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On identities of Bol-Moufang type *

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Abstract. (Left) Bol loops are usually introduced as loops in which (left) Bol condition is satisfied, and the existence of the two-sided inverse of any element as well as the left inverse property are deduced. It appears that some of the assumptions on the structure are superflous and can be omitted, or modified. Also, Bol loops can be presented in various settings as far as the family of operation symbols is concerned. First we give a short survey on main known results on identities of Bol-Moufang type in quasigroups, written in a unified notation, and try to employ only multiplication and left division for the equational theory of left Bol loops. Then we propose a rather non-traditional concept of the variety of left Bol loops in type $(2, 1, 0)$, with operation symbols $(\cdot, {}^{-1}, e)$ and with five-element defining set of identities, namely $xe = ex = x$, $(x^{-1})^{-1} = x$, $x^{-1}(xy) = y$, $x(y(xz)) = (x(yx))z$.

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1 Preliminaries

The set of all terms over an alphabet X is denoted $T_{(\tau)}(X)$. If $\mathcal{A} = (A; F)$ is an algebra with the carrier set A and the sequence $F = (f_i)_{i\in I}$ of operation symbols, and $\tilde{F} = (f_i)_{i\in\tilde{I}}$, $\tilde{I} \subset I$, a subsequence of the sequence F of operation symbols then $\tilde{\mathcal{A}} = (A; \tilde{F})$ is called a *reduct* of $\mathcal{A}$. $\underline{V}$ denotes a class of algebras defined by identities, i.e. a variety of algebras, and we write $\underline{V} = Mod(\Sigma)$ if Σ is the defining set of identities for $\underline{V}$. We will distinguish graphically between identities in a variety and equalities between elements in a particular algebra.

An algebra with one binary operation (of type $\tau = (1)$) is called a *groupoid* here, [4]. (The terminology varies in this respect. In [11] and in [10, p. 23] *magma* is used, and 1970' edition of N. Bourbaki's *Algebra* is mentioned as the first source. In [19], magma means a groupoid with two-sided neutral element.)

Convention. We use a common convention of nonassociative algebra that the symbol of binary operation can be omitted (to save space and brackets in formulas); if $\cdot$ is used it plays the role of parentheses, i.e. indicates priority of the "non-dotted" multiplication, or other operation.

Given a groupoid $(A; \cdot)$ and $a \in A$ then $L_a : A \to A$, $x \mapsto ax$ denotes the left translation by an element a, similarly $R_a : x \mapsto ax$ denotes the right translation by a. An element e of a groupoid $\mathcal{A} = (\mathcal{A}; \cdot)$ is called a *right* (respectively *left*, respectively

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two-sided) *neutral element* of $\mathcal{A}$ if for all elements $a \in A$, $ae = a$ (respectively $ea = a$, respectively both equalities) hold(s). The following easy observation is useful:

If e' is a left and e a right neutral element of $\mathcal{A}$ they coincide, $e' = e'e = e$.

If "$\circ$" is a binary operation on the carrier set A then so is its *dual* (opposite) operation $\tilde{\circ} : A \to A$, $(a, b) \mapsto a \tilde{\circ} b := b \circ a$. The groupoid $\tilde{\mathcal{A}} = (A, \tilde{\circ})$ is *dual* (opposite)[1] to $\mathcal{A} = (A, \circ)$, and left (right) translations of $\tilde{\mathcal{A}}$ ($\mathcal{A}$). If an identity is satisfied in $(A, \circ)$ then its *dual* ("mirror") is this same identity in $\tilde{\mathcal{A}} = (A, \tilde{\circ})$ (that is, "$\tilde{\circ}$" replaces "$\circ$" in each occurance), rewritten into an identity in $(A, \circ)$ [10, p. 2]. A groupoid $\mathcal{B}$ is *antiisomorphic* to $\mathcal{A}$ if it is isomorphic to $\tilde{\mathcal{A}}$. The advantage of dualisation is obvious: it saves space and time. If something is proven for an algebraic structure (which is not "self-dual"), the dual ("mirror") proof must clearly work for the dual structure, and it is sufficient to study one of both theories.

1.1 Quasigroups, one-sided and two-sided equasigroups. Quasigroups (in the "usual" sense) form a historically important class of groupoids in which groups can be considered as a subclass. A (two-sided) quasigroup is often characterized as a groupoid $\mathcal{A} = (A; \cdot)$ such that the following "quasigroup property" (Q) is satisfied, [16]:

(Q) the maps $L_a : A \to A$ and $R_a : A \to A$ are bijections for all $a \in A$.

Equivalently, for each of the equations $a \cdot x = b$, $y \cdot a = b$ with $a, b \in A$, there exists a uniquely determined solution in A, $x \in A$ or $y \in A$, respectively [4, 14]. Another speaking, for any triple of elements a, b, c from A such that $a \cdot b = c$, each couple of them determines the third one in A uniquely. Such a characterization might be sufficient in many aspects, but makes troubles in the infinite case as far as congruence relations, quotient algebras and homomorphisms are concerned. If a homomorphic image $\mathcal{A}'$ of a quasigroup $\mathcal{A}$ (in the above sense) is finite, or associative, then $\mathcal{A}'$ is a quasigroup. But there are infinite examples of groupoids (even loops) with the above "quasigroup property" (Q) admitting homomorphic maps onto (neither finite nor associative) groupoids in which (Q) fails, i.e. the image is no quasigroup [1, p. 1182–1183]. That is why the so called equasigroups were introduced (see [6]) defined via identities, i.e. forming a variety (see [7]) (primitive quasigroups in the sense of [2]). We prefer to keep the outline of an equational theory here.

Under a *left quasigroup* we will understand an algebra $(A; \cdot, \backslash)$ of type $(2, 2)$ in which the following identities hold (that guarantee existence and unicity of the solution):

$$(Q1_l): \quad x(x\backslash y) \approx y, \qquad (Q2_l): \quad x\backslash(xy) \approx y.$$

Left quasigroups form the variety $\underline{LQ} = Mod(\{(Q1_l), (Q2_l)\})$ in type $(2, 2)$, and an algebra $(A; \cdot, \backslash)$ belongs to $\underline{LQ}$ if and only if in the groupoid $(A; \cdot)$, the equations of the form $a \cdot u = b$ are uniquely solvable in A for any $a, b \in A$, with $u = a\backslash b$.

[1] sometimes also denoted by $\mathcal{A}^{\mathrm{op}}$

Similarly, we can introduce mirrors of $(Q1_l)$, $(Q2_l)$

$$(Q1_r): \quad (y/x)x \approx y, \qquad (Q2_r): \quad (yx)/x \approx y,$$

the variety of right quasigroups in type $(2,2)$ with operation symbols $(\cdot, /)$, $\underline{RQ} = Mod(\{(Q1_r), (Q2_r)\})$, and to give a dual characterization. Each left (right) quasigroup is left (right) cancellative, that is satisfies the following quasi-identity:

$(C_l) \qquad xz = xz' \Longrightarrow z = z' \qquad (left\ cancellation)$

respectively

$(C_r) \qquad zx = z'x \Longrightarrow z = z' \qquad (right\ cancellation).$

In fact, if $\mathcal{Q} = (Q; \cdot, \backslash) \in \underline{LQ}$ and $ab = ab'$ for $a, b, b' \in Q$ then $a\backslash ab = a\backslash ab'$. Applying $(Q2_l)$ we obtain $b = b'$ (and the dual proof works for right quasigroups).

We prefer "left" structures here.

If in a left quasigroup $\mathcal{Q} \in \underline{LQ}$, $q\backslash q = p\backslash p$ holds for any $p, q \in Q$, then the common value $e = q\backslash q$ is a right neutral element of $\mathcal{Q}$.

A *quasigroup* ("equasigroup") is an algebra $\mathcal{A} = (\mathcal{A}; \cdot, \backslash, /)$ of type $(2,2,2)$ satisfying all of $(Q1_l)$, $(Q2_l)$, $(Q1_r)$, $(Q2_r)$ (and $\mathcal{A} \in \underline{Q}$ iff equations of both types are solved uniquely in A), [7, 24], i.e. the reduct $(A; \cdot, \backslash)$ is a left quasigroup, and the reduct $(A; \cdot, /)$ is a right one). Obviously, quasigroups are left and right cancellative (equivalently, all left and right translations are bijections on A).

In the variety of quasigroups $\underline{Q} = Mod(\{(Q1_l), (Q2_l), (Q1_r), (Q2_r)\})$, the following identities are consequences of the defining ones:

$$(Q3): \qquad y/(x\backslash y) \approx x, \qquad\qquad (Q4): \qquad (y/x)\backslash y \approx x.$$

If a quasigroup possesses a neutral element we speak about a *loop*. If this is the case, we can identify e with a new nullary operation $e : \{\emptyset\} \to A$, $\emptyset \mapsto e$. Let us consider the following identities:

$$(U_r): \qquad xe \approx x, \qquad\qquad (U_l): \qquad ex \approx x,$$

$(AS) \qquad x(yz) \approx (xy)z \qquad$ (associativity),

$(CO) \qquad xy \approx yx \qquad$ (commutativity).

Associative groupoids are called *semigroups*. Commutative associative quasigroups are also called *abelian*.

1.2 Groups as quasigroups. It is well known that a *group* can be regarded as a quasigroup $\mathcal{G} = (\mathcal{G}; \cdot, \backslash, /)$ for which the reduct $(G; \cdot)$ is a semigroup (satisfies (AS)), [14, Chap. II], [4, p. 28], and that in the variety of "groups" in type $(2,2,2)$, $\underline{G'} = Mod(\{(Q1_l), (Q2_l), (Q1_r), (Q2_r), (AS)\})$, the following identities hold:

$$x/x \approx y/y, \qquad x\backslash x \approx y\backslash y, \qquad x/x \approx x\backslash x,$$

i.e. a uniquely determined neutral element is present in any $\mathcal{G} \in \underline{\mathcal{G}'}$, and $\mathcal{G}$ can be regarded as an associative loop.

1.3 Loops. A *loop* is often considered as "usual" quasigroup $(Q, \cdot)$, i.e. satisfying (Q), endowed with an identity element. For our purpose, let us consider a *variety of loops* $\underline{L}$ of type $(2,2,2,0)$ with the sequence of operation symbols $(\cdot, \backslash, /, e)$ as

$$\underline{L} = Mod(\{(Q1_l), (Q2_l), (Q1_r), (Q2_r), (U_r), (U_l)\});$$

now a *loop* $\mathcal{L} = (\mathcal{Q}; \cdot, \backslash, /, \rceil)$ will be an algebra belonging to the variety $\underline{L}$. Obviously, a reduct $(Q; \cdot, \backslash, /)$ of $\mathcal{L}$ is in $\underline{Q}$. Denote by $\underline{Gr}$ the subvariety of $\underline{L}$ determined by the additional identity (AS).

It is also reasonable to consider *left loops* with two-sided neutral element in type $(2,2,0)$ with operation symbols $(\cdot, \backslash, e)$ as elements from the variety

$$\underline{LL} = Mod(\{(Q1_l), (Q2_l), (U_r), (U_l)\})$$

and similarly *right loops* with two-sided neutral element and operation symbols $(\cdot, /, e)$ as elements from

$$\underline{RL} = Mod(\{(Q1_r), (Q2_r), (U_r), (U_l)\}).$$

Associativity is rather a strong property. Many kinds of "weak" associativity are studied in quasigroups, e.g. [8, 9, 12, 13, 17, 21, 22], as well as in the varieties $\underline{L}$, $\underline{LL}$ or $\underline{RL}$, respectively.

2 Identities of Bol-Moufang type in quasigroups

2.1 Identities of Bol-Moufang type. An identity $s \approx t$ where $s, t \in T_{(2)}(X)$ are binary terms, is said to be *of Bol-Moufang type* if the number of distinct variables occuring in s as well as in t is three, the total number of variables appearing in s is four, the same for t, and the order in which the variables appear in s is exactly the same as the order of these variables in the term t [17]. Historically, some of these identities have been discovered in connection with geometric closure conditions in webs.

2.2 Left Bol quasigroups. A (two-sided) quasigroup satisfying the so-called left Bol identity

(B_l) $\quad x(y(xz)) \approx (x(yx))z$

is called a *left Bol quasigroup* (after Geritt Bol [3]). Similarly, *right Bol quasigroups* satisfy its dual, the so called right Bol identity

(B_r) $\quad ((zx)y)x \approx z((xy)x)$.

Both theories are "mirror-symmetric" to each other, we prefer here the variety of left Bol quasigroups in type $(2,2,2)$

$$\underline{LBQ} = Mod(\{(Q1_l), (Q2_l), (Q1_r), (Q2_r), (B_l)\}).$$

Note that the mirror variety $\underline{RBQ}$ of right Bol quasigroups was investigated in [22].

Lemma 2.1 [22]. *Let $(A;\cdot,\backslash,/)$ be a left (right) Bol quasigroup. Then $(A;\cdot)$ has a unique right (respectively left) neutral element satisfying (U_r) (respectively (U_l)).*

Proof. Let $a \in A$ be a fixed element of a left Bol quasigroup. For any $b \in A$, $b(a\backslash a) \underset{(Q1_l)}{=} (a \cdot (a\backslash b)) \cdot (a\backslash a) \underset{(Q1_r)}{=} (a \cdot [((a\backslash b)/a) \cdot a]) \cdot (a\backslash a) \underset{(B_l)}{=} a((a\backslash b)/a) \cdot (a \cdot (a\backslash a))) \underset{(Q1_l)}{=} a(((a\backslash b)/a) \cdot a) \underset{(Q1_r)}{=} a(a\backslash b) \underset{(Q1_l)}{=} b$. So, indeed, $a\backslash a$ is right neutral, and (U_r) holds. Unicity follows from $b(a\backslash a) = b = b(c\backslash c)$, $c \in A$ by (C_l). Similarly for right Bol quasigroups. □

In any algebra $\mathcal{A} \in \underline{\mathcal{LBQ}}$, a new nullary operation $e^{\mathcal{A}}$ satisfying (U_r) can be introduced by $e^{\mathcal{A}} : \{\emptyset\} \to A$, $e^{\mathcal{A}}(\emptyset) = e \in A$, and a Bol quasigroup $\mathcal{A}$ can be regarded as a reduct of the algebra $\mathcal{A}' = (A;\cdot,\backslash,/,e)$ of type $(2,2,2,0)$ from the variety $\underline{LBQ'} = Mod(\{(Q1_l),(Q2_l),(Q1_r),(Q2_r),(U_r),(B_l)\})$ (of left Bol quasigroups with right neutral element). Consequently, in the variety $\underline{LBQ}$ (as well as in $\underline{LBQ'}$), the identity $x\backslash x \approx y\backslash y$ holds. (Analogously for the varieties of right Bol quasigroups $\underline{RBQ}$ and $\underline{RBQ'}$.)

Corollary 2.2. *Each Bol quasigroup (belonging to $\underline{LBQ}$, $\underline{LBQ'}$, $\underline{RBQ}$, or to $\underline{RBQ'}$) satisfies*

(H) $\quad (xx)x \approx x(xx)$ $\quad$ *(monoassociativity).*

Proof. In the left Bol case, we use (B_l) with $y = x$, $z = e$, and (U_r). □

If the right neutral element of a left Bol quasigroup is two-sided, i.e. satisfies $xe = ex = x$ for all x, the quasigroup is left alternative: setting $y = e$ in (B_l) we get $x(xz) = (xx)z$.

Note that a groupoid $\mathcal{A}$ is said to have the *left inverse property*, or is a LIP-groupoid, if for each $a \in A$ there is at least one $a' \in A$ such that $a'(ac) = c$ for every $c \in A$ [4, p. 111]. The right case is mirror again.

In a Bol quasigroup, each element has a (unique) two-sided inverse, and left (right) Bol quasigroups have left (right) inverse property:

Lemma 2.3 [22]. *In $\underline{LBQ'}$ the following identities hold:*

lip $\quad (e/x)\cdot(xy) \approx y$ $\quad$ *(left inverse property),*
lip''' $\quad x\cdot((x\backslash e)y) \approx y,$
bi $\quad e/x \approx x\backslash e$ $\quad$ *(two-sided inverse),*
lip'' $\quad (x\backslash e)\cdot(xy) \approx y,$
lip' $\quad x\cdot((e/x)y) \approx y,$

Proof. Let us evaluate $x((e/x)\cdot(xy)) \underset{(B_l)}{\approx} [x((e/x)\cdot x)]\,y \underset{(Q1_r)}{\approx} (xe)y \underset{(U_r)}{\approx} xy$. By left cancellation we obtain (lip). Similarly, $(x\backslash e)\cdot(x\cdot((x\backslash e)y)) \underset{(B_l)}{\approx} [(x\backslash e)(x\cdot(x\backslash e))]\,y \underset{(Q1_l)}{\approx} ((x\backslash e)\cdot e)y \underset{(U_r)}{\approx} (x\backslash e)y$, and (C_l) gives (lip'''). Let $\mathcal{A} \in \underline{\mathcal{LBQ}'}$ have the right neutral element e. For $a \in A$, $((e/a)(a\cdot e/a))a = (e/a)\cdot(a(e/a\cdot a)) = e/a\cdot(ae) = e/a\cdot a$. According to (C_r), $(e/a)\cdot(a(e/a)) = e/a = (e/a)e$, and we obtain $a(e/a) = e$ by (C_l). Since also $a(a\backslash e) = e$ holds it must be $e/a = a\backslash e$ again by (C_l). The rest is a consequence. $\square$

Hence for every (left) Bol quasigroup $\mathcal{B} \in \underline{\mathcal{LBQ}'}$, it is natural to introduce for any $b \in B$ an element $b^{-1} =: e/b = b\backslash e$, a both-sided inverse. In this way, a new unary operation $b \mapsto b^{-1}$ of "inverting" arises (on the carrier set B) satisfying $b^{-1}(bc) = c$ and $b(b^{-1}c) = c$ for all $b, c \in B$ ($\mathcal{B}$ has the left inverse property, is a LIP-quasigroup). Analogously, the same construction works for right Bol quasigroups, and $(cb)b^{-1} = (cb^{-1})b = c$ holds for all $b, c \in B$ (right Bol quasigroups have the right inverse property).

2.3 Moufang quasigroups (are loops). A bit stronger "weak associativity" conditions are conditions of Moufang type (after Ruth Moufang). Consider the following pairs of identities:

$$(M1): \quad (xy)(zx) \approx (x(yz))x, \qquad (N1): \quad ((zx)y)x \approx z(x(yx)),$$

$$(M2): \quad x((yz)x) \approx (xy)(zx), \qquad (N2): \quad ((xy)x)z \approx x(y(xz)).$$

Each of the identities is a mirror of the other one on the same row. By results of Bol and Bruck [3], [4, p. 115] all four identities are equivalent in the variety of loops. By results of [12], they are in fact equivalent even in the variety $\underline{Q}$ of quasigroups.

Call a quasigroup *Moufang* if it satisfies $(M1)$, and denote by $\underline{MQ}$ the variety of all Moufang quasigroups. If $\mathcal{Q}$ is a Moufang quasigroup let us choose a fixed $a \in Q$. Then $e =: a\backslash a$ is a left neutral element. In fact, for every $b \in Q$, $(ba)b = (b(a \cdot a\backslash a))b = (ba)(a\backslash a \cdot b)$, and $a\backslash a \cdot b = b$ follows by left cancellation. Similarly, $f =: (a\backslash a)/(a\backslash a)$ is a right neutral element since $[b((a\backslash a)/(a\backslash a)] \cdot (a\backslash a) = (a\backslash a)[[b((a\backslash a)/(a\backslash a)] \cdot (a\backslash a)] = [(a\backslash a)b][(a\backslash a)/(a\backslash a) \cdot (a\backslash a)] = b \cdot (a\backslash a)$. Now using (C_r) we conclude

Lemma 2.4 [12]. *Every Moufang quasigroup has an identity element e (and therefore can be regarded as a reduct of a Moufang loop).*

For quasigroups satisfying $(M2)$, a mirror proof of the same statement can be easily given. To prove that every quasigroup satisfying $(N1)$ (or $(N2)$, respectively) has a two-sided neutral element is rather more complicated, [12, p. 233].

Lemma 2.5. *Moufang quasigroups are left and right alternative and elastic, that is satisfy*

(ALT_l)	$(xx)y \approx x(xy)$	*(left alternativity),*
(ALT_r)	$(yx)x \approx y(xx)$	*(right alternativity),*
$(FLEX)$	$(xy)x \approx x(yx)$	*(flexibility, elasticity).*

Proof. From $(N2)$ with $y = e$ and (U_r), $(xx)z = x(xz)$. From $(N1)$ with $y = e$, (U_l) and (U_r), $(zx)x = z(xx)$. From $(N1)$ with $z = e$ and (U_l) (or from $(N2)$ with $z = e$ and (U_r)), $(xy)x = x(yx)$. $\square$

Note that if a groupoid satisfies a left Bol identity and possesses a two-sided neutral element then it is left alternative.

Every Moufang quasigroup is at the same time left and right Bol quasigroup:

Lemma 2.6. *The variety $\underline{MQ}$ is a subvariety in $\underline{LBQ}$ as well as in $\underline{RBQ}$.*

Proof. In $\underline{MQ}$, the identity (B_l) is a consequence of $(N2)$ and $(FLEX)$ since

$$(x(yx))z \underset{(FLEX)}{\approx} ((xy)x)z \underset{(N2)}{\approx} x(y(xz)),$$

and (B_r) is a consequence of (N_1) and $(FLEX)$,

$$z((xy)x) \underset{(FLEX)}{\approx} z(x(yx)) \underset{(N1)}{\approx} ((zx)y)x. \qquad \square$$

Hence in every Moufang quasigroup each element has a (unique) both-sided inverse, and both left and right inverse properties are satisfied.

Due to (AS), $\underline{G'}$ is a subvariety in $\underline{MQ}$. Now it is apparent that in each $\mathcal{G} \in \underline{G'}$ there is a (unique) identity element e satisfying (U_r), (U_l), and each $g \in G$ has a (unique) both-sided inverse $g^{-1} := e/g = g\backslash e$. More often, we take $(\cdot, {}^{-1}, e)$ as fundamental operations for groups.

3 Bol and Moufang loops

3.1 Left Bol loops. Left Bol loops are usually considered as a subvariety

$$\underline{LBL} = Mod(\{(Q1)_l, (Q2)_l, (Q1)_r, (Q2)_r, (U_r), (U_l), (B_l)\})$$

determined in $\underline{L}$ by the identity (B_l) (and belonging also to Bol quasigroups with right unit). Similarly, the subvariety $\underline{RBL}$ of right Bol loops is distinguished by the additional condition (B_r), and has mirror properties [20].

Lemma 3.1 [20]. *In the variety $\underline{LBL}$ of (left) Bol loops the following identities hold: (H), (lip), (lip'), (bi), (ALT_l),*

(inv)	$e/(e/x) \approx x, \qquad (x\backslash e)\backslash e \approx x,$	
(sa)	$(x(yx))\backslash e \approx (x\backslash e)\cdot(y\backslash e \cdot x\backslash e)$	*(semiautomorphic inverse).*

Proof. The first part was already proven. Left alternativity is an immediate consequence of (B_l) and (U_l) if we set $y = e$ and $z = y$, $x(xy) \approx x(e(xy)) \approx (x(ex))y \approx (xx)y$ (and monoassociativity follows for $y = x$). From (bi), $e/(e/x) \approx e/(x\backslash e) \underset{(Q3)}{\approx} x$. Now $(x(yx)) \cdot [(x\backslash e) \cdot (y\backslash e \cdot x\backslash e)] \underset{(Q3)}{\approx} x(y(x(x\backslash e(y\backslash e \cdot x\backslash e))) \underset{(lip')}{\approx} x(y(y\backslash e \cdot x\backslash e)) \underset{(lip')}{\approx} x(x\backslash e) \underset{(Q1_l)}{\approx} e$ which is equivalent to (sa). $\square$

In the more usual notation, $x^{-1}(xy) = y$, $x(x^{-1}y) = y$, $(x^{-1})^{-1} = x$, and $(x(yx))^{-1} = x^{-1}(y^{-1}x^{-1})$. Further, $x^{-1}x = xx^{-1} = e$ due to $(Q1_l)$, $(Q1_r)$.

3.2 Moufang loops. The variety of Moufang loops $\underline{ML}$ is introduced as the subvariety of $\underline{L}$ satisfying the identity of Moufang $(N2)$ (or equivalently, any one of the identities $(M1)$, $(M2)$, $(N1)$ [4, 12]), and $\underline{ML}$ is exactly a common part of $\underline{LBL}$

and $\underline{RBL}$. It can be easily checked that a left Bol loop is Moufang if and only if it is elastic, or equivalently, if and only if is right alternative:

Lemma 3.2 [2, p. 105, 1]. *In $\underline{L}$, a pair of identities (B_l), $(FLEX)$ is equivalent to $(N2)$, and a pair of identities (B_r), $(FLEX)$ is equivalent to $(N1)$.*

Lemma 3.3 [2, p. 105, 5]. *In the variety $\underline{L}$ of loops, a couple of identities (B_l), (ALT_r) is equivalent to $(N2)$, and a couple (B_r), (ALT_l) of identities is equivalent to $(N1)$.*

Proof. Let $\mathcal{B}$ be a Bol loop satisfying (ALT_r). Let us show that $\mathcal{B}$ is also elastic. Let $a, c \in B$. Then (B_l) with $x = z =: a$ and $y =: c$ and (ALT_r) give $(a(ca))a = a(c(aa)) = a((ca)a)$. Now let a, $b \in B$ be arbitrary elements. Then $(ab)a = (a((b/a) \cdot a))a = a(((b/a) \cdot a)a) = a(ba)$, that is, $(FLEX)$ holds in $\mathcal{B}$. Hence $\mathcal{B}$ is Moufang. The converse is obvious. $\square$

Corollary 3.4. *A left (right) Bol loop is a Moufang loop if and only if it is right (left) alternative.*

Lemma 3.5. *For Bol loops (particularly for Moufang loops), $(a^{-1})^2 = (a^2)^{-1}$.*

Proof. In a left Bol loop, $(a^{-1}a^{-1})(aa) \underset{(ALT)_l}{=} a^{-1}(a^{-1}(aa)) \underset{(lip)}{=} a^{-1}a \approx e$. In the case of a right Bol loop, a mirror proof works. $\square$

The common value can be denoted by $a^{-2} := (a^{-1})^2 = (a^2)^{-1}$. Similarly, $(a^n)^{-1} = (a^{-1})^n$ for any natural number n.

More generally, in a left Bol loop $\mathcal{B}$, a power a^n can be introduced for any element $a \in B$ and any integer: $a^0 = e$, $a^n = a \cdot a^{n-1}$, $a^{-n} = (a^{-1})^n$ for any natural $n \in \mathbb{N}$. In a left Bol loop, $b^n(b^m a) = b^{n+m}a$, in particular, $b^n b^m = b^{n+m}$ (Bol and Moufang loops are power-associative) [20].

The variety $\underline{CML}$ of commutative Moufang loops can be characterized as a subvariety of $\underline{L}$, which is characterized by the additional identity $x^2(yz) \approx (xy)(xz)$, a modification of $(M1)$.

The fact that most of the varieties of loops of Bol-Moufang type can be defined in several equivalent ways was a motivation for [17–19].

3.3 Left Bol left loops. We can investigate Bol conditions even in weaker structures, and await reasonable results.

Let us start with a left loop (with a two-sided neutral element e) which satisfies left Bol identity (B_l), and can be called a *left Bol left loop*. More formally, let $\mathcal{Q} = (Q; \cdot, \backslash, e)$ belong to

$$\underline{LBLL} = Mod(\{(Q1_l), (Q2_l), (U_l), (U_r), (B_l)\}).$$

Our aim is to show that on Q, a suitable binary operation $/$ can be introduced in such a way that our algebra $\mathcal{Q}$ is in fact a reduct of some left Bol loop from $\underline{LBL}$. Note that both concepts are different from the theoretical view-point of universal algebra, but as far as more practical purposes are concerned: their classes of examples are in a bijective correspondence, that is, essentially the same.

Lemma 3.6. *In any $\mathcal{Q} \in \underline{LBLL}$, the following identities are satisfied:*

(lip'): $x(x\backslash e \cdot y) \approx y$,
(lip''): $x\backslash e(xy) \approx y$,
$(*)$: $x\backslash e \cdot x \approx e$.

Proof. Let us start from the chain of identities

$$x\backslash e \cdot (x(x\backslash e \cdot y)) \underset{(B_l)}{\approx} [x\backslash e \cdot (x \cdot x\backslash e)]y \underset{(Q1_l),(U_r)}{\approx} x\backslash e \cdot y.$$

Due to left cancellation, (lip') is obtained. To prove (lip'') we proceed similarly, $x(x\backslash e(xy)) \underset{(B_l)}{\approx} x(x\backslash e \cdot x))y \underset{(lip')}{\approx} xy$, and we use (C_l) again. Setting $y = e$ in (lip') we get the rest. □

Given $\mathcal{Q} \in \underline{\mathcal{LBLL}}$ let us introduce a new binary operation "\" on Q by

$$a/b := b\backslash e(ba \cdot b\backslash e) \quad \text{for any} \quad a, b \in Q.$$

Let us check that $(Q1_r)$ and $(Q2_r)$ are satisfied:

$$xy/y = y\backslash e(y(xy)\cdot)y\backslash e \underset{(B_l)}{=} y\backslash e(y(x(y\cdot y\backslash e))) \underset{(Q1_l)}{=} y\backslash e(y(x\cdot e)) \underset{(U_r)}{=} y\backslash e(yx)x \underset{(lip')}{=} x,$$

$$x/y \cdot y = [y\backslash e(yx \cdot y\backslash e)]y \underset{(B_l)}{=} y\backslash e(yx \cdot (y\backslash e \cdot y)) \underset{(*)}{=} y\backslash e(yx \cdot e) \underset{(U_r)}{=} y\backslash e(yx) \underset{(lip')}{=} x.$$

4 Bol loops in signature $(\cdot, ^{-1}, e)$

As we have already seen, varieties of loops of Bol-Moufang type (including groups) can be introduced in various types. The fact that they belong to the class of the so called IP-loops is of considerable importance.

(Left) Bol loops are frequently introduced as loops in which (left) Bol condition is satisfied, and consequences of the axioms (some of which might be redundant) are derived: particularly the existence of two-sided inverse elements and the left inverse property, (LIP), are remarkable. We propose here a rather non-traditional way how to minimalize the axiomatic system.

Let us introduce a variety of left Bol loops $\underline{B}$ in type $(2, 1, 0)$ with operation symbols $(\cdot, ^{-1}, e)$. Considering the fact that Bol loops belong to the class of left inverse property loops we can choose

$$xe \approx x, \quad ex \approx x, \quad (x^{-1})^{-1} \approx x, \; x^{-1}(xy) \approx y, \quad x(y(xz)) \approx (x(yx))z$$

as a basis of identities. For some purposes, this definition in type $(2, 1, 0)$ seems to be quite convenient, and the fact that groups (in the usual setting) form a subvariety in $\underline{B}$ is transparent enough.

Let K denote a class of algebras in type $(2, 1, 0)$, with a sequence of operation symbols $(\cdot, ^{-1}, e)$. Let us consider the following set of identities:

$(IN_r): \quad x \cdot x^{-1} \approx e, \qquad (IN_l): \quad x^{-1} \cdot x \approx e,$
$(LIP): \quad x^{-1}(xy) \approx y, \qquad (LIP)': \quad x(x^{-1}y) \approx y,$
$(INV): \quad (x^{-1})^{-1} \approx x, \qquad (Lu): \quad (x^{-1}((xx)x^{-1}) \cdot x \approx x.$

The following can be easily checked.

Lemma 4.1. *In an algebra $\mathcal{B} = (Q; \cdot, ^{-1}, e) \in K$ the following implications are satisfied:*

(i) (U_r) *and* (LIP) *imply* (IN_l),
(ii) (U_r), (INV) *and* (LIP) *imply* (IN_r),
(iii) *if* (U_r), (INV) *and* (LIP) *are satisfied then also* $(LIP)'$ *holds,*
(iv) *if* (U_r) *and* $(N2)$ *hold then* $(FLEX)$ *is also satisfied,*
(v) (U_r), (INV), (LIP) *and* $(N2)$ *imply* (U_l),
(vi) (U_r) *and* $(N2)$ *imply* (B_l),
(vii) *if* (U_r), (INV), (LIP) *and* $(N2)$ *hold then* (ALT_l) *is satisfied.*

Proof. Under the respective assumptions, we get the following chains of identities:

$$x^{-1}x \underset{(U_r)}{\approx} x^{-1}(xe) \underset{(LIP)}{\approx} e,$$

$$xx^{-1} \underset{(INV)}{\approx} (x^{-1})^{-1}x^{-1} \underset{(U_r)}{\approx} (x^{-1})^{-1}(x^{-1}e) \underset{(LIP)}{\approx} e,$$

$$x(x^{-1}y) \underset{(INV)}{\approx} (x^{-1})^{-1}(x^{-1}y)x \underset{(LIP)}{\approx} e,$$

$$(x^{-1}((xx)x^{-1})) \cdot x \underset{(B_l)}{\approx} x^{-1} \cdot [(xx)(x^{-1}x)] \underset{(IN_l)}{\approx} x^{-1} \cdot [(xx)e] \underset{(U_r)}{\approx} x^{-1} \cdot (xx) \underset{(LIP)}{\approx} x,$$

$$x(yx) \underset{(U_r)}{\approx} x(y(xe)) \underset{(N2)}{\approx} (x(yx))e \underset{(U_r)}{\approx} (xy)x,$$

$$ex \underset{(IN_r)}{\approx} (xx^{-1})x \underset{(FLEX)}{\approx} x(x^{-1}x) \underset{(IN_l)}{\approx} xe \underset{(U_r)}{\approx} x,$$

$$x(y(xz)) \underset{(N2)}{\approx} ((xy)x)z \underset{(FLEX)}{\approx} (x(yx))z,$$

$$x(xz) \underset{(U_l)}{\approx} x(e(xz)) \underset{(N2)}{\approx} ((xe)x)z \underset{(U_r)}{\approx} (xx)z. \qquad \square$$

The condition (Lu) tells that the "left neutral" element e_b^l of $b \in B$, determined by the equation $e_b^l \cdot b = b$, is of the form $b^{-1}[(bb)b^{-1}]$. In general, left neutral elements e_a^l, e_b^l corresponding to different elements $a \neq b$ might be different.

Lemma 4.2. *Let $\mathcal{B} \in K$ be an algebra satisfying (LIP). Then $\mathcal{B}$ satisfies the left cancellation law (C_l).*

Proof. Let a, c, $c' \in B$ and suppose $ac = ac'$. Then we evaluate $c \underset{(LIP)}{=} a^{-1}(ac) = a^{-1}(ac') \underset{(LIP)}{=} c'$. $\square$

Lemma 4.3. *Let $\mathcal{B} \in K$ satisfy (U_r), (IN_r), (C_l) and (B_l). Then $\mathcal{B}$ satisfies also the right cancellation law (C_r).*

Proof. Let the assumptions hold in $\mathcal{B}$. Suppose $ca = c'a$. Then also $a(ca) = a(c'a)$. Calculate

$$ac \underset{(U_r)}{=} a(ce) \underset{(IN_r)}{=} a(c(aa^{-1})) \underset{(B_l)}{=} (a(ca)) \cdot a^{-1} =$$

$$= (a(c'a)) \cdot a^{-1} \underset{(B_l)}{\approx} a(c'(aa^{-1})) \underset{(IN_r)}{\approx} a(c'e) = ac'.$$

Hence $ac = ac'$, and by (C_l), also $c = c'$. □

In the class K, distinguish the variety

$$\underline{B} = Mod(\{(U_r), (U_l), (INV), (LIP), (B_l)\}),$$

and call its algebras again left Bol loops.

Corollary 4.4. *Algebras from $\underline{B}$ satisfy (IN_r), (IN_l), and are both left and right cancellative.*

Let us consider also the varieties

$$\underline{M} = Mod(\{(U_r), (INV), (LIP), (N2)\}), \quad \underline{G} = Mod(\{(U_r), (IN_r), (AS).$$

In $\underline{G}$, the identities (U_l), (IN_l), (IN_r), (INV) are satisfied (now we have "usual" groups). Obviously, we obtain the chain of subvarieties $\underline{G} \subset \underline{M} \subset \underline{B}$.

Proposition 4.5. *Given an algebra $\mathcal{B} = (Q; \cdot, ^{-1}, e) \in \underline{B}$ of type $(2,1,0)$ let us introduce a couple of binary operations $\backslash$, $/$ by $a\backslash b := a^{-1}b$, $b/a := a^{-1}((ab)a^{-1})$, $a, b \in Q$ [15]. Then $\mathcal{B}' = (Q; \cdot, \backslash, /, e)$ is a Bol loop belonging to the variety $\underline{BL}$.*

Proof. Let us verify that in $\mathcal{B}'$, $(Q1_l)$, $(Q2_l)$, $(Q1_r)$ and $(Q2_r)$ hold. Given $a, b \in Q$ let us evaluate

$$a(a\backslash b) = a(a^{-1}b) \underset{(LIP)'}{=} b, \qquad a\backslash(ab) = a^{-1}(ab) \underset{(LIP)}{=} b,$$

$$(b/a)a = (a^{-1}((ab)a^{-1})a \underset{(B_l)}{=} a^{-1}((ab)(a^{-1}a)) \underset{(IN_l)}{=} a^{-1}((ab)e) \underset{(U_r)}{=} a^{-1}(ab) \underset{(LIP)}{=} b,$$

$$(ba)/a = a^{-1}((a(ba))a^{-1}) \underset{(B_l)}{=} a^{-1}(a(b(aa^{-1}))) \underset{(LIP)}{=} b(aa^{-1}) \underset{(IN_r)}{=} be \underset{(U_r)}{=} b.$$

Since (U_r), (U_l) and (B_l) are among the defining identities of $\underline{B}$ the rest follows. □

The varieties $\underline{B}$ and $\underline{BL}$ are term equivalent, the same for the varieties of groups $\underline{G}$ and $\underline{Gr}$, or for Moufang loops.

For any $\mathcal{B} \in \underline{B}$ the map $J : Q \to Q$, $x \mapsto x^{-1}$ is an involutive permutation of the underlying set Q, and moreover a semiautomorphism of the loop $\mathcal{B}$ ([20, p. 344]), that is, $J(x(yx)) = J(x)(J(y)J(x))$ holds.

Lemma 4.6. *In the variety $\underline{B}$ the following identities are satisfied:*

$$(SA)' : \quad [x(y^{-1}x)]^{-1} \approx x^{-1}(yx^{-1}), \qquad (SA) : \quad [x(yx)]^{-1} \approx x^{-1}(y^{-1}x^{-1}).$$

Proof. The first assertion $(SA)'$ follows by right cancellation (C_r) from

$$x^{-1}(yx^{-1})\cdot x(y^{-1}x) \underset{(B_l)}{\approx} x^{-1}(y(x^{-1}[x(y^{-1}x)])) \underset{(LIP)}{\approx} x^{-1}(y(y^{-1}x)) \underset{(LIP)'}{\approx}$$

$$\underset{(LIP)'}{\approx} x^{-1}x \underset{(IN_l)}{\approx} \approx [x(y^{-1}x)]^{-1}\cdot x(y^{-1}x),$$

and the second is a consequence. □

Lemma 4.7. *In the variety $\underline{B}$, the following identities hold for $n \geq 2$:*

$$\text{(i)} \qquad x_n(x_{n-1}(\dots(x_3(x_2(x_1(x_2(x_3(\dots(x_{n-1}(x_nz))\dots)\approx$$

$$\approx x_n(x_{n-1}(\dots[x_3([x_2(x_1x_2)]x_3)]\dots)x_{n-1}))x_n)\cdot z,$$

$$\text{(ii)} \qquad x_n(x_{n-1}(\dots(x_2(x_1(x_2(\dots(x_{n-1}x_n)\dots)\approx$$

$$\approx x_n((x_{n-1}(\dots[\dots[x_2(x_1x_2)]\dots]x_{n-1})x_n),$$

$$\text{(iii)} \qquad [x_n(x_{n-1}(\dots(x_2(x_1(x_2(\dots(x_{n-1}x_n)\dots)]^{-1}\approx$$

$$\approx x_n^{-1}(x_{n-1}^{-1}(\dots(x_2^{-1}(x_1^{-1}(x_2^{-1}(\dots(x_{n-1}^{-1}x_n^{-1})\dots).$$

Proof. We use the idenity (B_l) $(n-1)$-times to prove the first identity,

$$[x_n((x_{n-1}([\dots[x_3([x_2(x_1x_2)]x_3)]\dots]x_{n-1}))x_n)]\cdot z \underset{(B_l)}{\approx}$$

$$\underset{(B_l)}{\approx} x_n((x_{n-1}([\dots[x_3([x_2(x_1x_2)]x_3)]\dots]x_{n-1}))(x_nz)) \underset{(B_l)}{\approx}\dots$$

$$\dots\underset{(B_l)}{\approx} x_n(x_{n-1}(\dots(x_3(x_2(x_1(x_2(x_3(\dots(x_{n-1}(x_nz))\dots).$$

For $z=e$, the second identity is obtained, and (iii) is a consequence of (SA). □

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