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An application of Briot-Bouquet differential subordinations

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Abstract. Let $f \in \mathcal{A}$. We consider the following integral operator:

$$F(z) = \frac{2}{z} \int_0^z f(t)dt. \tag{1}$$

By using this integral operator we obtain a Briot-Bouquet differential subordination.

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1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots, \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

A function $f \in \mathcal{H}[a, 1]$ is convex in U if it is univalent and $f(U)$ is convex.

It is well known that f is convex if and only if $f'(0) \neq 0$ and

$$\mathrm{Re}\, \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U.$$

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

In order to prove the new results we shall use the following lemma.

Lemma A [1, Theorem 3.2b, p. 83]. *Let h be a convex function in U, with $h(0) = a$, and let n be a positive integer. Suppose that the Briot-Bouquet differential equation*

$$q(z) + \frac{nzq'(z)}{q(z) + 1} = h(z)$$

has a univalent solution q that satisfies $q(z) \prec h(z)$.

If $p \in \mathcal{H}[a, n]$ satisfies

$$p(z) + \frac{zp'(z)}{p(z)+1} \prec h(z) \tag{2}$$

then $p(z) \prec q(z)$, and q is the best (a, n) dominant of (2).

2 Main results

Lemma B. *The function $h(z) = 1 + Rz + \dfrac{Rz}{2+Rz}$ is convex.*

Proof. We show that the function $h(z) = 1+Rz+\dfrac{Rz}{2+Rz}$ is convex for all $R \in (0, 1]$.

We study the function

$$h(z) = 1 + z + \frac{z}{2+z}, \quad h'(z) = 1 + \frac{2}{(2+z)^2}, \quad h''(z) = -\frac{4}{(2+z)^3}.$$

We calculate

$$\text{Re}\ \left[1 + \frac{zh''(z)}{h'(z)}\right] = \text{Re}\ \left[1 - 4\frac{z}{(2+z)(z^2+4z+6)}\right].$$

We take $z = e^{i\theta}$, $\theta \in [0, 2\pi]$ and we obtain

$$\text{Re}\ \left[1 + \frac{zh''(z)}{h'(z)}\right] = \text{Re}\ \left[1 - 4\frac{e^{i\theta}}{(2+e^{i\theta})(e^{2i\theta}+4e^{i\theta}+6)}\right] =$$

$$= \text{Re}\ \left[1 - 4\frac{\cos\theta + i\sin\theta}{(2+\cos\theta+i\sin\theta)(\cos 2\theta + i\sin 2\theta + 4\cos\theta + 4i\sin\theta + 6)}\right] =$$

$$= 1 - 4\frac{(2\cos\theta+1)(2\cos^2\theta+4\cos\theta+5) + 4\sin^2\theta(\cos\theta+2)}{(4\cos\theta+5)[(2\cos^2\theta+4\cos\theta+5)^2 + 4\sin^2\theta(\cos\theta+2)^2]}.$$

We let $\cos\theta = t$, $t \in [-1, 1]$. Then

$$\text{Re}\ \left[1 + \frac{zh''(z)}{h'(z)}\right] = \frac{96t^3 + 336t^2 + 372t + 153}{96t^3 + 344t^2 + 444t + 205} > 0 \text{ for all } t \in [-1, 1]$$

which shows that h is a convex function for $R = 1$, hence it is convex for any $0 \le R \le 1$.

Remark 1. The equation $32t^3 + 112t^2 + 126t + 51 = 0$ has the root $t = -1,905$.

Theorem. *Let $0 < R \le 1$, $q(z) = 1 + Rz$, with $\text{Re}\ q(z) > 0$ and*

$$h(z) = 1 + Rz + \frac{Rz}{2+Rz} \tag{3}$$

be convex in U.

If $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec h(z) \tag{4}$$

then

$$\frac{zF'(z)}{F(z)} \prec 1 + Rz$$

where F is given by (1).

Proof. From (1) we have

$$zF(z) = 2\int_0^z f(t)dt, \quad z \in U.$$

By using the derivate of this equality, with respect to z, after a short calculation, we obtain $zF'(z) + F(z) = 2f(z)$. This equality is equivalent to

$$F(z)\left[1 + \frac{zF'(z)}{F(z)}\right] = 2f(z). \tag{5}$$

If we let

$$p(z) = \frac{zF'(z)}{F(z)} \tag{6}$$

then (5) becomes

$$F(z)[1 + p(z)] = 2f(z). \tag{7}$$

By using the derivate of (7) with respect to z, after a short calculation, we obtain

$$\frac{zF'(z)}{F(z)} + \frac{zp'(z)}{1+p(z)} = \frac{zf'(z)}{f(z)}$$

which, using (6), is equivalent to

$$p(z) + \frac{zp'(z)}{1+p(z)} = \frac{zf'(z)}{f(z)}.$$

Using (4), we have

$$p(z) + \frac{zp'(z)}{1+p(z)} \prec h(z).$$

According to Lemma B the function h given by (3) is convex and by applying Lemma A we deduce that $p(z) \prec q(z)$, which shows that F satisfies

$$\frac{zF'(z)}{F(z)} \prec 1 + Rz$$

and $q(z) = 1 + Rz$, is the best dominant.

From our theorem we deduce the following result:

Corollary. *Let n be a positive integer, $0 < R \leq 1$, $q(z) = 1 + Rz$, with* $\mathrm{Re}\, q(z) > 0$, *and*

$$h(z) = 1 + Rz + \frac{Rz}{2 + Rz},$$

be convex in U.

If $f \in \mathcal{H}[0, n]$ and

$$\frac{zf'(z)}{f(z)} \prec h(z)$$

then

$$\left|\frac{zF'(z)}{F(z)} - 1\right| < R,$$

where F is given by (1).

Remark. For $R = 1$, $n = 1$, the Corollary was obtained in [2].

References

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