euler*** Δ***-cycle*** *of* V^n*. If the simplexes* S_i^m *and* $S_j^m, i, j \in \{1, 2, \ldots, , \alpha_m\}$, $i \neq j$, *are also coherent, where* $g_i = 1$ *or* $g_i = -1$, $\forall i \in \{1, 2, \ldots, \alpha_m\}$ *and* $\Delta l^m = 0$*, then this* Δ*-chain will be called a* ***m-dimensional Euler*** Δ***-contour*** *of the manifold* V^n*.*

Let's remind that the *manifold* V^n is called *directed* if there exists a n-dimensional Δ-cycle, $\Delta l^n = 0$.

It is obvious that the manifold V^n determines the existence of the n-dimensional Euler Δ-cycle. Every n-dimensional simplex takes part in this cycle and repetitions are not admitted. It is obvious that the existence of a n-dimensional Euler Δ-contour is determined on a manifold with coherent simplexes.

This generalization of the directed Euler tour in the graph theory is trivial and is not far-reaching for obtaining other information, with exception of the classical results [1]. It is more important to examine the next generalization.

b) Let $\mathcal{S}^m = \{S_1^m, S_2^m, \ldots, S_{\alpha_m}^m\}$, $0 \leq m \leq n$, be the set of all m-dimensional simplexes of the manifold V^n.

Definition 4 [3]. *The sequence of simplexes* $S_1^m, S_2^m, \ldots, S_{\alpha_m}^m$ *of the manifold* V^n *is called a* ***linear m-dimensional Euler cycle*** *of this manifold if for* $\forall S_r^m, S_{r+1}^m \in V^n$, $r \in \{1, 2, \ldots, \alpha_m\}$*, it holds that* $S_r^m \cap S_{r+1}^m = S^{m-1} \in \mathcal{S}^{m-1}, S_1^m \equiv S_{\alpha_m}^m$*. Moreover, if* $\forall S_r^m, S_{r+1}^m \in V^n$ *are coherent,* $r \in \{1, 2, \ldots, \alpha_{m-1}\}$*, the sequence mentioned above will be called a* ***linear m-dimensional Euler contour*** *of the manifold* V^n*.*

It is necessarily to define the next new notion to achieve our goal. We define the next notion by induction.

Definition 5.

1^o. *The* **abstract 0-dimensional** *(**1-dimensional**, respectively)* ***cube*** *and the abstract simplex of the same dimension are the same. The* ***vacuum of 0-dimensional (1-dimensional**, respectively)* ***cube*** *and the simplex' vacuum of the same dimension are the same.*

2^o. *Let consider two pairs of* 0*-dimensional cubes* S_1^0, S_2^0 *and* S_3^0, S_4^0*. The* 2*-ary relations of the pairs of cubes* S_1^0, S_2^0 *and* S_3^0, S_4^0 *determine the existence of the* 1*-dimensional cubes* $S_1^1 = (S_1^0, S_2^0)$, $S_2^1 = (S_3^0, S_4^0)$, $S_3^1 = (S_1^0, S_3^0)$ *and* $S_4^1 = (S_2^0, S_4^0)$*. We construct between these pairs of cubes only the* 2 *and* 3*-ary relations which determine a simplicial complex. So we have the existence of the simplexes* $S_5^1 = (S_1^0, S_4^0), S_1^2 = (S_1^0, S_3^0, S_4^0)$ *and* $S_2^2 = (S_1^0, S_2^0, S_4^0)$*.*
To define the notion of a 2-dimensional abstract cube, we define the vacuum of the respective cube, denoted $\overset{\circ}{I}{}^2$*. The* ***vacuum of 2-dimensional cube*** *is the union of the vacuums of simplexes* [4] $\overset{\circ}{I}{}^2 = \overset{\circ}{S}{}_1^2 \cup \overset{\circ}{S}{}_2^2 \cup \overset{\circ}{S}{}_5^1$*. The* ***abstract 2-dimensional cube*** *is defined from its vacuum by the following relation* $I^2 = \bigcup_{i=1}^{4} S_i^1 \bigcup \overset{\circ}{I}{}^2$*.*

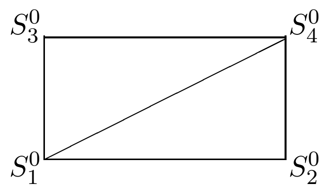

Figure 1

In Figure 1 so called **procube**, which will be denoted $I^2(\Delta)$, is represented.

3^o. *Let's consider, as axiomatic principle, that* I^i *is an* ***abstract i-dimensional cube***, $1 \le i \le n-1$ *and* $I^i(\Delta)$ *is the respective* ***procube****, but* $\overset{\circ}{I}{}^i$ *is its* ***vacuum****.*

4^o. *Let's inductively define* n*-dimensional abstract cubes by the cubes of dimension* $n-1$*. So let consider* $2n$ *cubes of dimension* $n-1$*:* $I_1^{n-1}, I_2^{n-1}, \ldots, I_{2n}^{n-1}$*, and* $I_1^{n-1}(\Delta), I_2^{n-1}(\Delta), \ldots, I_{2n}^{n-1}(\Delta)$ *are respective procubes. It is considered only the* i*-ary relations between* 0*-dimensional cubes of their procubes,* $2 \le i \le n$*, which determine a simplicial complex* [7]*. The* ***vacuum of n-dimensional cube****, denoted* $\overset{\circ}{I}{}^n$*, is the union of all the vacuums of simplexes which do not intersect the procubes* $I_j^{n-1}(\Delta)$, $1 \le j \le 2n$*. The* ***abstract n-dimensional cube*** *is the union* $I^n = \bigcup_{i=1}^{2n} I_i^{n-1} \bigcup \overset{\circ}{I}{}^n$*. Let's denote the* ***procube*** *of the cube* I^n *by* $I^n(\Delta)$*, which represents the simplicial complex mentioned above.*

Definition 6. *Any abstract cube I^m, $0 \leq m \leq n-1$, which takes part in the building of the cube I^n, in concordance with Definition 5, is called a* ***proper face of the cube*** *I^n.*

Example 1. *The cube I^3, plotted in Figure 2, is the union $I^3 = \bigcup_{i=1}^{6} I_i^2 \bigcup \overset{\circ}{I^3}$, where*
$I_1^2 = (S_1^0, S_2^0, S_7^0, S_6^0)$, $I_2^2 = (S_2^0, S_3^0, S_8^0, S_7^0)$, $I_3^2 = (S_4^0, S_3^0, S_8^0, S_5^0)$,
$I_4^2 = (S_1^0, S_4^0, S_5^0, S_6^0)$, $I_5^2 = (S_6^0, S_7^0, S_8^0, S_5^0)$, $I_6^2 = (S_1^0, S_2^0, S_3^0, S_4^0)$.
The vacuum of the 3-dimensional cube is the union of the vacuums of simplexes
$\overset{\circ}{I^3} = (\bigcup_{i=1}^{2} \overset{\circ}{S_i^3}) \bigcup (\bigcup_{i=1}^{8} \overset{\circ}{S_i^2}) \bigcup S_1^1$, *where*
$S_1^3 = (S_2^0, S_6^0, S_7^0, S_8^0)$, $S_2^3 = (S_1^0, S_3^0, S_4^0, S_5^0)$,
$S_1^2 = (S_1^0, S_2^0, S_3^0, S_8^0)$, $S_2^2 = (S_1^0, S_6^0, S_5^0, S_8^0)$, $S_3^2 = (S_1^0, S_2^0, S_8^0, S_6^0)$,
$S_4^2 = (S_1^0, S_3^0, S_8^0, S_5^0)$, $S_5^2 = (S_1^0, S_6^0, S_8^0)$, $S_6^2 = (S_1^0, S_8^0, S_3^0)$, $S_7^2 = (S_1^0, S_5^0, S_3^0)$,
$S_8^2 = (S_6^0, S_8^0, S_2^0)$, $S_1^1 = (S_1^0, S_8^0)$.

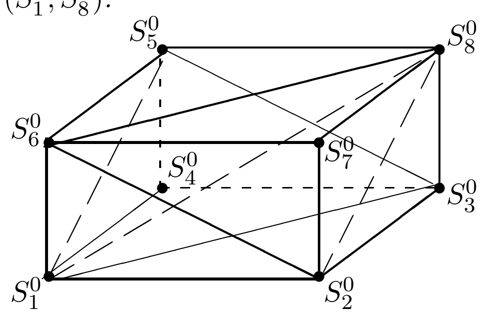

Figure 2

Let's denote I^m, $1 \leq m \leq n$, the abstract m-dimensional cube, constructed by the set of vertices $X = \{x_1, x_2, \ldots, x_{2^m}\}$.

Definition 7. *The finite and nonempty sequence of abstract cubes, denoted $\mathcal{I}^n = \{I^m, 0 \leq m \leq n\}$, is called* ***abstract n-dimensional cubic complex*** *if are satisfied the following postulates:*

1. *for $\forall I^s, I^t \in \mathcal{I}^n$, $0 \leq s, t \leq n$, it holds $I^s \cap I^t \in \mathcal{I}^n$ or $I^s \cap I^t = \emptyset$;*
2. *any face I^k of $\forall I^n \in \mathcal{I}^n$, $k < n$, is an element of $\mathcal{I}^n$;*
3. *$\exists I^n \in \mathcal{I}^n$.*

We need to define the orientation for an abstract cube $I^r \in \mathcal{I}^n$. Thus, we give an analogy with the classical geometrical situation. Considering an arbitrary unit cube given of his vertices, $I^r = (x_1, x_2, \ldots, x_{2^r})$, from the linear space R^n. Let's fix a 0-dimensional face x_{i_0} of this cube, as the origin of coordinates of the linear space R^n. Using all other 0-dimensional faces, x_j, $1 \leq j \leq 2^r$, which are adjacent with the origin x_0, we will represent by arcs the 2-ary relations of the examined coordinate system. The sequence of 0-dimensional faces shows a positive oriented cube, because it belongs to the 1-dimensional simplexes of positive orientation of the examined coordinate system, which respects the counterclockwise order (see Figure 3, on the left), having an even number of transpositions' arrangement of the order indicated by their simplexes. Let's denote this geometrical cube $+I^r = +(x_1, x_2, \dots, x_{2^r})$.

If the sequence of 0-dimensional faces of the examined cube belongs to 1-dimensional simplexes with negative orientation of examined coordinate system, which respect the counterclockwise order (see Figure 3, on the right), having an odd number of transpositions' arrangement of the order indicated by their simplexes, we will have a cube with negative orientation. In this case we will denote it $-I^r = -(x_1, x_2, \dots, x_{2^r})$.

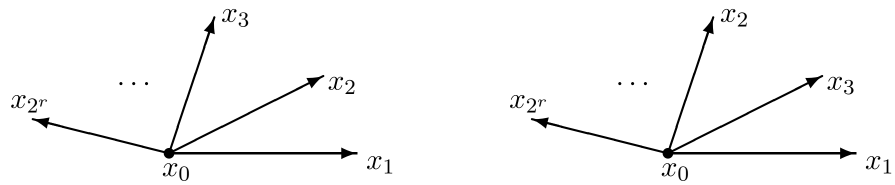

Figure 3

Is obvious

Theorem 1. *Let $\mathcal{I}^n$ be a complex made of abstract cubes and consider an arbitrary r-dimensional cube $I^m = (x_{i_1}, x_{i_2}, \dots, x_{i_r}) \in \mathcal{I}^n$, $1 \le r \le n$, having a set of 0-dimensional faces $\{x_{i_0}, x_{i_1}, \dots, x_{i_{2^r}}\}$. For a fixed $x_{i_t} \in \{x_{i_1}, x_{i_2}, \dots, x_{i_{2^r}}\}$, all the pairs (x_{i_t}, x_{i_j}), $j \in \{1, \dots, 2^r\} \backslash \{t\}$, form binary relations, so determine graphically some arcs.*

This analogy allows us to define the *orientation* of an abstract cube I^r.

Definition 8. *If the number of transpositions of the sequence $(i_1, i_2, \dots, i_{2^r})$ is even, the abstract cube mentioned above is called* ***positively oriented cube*** *and is denoted $+I^r$. Otherwise, if the number of transpositions is odd, the abstract cube I^r is* ***negatively oriented*** *and is denoted $-I^r$.*

Holds

Theorem 2. *Let V^n_Δ be a manifold defined by abstract simplexes (see Definition 1) and $V^n_\square$ be a manifold defined by respective abstract cubes (see Definition 5). If V^n_Δ and $\mathcal{V}^n_\square$ have the same genus, then they belong to the same class* [4].

The proof is simple if we rely on the procubes of the abstract cubes which make the manifold $V^n_\square$. In this case, the last one is built from abstract simplexes (see Definition 5).

Let's consider the collection of all single-valued maps, $H : \mathcal{I}^n \to Z$, that satisfies the propriety: if $I^r_i \in \mathcal{I}^n$, $0 \le r \le n$, is an abstract negative oriented cube, $-I^r_i$, then $h(-I^r_i) = -h(I^r_i)$, for $\forall h \in H$. We will denote $h(I^r_i) = g_i$. For simplicity, we will use $g_i I^r_i$ for $h(I^r_i)$, and so $-g_i I^r_i$ for $-I^r_i$.

By analogy with the elements of the complex of multi-ary relations, the following notions allow us to determine the *coherence*, the *noncoherence* and the *orientability* of the abstract cube $I^r \in \mathcal{I}^n$ and of the cubic manifold.

Definition 9. *The sum of all r-dimensional cubes of cubic complex* $\mathcal{I}^n$,

$$L_C^r = g_1 I_1^r + g_2 I_2^r + \, \dots \, + g_{\beta_r} I_{\beta_r}^r \,, 2 \leq r \leq n, \tag{2}$$

where β_r *is the cardinal of the set of all r-dimensional abstract cubes from* $\mathcal{I}^n$, *is called a* ***□-chain of dimension r***, *formed by the cubes of the complex* $\mathcal{I}^n$.

Definition 10. *The expression*

$$L_{C_1}^r + L_{C_2}^r = \sum_{i=1}^{\beta_r} (g_i^1 + g_i^2) I_i^r \tag{3}$$

is called the ***sum of the □-chains*** $L_{C_1}^r = \sum\limits_{i=1}^{\beta_r} g_i^1 I_i^r$ *and* $L_{C_2}^r = \sum\limits_{i=1}^{\beta_r} g_i^2 I_i^r$.

Analogously with the case from [6], it is easy to verify the following affirmation:

Theorem 3. *The set of all r-dimensional □-chains of the complex of cubes* $\mathcal{I}^n$, *denoted* $\mathcal{L}^r, 0 \leq r \leq n$, *forms a commutative group with respect to the operation defined by* (3).

Definition 11. *The abstract cubes* $I^r, I^{r-1} \in \mathcal{I}^n$ *are called* ***coherent*** *if they have the same orientation, otherwise they are called* ***noncoherent***. *If the cubes* I^r *and* I^{r-1}, *where* I^{r-1} *is a face of* I^r, *have the same orientation (the opposite orientation, respectively), the ratio* $[I^r : I^{r-1}]$ $(-[I^{r-1} : I^r]$, *respectively) is called their* ***incidence coefficient***.

Let's mention that $\forall I_i^r \in \mathcal{I}^n$, $I_i^r = (x_{i_1}, x_{i_2}, \, \dots \, , x_{i_{2^r}})$, have r pairs of opposite $(r-1)$-dimensional faces. Let $I_{i_j 0}^{r-1}$ and $I_{i_j 1}^{r-1}$ be a pair of opposite faces of I_i^r, $j \in \{1, 2, \, \dots \, , 2^r\}$.

It is obvious that $I_{i_j 0}^{r-1}$ and $I_{i_j 1}^{r-1}$ are noncoherent (see Definition 11). Using the *algebraical border* [6] of I_i^r, we can determine which faces of the cube I_i^r are coherent:

$$\square I_i^r = \sum_{j=0}^{r} (-1)^j (I_{i_j 0}^{r-1} - I_{i_j 1}^{r-1}) \tag{4}$$

Definition 12. *The cubic complex* $\mathcal{I}^n$ *is called an* ***abstract cubic n-dimensional manifold*** *if the following proprieties are satisfied:*

A. any $(n-1)$*-dimensional cube is a common face exactly for two n-dimensional cubes from* $\mathcal{I}^n$;

B. for $\forall I_i^n, I_j^n \in \mathcal{I}^n$, $i \neq j$, *it exists a sequence of n-dimensional cubes from* $\mathcal{I}^n, I_{i_1}^n = I_i^n, I_{i_2}^n \, \dots \, , I_{i_q}^n = I_j^n$, *where* $I_r^n \cap I_{r+1}^n = I_{r,r+1}^{n-1}$, $r \in \{i_1, i_2, \, \dots \, , i_{q-1}\}$;

C. for $\forall I^p \in \mathcal{I}^n$, $0 \leq p \leq n-1$, $\exists I^n \in \mathcal{I}^n$, *where* I^p *is a face of* I^n;

D. for disjoint cubes $\forall I_i^n, I_j^n \in \mathcal{I}^n$, $I_i^n \cap I_j^n = I^p$, $2 \leq p < n$, *it exists a sequence of abstract cubes,* $I_{i_1}^n = I_i^n, I_{i_2}^n, \, \dots \, , I_{i_q}^n = I_j^n$, *such that* $\bigcap\limits_{j=1}^{q} I_{i_j}^n = I^p$.

Let's denote the n-dimensional cubic manifold by $\mathcal{V}^n_\square$. From A it results that any n-dimensional face takes part in the building of $\mathcal{V}^n_\square$ just one time.

Is obvious

Theorem 4. *Let's have two coherent cubes $I^r_i, I^r_j \in \mathcal{I}^n$ and their common face I^{r-1}_{ij} of dimension $r-1$. Their incident coefficients have opposite signs, $[I^r_i : I^{r-1}_{ij}] = -[I^r_j : I^{r-1}_{ij}]$.*

Definition 13. *Let $L^n_C \in \mathcal{L}^n$ be an arbitrary $\square$-chain, $L^n_C = \sum\limits_{i=1}^{\beta_n} g_i I^n_i$. The sum*

$$\square L^n_C = \sum_{i=1}^{\beta_n} g_i \square I^n_i,\ g_i \in Z,\ 1 \le i \le \beta_n \tag{5}$$

is called a ***$\square$-border*** *or* ***algebraical border*** *of the $\square$-chain L^n_C.*

Using the formula (5) we will define the *orientability* for an abstract cubic manifold, $\mathcal{V}^n_\square$.

Definition 14. *If there exists a n-dimensional $\square$-chain of the cubic manifold $\mathcal{V}^n_\square$, $\square L^n_C \in \mathcal{L}^n$, such that $\square L^n_C = 0$ (so there exists a n-dimensional $\square$-chain), then the* ***manifold*** *$\mathcal{V}^n_\square$ is called* ***directed****. The directed manifold $V^n_\square$ is called* ***totally coherent*** *if all its n-dimensional cubes have the same orientation.*

Holds

Theorem 5. *An abstract directed cubic manifold $\mathcal{V}^n_\square$ is totally coherent.*

The proof results from the fact that an abstract directed manifold V^n_Δ, built from simplexes, is totally coherent [3] and the cubes are built from procubes (see Definition 5), but the procubes – from abstract simplexes.

Holds

Theorem 6. *The abstract totally coherent directed cubic n-dimensional manifold, where $n = 2m$, has a linear Euler $\square$-contour of dimension $n-1$.*

Proof. Using Theorem 5, we put into correspondence to the cubic manifold $\mathcal{V}^n_\square$ an oriented graph $G = (X, U)$, where $X = \{I^n_1, I^n_2 \ldots, I^n_{\beta_n}\}$ and $U = \{I^{n-1}_1, I^{n-1}_2 \ldots, I^{n-1}_{\beta_{n-1}}\}$. X is the set of all n-dimensional cubes of $\mathcal{V}^n_\square$, which represent vertexes of the graph G, denoted $x_i = I^n_i, 1 \le i \le \beta_n$. U is the set of all $(n-1)$-dimensional cubes of manifold $\mathcal{V}^n_\square$, which represent arcs of the graph G, denoted $u_j = I^{n-1}_j$, $1 \le j \le \beta_{n-1}$. We can fix arcs orientation: $(x_p, x_q) \in U$, if I^n_p and I^n_q are coherent, respecting the indicated order and having only one common face (it follows from Theorem 4 and Definition 11).

Every n-dimensional cube component of the cubic manifold $\mathcal{V}^n_\square$ has pairs of opposite $(n-1)$-dimensional faces, with opposite orientation. Thus, if x_i is the tail of an arc in G, it exists exactly one arc which head is x_i. The valency of every vertex in $G = (X, U)$ is even, and the graph G is pseudosymmetrical [1]. So we are in the conditions of classical theorem of the graph theory, where any conex and pseudosymmetrical graph has a directed Euler tour. Translating this result in the language of used notations, we obtain that the abstract cubic manifold $\mathcal{V}_{\square}^{n}$ has a linear $(n-1)$-dimensional Euler $\square$-contour. Theorem 6 is proved.

Now we can show the conditions for existence of a linear $(n-1)$-dimensional Euler contour for an abstract even dimensional manifold, V_{Δ}^{n}.

Holds

Theorem 7. *Let V_{Δ}^{2m}, $m \geq 1$, be an abstract directed manifold of odd dimension. It has a directed linear $(2m-1)$-dimensional Euler tour if the simplexes of V_{Δ}^{2m} determine an cubic manifold.*

The proof results from Theorems 5 and 6.

The author plan to publish some essential results for existence and applications of the directed linear Euler tours on $\mathcal{V}_{\square}^{n}$ with different dimensions.

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