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Loops of Bol-Moufang type with a subgroup of index two

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Abstract. We describe all constructions for loops of Bol-Moufang type analogous to the Chein construction $M(G,*,g_0)$ for Moufang loops.

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1 Introduction

Due to the specialized nature of this paper we assume that the reader is already familiar with the theory of quasigroups and loops. We therefore omit basic definitions and results (see [1],[6]).

In a sense, a nonassociative loop is closest to a group when it contains a subgroup of index two. Such loops proved useful in the study of Moufang loops, and it is our opinion that they will also prove useful in the study of other varieties of loops.

Here is the well-known construction of Moufang loops with a subgroup of index two:

Theorem 1.1 (Chein [3])**.** *Let G be a group, $g_0 \in Z(G)$, and $*$ an involutory antiautomorphism of G such that $g_0^* = g_0$, $gg^* \in Z(G)$ for every $g \in G$. For an indeterminate u, define multiplication $\circ$ on $G \cup Gu$ by*

$$g \circ h = gh, \quad g \circ (hu) = (hg)u, \quad gu \circ h = (gh^*)u, \quad gu \circ hu = g_0 h^* g, \tag{1}$$

where g, $h \in G$. Then $L = (G \cup Gu, \circ)$ is a Moufang loop. Moreover, L is associative if and only if G is commutative.

It has been shown in [9] that (1) is the only construction of its kind for Moufang loops. (This statement will be clarified later.) In [10], all constructions similar to (1) were determined for Bol loops.

The purpose of this paper is to give a complete list of all constructions similar to (1) for all loops of Bol-Moufang type. A groupoid identity is of *Bol-Moufang type* if it has three distinct variables, two of the variables occur once on each side, the third variable occurs twice on each side, and the variables occur in the same order on both sides. A loop is of *Bol-Moufang type* if it belongs to a variety of loops defined by

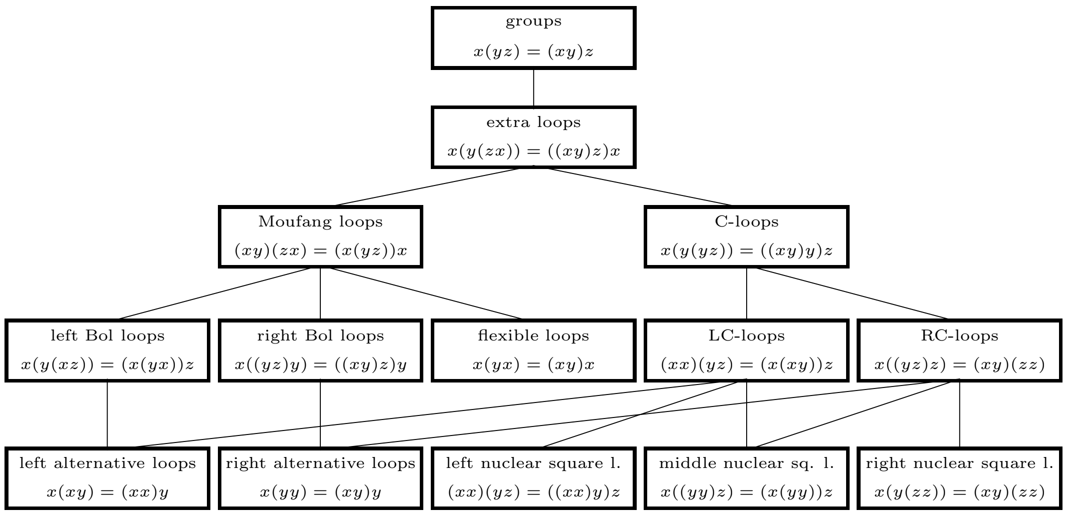

Figure 1. The varieties of loops of Bol-Moufang type.

a single identity of Bol-Moufang type. Figure 1 shows all varieties of loops of Bol-Moufang type and all inclusions among them (cf.[4],[8]). Some varieties of Figure 1 can be defined equivalently by other identities of Bol-Moufang type. For instance, Moufang loops are equivalently defined by the identity $x(y(xz)) = ((xy)x)z$. See [8] for all such equivalences. Furthermore, although some defining identities of Figure 1 do not appear to be of Bol-Moufang type, they are in fact equivalent to some Bol-Moufang identity. For instance, the flexible law $x(yx) = (xy)x$ is equivalent to the Bol-Moufang identity $(x(yx))z = ((xy)x)z$ in any variety of loops.

As we shall see, the computational complexity of our programme is overwhelming (for humans). We therefore first carefully define what we mean by a construction similar to (1) (see Section 2), and then identify situations in which two given constructions are "the same" (see Sections 3, 4, 5). Upon showing which constructions yield loops, we work out one construction by hand (see Section 6), and then switch to a computer search, described in Section 7. The results of the computer search are summarized in Section 8.

2 Similar Constructions

Throughout the paper, we assume that G is a finite group, $g_0 \in Z(G)$, and $*$ is an involutory automorphism of G such that $g_0^* = g_0$ and $gg^* \in Z(G)$ for every $g \in G$.

The following property of $*$ will be used without reference:

Lemma 2.1. *Let G be a group and $* : G \to G$ an involutory map such that $gg^* \in Z(G)$ for every $g \in G$. Then $g^*g = gg^* \in Z(G)$ for every $g \in G$.*

Proof. For $g \in G$, we have $g^*g = g^*(g^*)^* \in Z(G)$. Then $(g^*g)g^* = g^*(g^*g)$, and $gg^* = g^*g$ follows upon canceling g^* on the left. $\square$

Consider the following eight bijections of $G \times G$:

$$\begin{array}{ll} \theta_{xy}(g,h) = (g,h), & \theta_{xy^*}(g,h) = (g,h^*), \\ \theta_{x^*y}(g,h) = (g^*,h), & \theta_{x^*y^*}(g,h) = (g^*,h^*), \\ \theta_{yx}(g,h) = (h,g), & \theta_{yx^*}(g,h) = (h,g^*), \\ \theta_{y^*x}(g,h) = (h^*,g), & \theta_{y^*x^*}(g,h) = (h^*,g^*). \end{array}$$

They form a group Θ under composition, isomorphic to the dihedral group D_8 (unless G or $*$ are trivial). It is generated by $\{\theta_{yx}, \theta_{xy^*}\}$, say. Let Θ_0 be the group generated by Θ and θ_{g_0}, where $\theta_{g_0}(g,h) = (g_0g,h)$.

Let $\Delta : G \times G \to G$ be the map $\Delta(g,h) = gh$, and u an indeterminate. Given α, β, γ, $\delta \in \Theta_0$, define multiplication $\circ$ on $G \cup Gu$ by

$$g \circ h = \Delta\alpha(g,h), \quad g \circ hu = (\Delta\beta(g,h))u, \quad gu \circ h = (\Delta\gamma(g,h))u, \quad gu \circ hu = \Delta\delta(g,h),$$

where g, $h \in G$. The resulting groupoid $(G \cup Gu, \circ)$ will be denoted by

$$Q(G, *, g_0, \alpha, \beta, \gamma, \delta),$$

or by $Q(G, \alpha, \beta, \gamma, \delta)$, when g_0, $*$ are known from the context or if they are not important. It is easy to check that $Q(G, *, g_0, \alpha, \beta, \gamma, \delta)$ is a quasigroup.

We also define

$$\mathcal{Q}(G, *, g_0) = \{Q(G, *, g_0, \alpha, \beta, \gamma, \delta) \mid \alpha,\ \beta,\ \gamma,\ \delta \in \Theta_0\},$$

and

$$\mathcal{Q}(G) = \bigcup_{*, g_0} \mathcal{Q}(G, *, g_0),$$

where the union is taken over all involutory antiautomorphisms $*$ satisfying $gg^* \in Z(G)$ for every $g \in G$, and over all elements g_0 such that $g_0^* = g_0 \in Z(G)$. By definition, we call elements of $\mathcal{Q}(G)$ *quasigroups obtained from G by a construction similar to* (1).

3 Reductions

The goal of this section is to show that one does not have to take all elements of Θ_0 into consideration in order to determine $\mathcal{Q}(G, *, g_0)$.

Note that $g_0^n = (g_0^n)^* \in Z(G)$ for every integer n. Therefore

$$g_0^n \Delta\theta_0(g,h) = \Delta\theta_{g_0}^n \theta_0(g,h) = \Delta\theta_0\theta_{g_0}^n(g,h) \tag{2}$$

for every $\theta_0 \in \Theta_0$ and every g, $h \in G$.

Lemma 3.1. *For every integer n, the quasigroup $Q(G, \theta_{g_0}^n\alpha, \theta_{g_0}^n\beta, \theta_{g_0}^n\gamma, \theta_{g_0}^n\delta)$ is isomorphic to $Q(G, \alpha, \beta, \gamma, \delta)$.*

Proof. We use (2) freely in this proof. Let $t = g_0^n$. Denote by $\circ$ the multiplication in $Q(G,\alpha,\beta,\gamma,\delta)$, and by $\bullet$ the multiplication in $Q(G,\theta_{g_0}^n\alpha,\ \theta_{g_0}^n\beta,\ \theta_{g_0}^n\gamma,\ \theta_{g_0}^n\delta)$. Let f be the bijection of $G\cup Gu$ defined by $g\mapsto t^{-1}g$, $gu\mapsto (t^{-1}g)u$, for $g\in G$. Then for g, $h\in G$, we have

$$f(g\circ h) = t^{-1}\Delta\alpha(g,h) = t\Delta\alpha(t^{-1}g,t^{-1}h) = t^{-1}g\bullet t^{-1}h = f(g)\bullet f(h),$$
$$f(g\circ hu) = t^{-1}\Delta\beta(g,h)u = t\Delta\beta(t^{-1}g,t^{-1}h)u = t^{-1}g\bullet (t^{-1}h)u = f(g)\bullet f(hu),$$

and similarly for γ, δ. Hence f is the desired isomorphism. □

Therefore, if we only count the quasigroups in $\mathcal{Q}(G,*,g_0)$ up to isomorphism, we can assume that $\mathcal{Q}(G,*,g_0) = \{Q(G,*,g_0,\alpha,\beta,\gamma,\delta) \mid \alpha\in\Theta$, and β, γ, δ are of the form $\theta\theta_{g_0}^n$ for some $n\in\mathbb{Z}$ and $\theta\in\Theta\}$.

Given a groupoid $(A,\cdot)$, the *opposite groupoid* $(A,\cdot^{\mathrm{op}})$ is defined by $x\cdot^{\mathrm{op}} y = y\cdot x$.

Lemma 3.2. *The quasigroups $Q(G,\alpha,\beta,\gamma,\delta)$ and $Q(G,\theta_{yx}\alpha,\ \theta_{yx}\gamma,\ \theta_{yx}\beta,\ \theta_{yx}\delta)$ are opposite to each other.*

Proof. Let $\circ$ denote the multiplication in $Q(G,\alpha,\beta,\gamma,\delta)$, and $\bullet$ the multiplication in $Q(G,\theta_{yx}\alpha,\ \theta_{yx}\gamma,\ \theta_{yx}\beta,\ \theta_{yx}\delta\}$. For g, $h\in G$ we have

$$\begin{aligned}
g\circ h &= \Delta\alpha(g,h) = \Delta\theta_{yx}\alpha(h,g) = h\bullet g,\\
g\circ hu &= \Delta\beta(g,h)u = \Delta\theta_{yx}\beta(h,g)u = hu\bullet g,\\
gu\circ h &= \Delta\gamma(g,h)u = \Delta\theta_{yx}\gamma(h,g)u = h\bullet gu,\\
gu\circ hu &= \Delta\delta(g,h) = \Delta\theta_{yx}\delta(h,g) = hu\bullet gu.
\end{aligned}$$

□

Therefore, if we only count the quasigroups in $\mathcal{Q}(G,*,g_0)$ up to isomorphism and opposites, we can assume that $\mathcal{Q}(G,*,g_0) = \{Q(G,*,g_0,\alpha,\beta,\gamma,\delta) \mid \alpha\in\{\theta_{xy}$, θ_{xy^*}, θ_{x^*y}, $\theta_{x^*y^*}\}$, and β, γ, δ are of the form $\theta\theta_{g_0}^n$ for some $n\in\mathbb{Z}$ and $\theta\in\Theta\}$.

Assumption 3.3. From now on we assume that $\alpha\in\{\theta_{xy},\ \theta_{xy^*},\ \theta_{x^*y},\ \theta_{x^*y^*}\}$, and that β, γ, δ are of the form $\theta\theta_{g_0}^n$ for some $n\in\mathbb{Z}$ and $\theta\in\Theta$.

4 When $*$ is identical on G

Assume for a while that $g = g^*$ for every $g\in G$. Then $gh = (gh)^* = h^*g^* = hg$ shows that G is commutative. In particular, $\Theta = \{\theta_{xy}\}$, and $\Theta_0 = \bigcup_n \theta_{g_0}^n$. We show in this section that loops $Q(G,*,g_0,\alpha,\beta,\gamma,\delta)$ obtained with identical $*$ are not interesting.

Let ψ be a groupoid identity, and let $\operatorname{var}\psi$ be all the variables appearing in ψ. Assume that for every $x\in\operatorname{var}\psi$ a decision has been made whether x is to be taken from G or from Gu. Then, while evaluating each side of the identity ψ in $G\cup Gu$, we have to use the multiplications α, β, γ and δ a certain number of times.

Example 4.1. Consider the left alternative law $x(xy) = (xx)y$. With $x\in G$, $y\in Gu$, we see that we need β twice to evaluate $x\circ(x\circ y)$, while we need α once and β once to evaluate $(x\circ x)\circ y$.

A groupoid identity is said to be *strictly balanced* if the same variables appear on both sides of the identity the same number of times and in the same order. For instance $(x(y(xz)))(yx) = ((xy)x)(z(yx))$ is strictly balanced.

The above example shows that the same multiplications do not have to be used the same number of times even while evaluating a strictly balanced identity. However:

Lemma 4.2. *Let ψ be a strictly balanced identity. Assume that for $x \in \operatorname{var}\psi$ a decision has been made whether $x \in G$ or $x \in Gu$. Then, while evaluating ψ in $Q(G, *, g_0, \alpha, \beta, \gamma, \delta)$, δ is used the same number of times on both sides of ψ.*

Proof. Let k be the number of variables on each side of ψ, with repetitions, whose value is assigned to be in Gu. The number k is well-defined since ψ is strictly balanced.

While evaluating the identity ψ, each multiplication reduces the number of factors by 1. However, only δ reduces the number of factors from Gu (by two). Since the coset multiplication in $G \cup Gu$ modulo G is associative, and since ψ is strictly balanced, either both evaluated sides of ψ will end up in G (in which case δ is applied $k/2$ times on each side), or both evaluated sides of ψ will end up in Gu (in which case δ is applied $(k-1)/2$ times on each side). $\square$

Lemma 4.3. *If $\alpha \in \Theta$ and $L = Q(G, *, g_0, \alpha, \beta, \gamma, \delta)$ is a loop, then the neutral element of Q coincides with the neutral element of G.*

Proof. Let e be the neutral element of L and 1 the neutral element of G. Since $1 = 1^*$, we have $1 \circ 1 = \Delta\alpha(1, 1) = 1 = 1 \circ e$, and the result follows from the fact that L is a quasigroup. $\square$

Proposition 4.4. *Assume that $g^* = g$ for every $g \in G$, and let α, β, γ, $\delta \in \Theta_0$. If $L = Q(G, *, g_0, \alpha, \beta, \gamma, \delta)$ happens to be a loop, then every strictly balanced identity holds in L. In particular, L is an abelian group.*

Proof. Since $*$ is identical on G, we have $\Theta_0 = \{\theta_{g_0}^n \mid n \in \mathbb{Z}\}$. By Assumption 3.3, we have $\alpha = \theta_{xy}$. Then by Lemma 4.3, L has neutral element 1. Assume that $\beta = \theta_{g_0}^n$ for some n. Then $gu = 1 \circ gu = (\Delta\beta(1, g))u = (g_0^n g)u$, which means that $n = 0$. Similarly, if $\gamma = \theta_{g_0}^m$ then $m = 0$.

Let $\delta = \theta_{g_0}^k$. Let ψ be a strictly balanced identity. For every $x \in \operatorname{var}\psi$, decide if $x \in G$ or $x \in Gu$. By Lemma 4.2, while evaluating ψ in L, the multiplication δ is used the same number of times on the left and on the right, say t times. Since $\alpha = \beta = \gamma = \theta_{xy}$, we conclude that ψ reduces to $g_0^{kt}z = g_0^{kt}z$, for some $z \in G \cup Gu$.

Since the associative law is strictly balanced, L is associative. We have already noticed that identical $*$ forces G to be abelian. Then L is abelian too, as $gu \circ h = (gh)u = (hg)u = h \circ gu$ and $gu \circ hu = g_0^k gh = g_0^k hg = hu \circ gu$ for every g, $h \in G$. $\square$

We have just seen that if $g = g^*$ for every $g \in G$ then our constructions do not yield nonassociative loops. Therefore:

Assumption 4.5. From now on, we assume that there exists $g \in G$ such that $g^* \neq g$.

5 Loops

In this section we further narrow the choices of α, β, γ, δ when $Q(G,\alpha,\beta,\gamma,\delta)$ is supposed to be a loop.

Proposition 5.1. *Let $L = Q(G,*,g_0,\alpha,\beta,\gamma,\delta)$. Then L is a loop if and only if $\alpha = \theta_{xy}$, $\beta \in \{\theta_{xy},\ \theta_{x^*y},\ \theta_{yx},\ \theta_{yx^*}\}$, $\gamma \in \{\theta_{xy},\ \theta_{xy^*},\ \theta_{yx},\ \theta_{y^*x}\}$, and δ is of the form $\theta\theta_{g_0}^n$ for some integer n and $g_0 \in G$.*

Proof. If L is a loop then $\alpha \in \{\theta_{xy},\ \theta_{xy^*},\ \theta_{x^*y},\ \theta_{x^*y^*}\}$ and Lemma 4.3 imply that 1 is the neutral element of L.

The equation $g = 1 \circ g$ holds for every $g \in G$ if and only if $\Delta\alpha(1,g) = g$ for every $g \in G$, which happens if and only if $\alpha \in \{\theta_{xy},\ \theta_{x^*y}\}$. (Note that we use Assumption 4.5 here.) Similarly, $g = g \circ 1$ holds for every $g \in G$ if and only if $\Delta\alpha(g,1) = g$ for every $g \in G$, which happens if and only if $\alpha \in \{\theta_{xy},\ \theta_{xy^*}\}$. Therefore $g = 1 \circ g = g \circ 1$ holds for every $g \in G$ if and only if $\alpha = \theta_{xy}$.

Now, $gu = 1 \circ gu$ holds for every $g \in G$ if and only if $\Delta\beta(1,g) = g$ for every $g \in G$, which happens if and only if $\beta \in \{\theta_{xy},\ \theta_{x^*y},\ \theta_{yx},\ \theta_{yx^*}\}$. Similarly, $gu = gu \circ 1$ holds for every $g \in G$ if and only if $\Delta\gamma(g,1) = g$ for every $g \in G$, which happens if and only if $\gamma \in \{\theta_{xy},\ \theta_{xy^*},\ \theta_{yx},\ \theta_{y^*x}\}$. □

We are only interested in loops, and we have already noted that $(g_0^n)^* = g_0^n \in Z(G)$. Since we allow $g_0 = 1$, we can agree on:

Assumption 5.2. From now on, we assume that $\alpha = \theta_{xy}$, $\beta \in \{\theta_{xy},\ \theta_{x^*y},\ \theta_{yx},\ \theta_{yx^*}\}$, $\gamma \in \{\theta_{xy},\ \theta_{xy^*},\ \theta_{yx},\ \theta_{y^*x}\}$, and $\delta \in \theta_{g_0}\Theta$.

Our last reduction concerns the maps β and γ.

Lemma 5.3. *We have $\Delta\theta_{x^*y^*}\theta_0 = \Delta\theta_0\theta_{x^*y^*}$ for every $\theta_0 \in \Theta_0$.*

Proof. The group Θ_0 is generated by θ_{yx}, θ_{xy^*} and θ_{g_0}. It therefore suffices to check that $\Delta\theta_{x^*y^*}\theta_0 = \Delta\theta_0\theta_{x^*y^*}$ holds for $\theta_0 \in \{\theta_{yx},\ \theta_{xy^*},\ \theta_{g_0}\}$, which follows by straightforward calculation. □

Lemma 5.4. *The quasigroups $Q(G,*,g_0,\alpha,\beta,\gamma,\delta)$, $Q(G,*,g_0,\alpha,\beta',\gamma',\theta_{x^*y^*}\delta)$ are isomorphic if*

$$\begin{aligned}\{\beta,\beta'\} &\in \{\{\theta_{xy},\theta_{yx^*}\},\ \{\theta_{yx},\theta_{x^*y}\}\},\\ \{\gamma,\gamma'\} &\in \{\{\theta_{xy},\theta_{y^*x}\},\ \{\theta_{yx},\theta_{xy^*}\}\}.\end{aligned}$$

Proof. Let $\circ$ denote the multiplication in $Q(G,*,g_0,\alpha,\beta,\gamma,\delta)$, and $\bullet$ the multiplication in $Q(G,*,g_0,\alpha,\beta',\gamma',\theta_{x^*y^*}\delta)$. Consider the permutation f of G defined by $f(g) = g$, $f(gu) = g^*u$, for $g \in G$.

We show that f is an isomorphism of $(G \cup Gu, \circ)$ onto $(G \cup Gu, \bullet)$ if and only if

$$(\Delta\beta(g,h))^* = \Delta\beta'(g,h^*), \quad (\Delta\gamma(g,h))^* = \Delta\gamma'(g^*,h). \tag{3}$$

Once we establish this fact, the proof is finished by checking that the pairs (β,β'), (γ,γ') in the statement of the Lemma satisfy (3).

Let g, $h\in G$. Then

$$\begin{aligned}
f(g\circ h) &= f(\Delta\alpha(g,h)) = \Delta\alpha(g,h),\\
f(g\circ hu) &= f(\Delta\beta(g,h)u) = (\Delta\beta(g,h))^*u,\\
f(gu\circ h) &= f(\Delta\gamma(g,h)u) = (\Delta\gamma(g,h))^*u,\\
f(gu\circ hu) &= f(\Delta\delta(g,h)) = \Delta\delta(g,h),
\end{aligned}$$

while

$$\begin{aligned}
f(g)\bullet f(h) &= g\bullet h = \Delta\alpha(g,h),\\
f(g)\bullet f(hu) &= g\bullet h^*u = \Delta\beta'(g,h^*)u,\\
f(gu)\bullet f(h) &= g^*u\bullet h = \Delta\gamma'(g^*,h)u,\\
f(gu)\bullet f(hu) &= g^*u\bullet h^*u = \Delta\theta_{g^*h^*}\delta(g^*,h^*).
\end{aligned}$$

We see that $f(g\circ h)=f(g)\bullet f(h)$ always holds. By Lemma 5.3, $f(gu\circ hu)=f(gu)\bullet f(hu)$ always holds. Finally, $f(g\circ hu)=f(g)\bullet f(hu)$, $f(gu\circ h)=f(gu)\bullet f(h)$ hold if and only if (β,β'), (γ,γ') satisfy (3). □

Assume that $Q(G,*,g_0,\theta_{xy},\beta,\gamma,\delta)$ is a loop (satisfying Assumption 5.2). Then Lemma 5.4 provides an isomorphism of $Q(G,*,g_0,\theta_{xy},\beta,\gamma,\delta)$ onto some loop $Q(G,*,g_0,\theta_{xy},\beta',\gamma',\delta')$ such that if $\gamma=\theta_{xy^*}$ then $\gamma'=\theta_{yx}$, and if $\gamma=\theta_{y^*x}$ then $\gamma'=\theta_{xy}$. We can therefore assume:

Assumption 5.5. From now on, we assume that $\alpha=\theta_{xy}$, $\beta\in\{\theta_{xy},\ \theta_{x^*y},\ \theta_{yx},\ \theta_{yx^*}\}$, $\gamma\in\{\theta_{xy},\ \theta_{yx}\}$, and $\delta\in\theta_{g_0}\Theta$.

In order to find all loops $Q(G,*,g_0,\alpha,\beta,\gamma,\delta)$ that satisfy a given groupoid identity ψ, we only have to consider $1\cdot4\cdot2\cdot8=64$ choices for $(\alpha,\beta,\gamma,\delta)$. (To appreciate the reductions, compare this with the unrestricted case α, β, γ, $\delta\in\Theta_0$.) Once $(\alpha,\beta,\gamma,\delta)$ is chosen, we must verify 2^k equations in G, where k is the number of variables in ψ (since each variable can be assigned value in G or in Gu).

We work out the calculation for one identity ψ and one choice of multiplication $(\alpha,\beta,\gamma,\delta)$. After seing the routine nature of the calculations, we gladly switch to a computer search.

6 C-loops arising from the construction of de Barros and Juriaans

C-loops are loops satisfying the identity $((xy)y)z=x(y(yz))$. In [2], de Barros and Juriaans used a construction similar to (1) to obtain loops whose loop algebras are flexible. In our systematic notation, their construction is

$$Q(G,*,g_0,\theta_{xy},\theta_{xy},\theta_{y^*x},\theta_{g_0}\theta_{xy^*}), \tag{4}$$

with the usual conventions on g_0 and $*$. The construction (4) violates Assumption 5.5 but, by Lemma 5.4, it is isomorphic to

$$Q(G, *, g_0, \theta_{xy}, \theta_{yx^*}, \theta_{xy}, \theta_{g_0}\theta_{x^*y}),$$

which complies with all assumptions we have made.

Theorem 6.1. *Let G be a group and let L be the loop defined by (4). Then L is a flexible loop, and the following conditions are equivalent:*

(i) L is associative,

(ii) L is Moufang,

(iii) G is commutative.

Furthermore, L is a C-loop if and only if $G/Z(G)$ is an elementary abelian 2-group. When L is a C-loop, it is diassociative.

Proof. Throughout the proof, we use $g_0 = g_0^* \in Z(G)$, $gg^* = g^*g \in Z(G)$, $(g^*)^* = g$ and $(gh)^* = h^*g^*$ without warning.

By Proposition 5.1, L is a loop.

Flexibility. For x, $y \in G$ we have:

$$\begin{aligned}
&(x \circ y) \circ x = (xy)x = x(yx) = x \circ (y \circ x),\\
&(x \circ yu) \circ x = (xy)u \circ x = x^*xyu = xx^*yu = x \circ x^*yu = x \circ (yu \circ x),\\
&(xu \circ y) \circ xu = y^*xu \circ xu = g_0y^*xx^* = g_0xx^*y^* = xu \circ (yx)u = xu \circ (y \circ xu),\\
&(xu \circ yu) \circ xu = g_0xy^* \circ xu = g_0xy^*xu = xu \circ g_0yx^* = xu \circ (yu \circ xu).
\end{aligned}$$

Thus L is flexible.

Associativity. For x, y, $z \in G$ we have:

$$\begin{aligned}
&x \circ (y \circ z) = x(yz) = (xy)z = (x \circ y) \circ z,\\
&x \circ (y \circ zu) = x(yz)u = (xy)zu = (x \circ y) \circ zu,\\
&xu \circ (y \circ z) = xu \circ yz = z^*y^*xu = y^*xu \circ z = (xu \circ y) \circ z,\\
&x \circ (yu \circ zu) = x \circ g_0yz^* = g_0xyz^* = xyu \circ zu = (x \circ yu) \circ zu,\\
&xu \circ (yu \circ z) = xu \circ z^*yu = g_0xy^*z = g_0xy^* \circ z = (xu \circ yu) \circ z.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&x \circ (yu \circ z) = x \circ z^*yu = xz^*yu, && (x \circ yu) \circ z = xyu \circ z = z^*xyu,\\
&xu \circ (y \circ zu) = xu \circ yzu = g_0xz^*y^*, && (xu \circ y) \circ zu = y^*xu \circ zu = g_0y^*xz^*,\\
&xu \circ (yu \circ zu) = xu \circ g_0yz^* = g_0zy^*xu, && (xu \circ yu) \circ zu = g_0xy^* \circ zu = g_0xy^*zu.
\end{aligned}$$

Thus L is associative if and only if G is commutative. (Sufficiency is obvious. For necessity, note that $*$ is onto, and substitute 1 for one of x, y, z if needed.)

Moufang property. Let x, y, $z \in G$. Then

$$x \circ (yu \circ (x \circ z)) = x \circ (yu \circ xz) = x \circ z^*x^*yu = xz^*x^*yu,$$
$$((x \circ yu) \circ x) \circ z = (xyu \circ x) \circ z = x^*xyu \circ z = z^*x^*xyu.$$

Therefore, this particular form of the Moufang identity holds if and only if $xz^*x^* = z^*x^*x$. Now, given x, $y \in G$, there is $z \in G$ such that $z^*x^* = y$. Therefore $xz^*x^* = z^*x^*x$ holds in G if and only if G is commutative. However, when G is commutative, then L is associative, and we have proved the equivalence of (i), (ii), (iii).

C property. Let x, y, $z \in G$. Then

$$\begin{aligned}
&x \circ (y \circ (y \circ z)) = x(y(yz)) = ((xy)y)z = ((x \circ y) \circ y) \circ z,\\
&x \circ (y \circ (y \circ zu)) = (x(y(yz))u = ((xy)y)z)u = ((x \circ y) \circ y) \circ zu,\\
&x \circ (yu \circ (yu \circ z)) = x \circ (yu \circ z^*yu) = x \circ g_0yy^*z = g_0xyy^*z = g_0xyy^* \circ z\\
&\qquad = (xyu \circ yu) \circ z = ((x \circ yu) \circ yu) \circ z,\\
&xu \circ (y \circ (y \circ z)) = xu \circ yyz = z^*y^*y^*xu = y^*y^*xu \circ z = (y^*xu \circ y) \circ z\\
&\qquad = ((xu \circ y) \circ y) \circ z,\\
&x \circ (yu \circ (yu \circ zu)) = x \circ (yu \circ g_0yz^*) = x \circ g_0zy^*yu = g_0xzy^*yu\\
&\qquad = g_0xyy^*zu = g_0xyy^* \circ zu = (xyu \circ yu) \circ zu = ((x \circ yu) \circ yu) \circ zu,\\
&xu \circ (yu \circ (yu \circ z)) = xu \circ (yu \circ z^*yu) = xu \circ g_0yy^*z = g_0z^*yy^*xu = g_0z^*xy^*yu\\
&\qquad = g_0xy^*yu \circ z = (g_0xy^* \circ yu) \circ z = ((xu \circ yu) \circ yu) \circ z,\\
&xu \circ (yu \circ (yu \circ zu)) = xu \circ (yu \circ g_0yz^*) = xu \circ g_0zy^*yu = g_0^2xy^*yz^*\\
&\qquad = g_0xy^*yu \circ zu = (g_0xy^* \circ yu) \circ zu = ((xu \circ yu) \circ yu) \circ zu.
\end{aligned}$$

While verifying the remaining form of the C identity, we obtain

$$xu \circ (y \circ (y \circ zu)) = xu \circ yyzu = g_0xz^*y^*y^*,$$
$$((xu \circ y) \circ y) \circ zu = (y^*xu \circ y) \circ zu = y^*y^*xu \circ zu = g_0y^*y^*xz^*.$$

The identity therefore holds if and only if y^*y^* commutes with all elements of G, which happens if and only if $G/Z(G)$ is an elementary abelian 2-group.

Finally, by Lemma 4.4 of [7], flexible C-loops are diassociative. □

7 The Algorithm

7.1 Collecting Identities

Let G be a group, ψ a groupoid identity and $(\alpha, \beta, \gamma, \delta)$ a multiplication. Then the following algorithm will output a set Ψ of group identities such that $Q(G, *, g_0, \alpha, \beta, \gamma, \delta)$ satisfies ψ if and only if G satisfies all identities of Ψ:

(i) Let $f : \operatorname{var}\psi \to \{0, 1\}$ be a function that decides whether $x \in \operatorname{var}\psi$ is to be taken from G or from Gu.

(ii) Upon assigning the variables of ψ according to f, let $\psi_f = (u, v)$ be the identity ψ evaluated in $Q(G, *, g_0, \alpha, \beta, \gamma, \delta)$.

(iii) Let $\Psi = \{\psi_f \mid f : \operatorname{var}\psi \to \{0, 1\}\}$.

This algorithm is straightforward but not very useful, since it typically outputs a large number of complicated group identities.

7.2 Understanding the identities in the Bol-Moufang case

We managed to decipher the meaning of Ψ for all multiplications $(\alpha, \beta, \gamma, \delta)$ and for all identities of Bol-Moufang type by another algotihm. First, we reduced the identity $\psi_f = (u, v)$ to a canonical form as follows:

(a) replace g_0^* by g_0,

(b) move all g_0 to the very left,

(c) replace x^*x by xx^*,

(d) move all substrings xx^* immediately to the right of the power g_0^m, and order the substrings xx^*, yy^*, ... lexicographically,

(e) cancel as much as possible on the left and on the right of the resulting identity.

Then we used Lemmas 7.1–7.5 to understand what the canonical identities collected in Ψ say about the group G:

Lemma 7.1. *If an identity of Ψ reduces to $x^* = x$ then it does not hold in any group.*

Proof. This follows since we assume that $*$ is not identical on G. $\square$

Lemma 7.2. *The following conditions are equivalent:*

(i) $G/Z(G)$ is an elementary abelian 2-group,

(ii) $xxy = yxx$,

(iii) $xyx^* = x^*yx$.

Proof. We have $xyx^* = x^*yx$ if and only if $x^*xyx^*x = x^*x^*yxx$. Since $x^*x \in Z(G)$, the latter identity is equivalent to $x^*xx^*xy = x^*x^*yxx$. Since $xx^* = x^*x$, we can rewrite it equivalently as $x^*x^*xxy = x^*x^*yxx$, which is by cancellation equivalent to $xxy = yxx$. $\square$

Lemma 7.3. *The following conditions are equivalent:*

(i) G is commutative,

(ii) $xx^*y = x^*yx$.

Proof. If $xx^*y = x^*yx$ then $x^*xy = x^*yx$ and so $xy = yx$. $\square$

Lemma 7.4. *If ψ is a strictly balanced identity that reduces to $xy = yx$ upon substituting 1 for some of the variables of ψ, then ψ is equivalent to commutativity.*

Proof. ψ implies commutativity. Once commutativity holds, we can rearrange the variables of ψ so that both sides of ψ are the same, because ψ is strictly balanced. $\square$

Lemma 7.5. *The following conditions are equivalent:*

(i) $xxy = yx^*x^*$ *holds in* G,

(ii) $(xx)^* = xx$ *and* $G/Z(G)$ *is an elementary abelian* 2*-group.*

Proof. Condition (ii) clearly implies (i). If (i) holds, we have $xx = x^*x^*$ (with $y = 1$) and so $(xx)^* = xx$. Also $xxy = yx^*x^* = yxx$. $\square$

7.3 What the identities mean in the Bol-Moufang case

Lemmas 7.1–7.5 are carefully tailored to loops of Bol-Moufang type, and we discovered them upon studying the canonical identities Ψ obtained by the computer search.

It just so happens that every identity ψ_f of Ψ is equivalent to a combination of the following properties of G:

(PN) No group satisfies ψ_f.

(PA) All groups satisfy ψ_f.

(PC) G is commutative.

(PB) $G/Z(G)$ is an elementary abelian 2-group.

(PS) $(gg)^* = gg$ for every $g \in G$.

A prominent example of $*$ is the inverse operation $^{-1}$ in G. Then (PB) says that G is of exponent 4, and it is therefore not difficult to obtain examples of groups satisfying any possible combination of (PN), (PA), (PC), (PB) and (PS).

We have implemented the algorithm in GAP [5], and made it available online at

`http://www.math.du.edu/~petr`

in section Research. The algorithm is not safe for identities that are not strictly balanced.

8 Results

We now present the results of the computer search. In order to organize the results, observe that if $L = Q(G, *, g_0, \alpha, \beta, \gamma, \delta)$ is associative, it satisfies all identities of Bol-Moufang type. Since we do not want to list the multiplications and properties of G repeatedly, we first describe all cases when L is associative, then all cases when L is an extra loop, then all cases when L is a Moufang loop, etc., guided by the inclusions of Figure 1.

All results of this section are computer generated. To avoid errors in transcribing, the TEX source of the statements of the results is also computer generated. In the statements, we write xy instead of θ_{xy}, g_0yx^* instead of $\theta_{g_0}\theta_{yx^*}$, etc., in order to save space and improve legibility. Some results are mirror versions of others (cf. Theorem 8.5 versus Theorem 8.6), but we decided to include them anyway for quicker future reference. Finally, when G is commutative, $\Delta(\Theta \cup \theta_{g_0}\Theta)$ coincides with $\Delta(S \cup \theta_{g_0}S)$, where $S = \{\theta_{xy},\ \theta_{xy^*},\ \theta_{x^*y},\ \theta_{x^*y^*}\}$. We therefore report only maps α, β, γ, δ from $S \cup \theta_{g_0}S$ in the commutative case.

In Theorems 8.1 – 8.14, G is a group, $*$ is a nonidentical involutory antiautomorphism of G satisfying $gg^* \in Z(G)$ for every $g \in G$, the element $g_0 \in Z(G)$ satisfies $g_0^* = g_0$, and the maps α, β, γ, δ are as in Assumption 5.5.

Theorem 8.1. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is associative iff the following conditions are satisfied:*

(β, γ, δ) is equal to
(xy, xy, g_0xy), or

G is commutative and (β, γ, δ) is equal to (x^*y, xy, g_0x^*y).

Theorem 8.2. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is extra iff it is associative or if the following conditions are satisfied:*

$G/Z(G)$ is an elementary abelian 2-group and (β, γ, δ) is equal to
(x^*y, yx, g_0yx^*).

Theorem 8.3. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is Moufang iff it is extra or if the following conditions are satisfied:*

(β, γ, δ) is equal to
(x^*y, yx, g_0yx^*).

Theorem 8.4. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is a C-loop iff it is extra or if the following conditions are satisfied:*

$G/Z(G)$ is an elementary abelian 2-group and (β, γ, δ) is among
(yx, yx, g_0yx), (yx^*, xy, g_0x^*y).

Theorem 8.5. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is left Bol iff it is Moufang or if the following conditions are satisfied:*

$G/Z(G)$ is an elementary abelian 2-group and (β,γ,δ) is among
(xy, yx, g_0yx), (x^*y, xy, g_0x^*y), *or*

G is commutative, $(xx)^* = xx$ for every $x \in G$ and (β,γ,δ) is among
(xy, xy, g_0x^*y), (x^*y, xy, g_0xy), *or*

$G/Z(G)$ is an elementary abelian 2-group, $(xx)^* = xx$ for every $x \in G$ and (β,γ,δ) is among
(xy, xy, g_0x^*y), (xy, yx, g_0yx^*), (x^*y, xy, g_0xy), (x^*y, yx, g_0yx).

Theorem 8.6. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is right Bol iff it is Moufang or if the following conditions are satisfied:*

$G/Z(G)$ is an elementary abelian 2-group and (β,γ,δ) is among
(yx, xy, g_0yx), (yx^*, yx, g_0x^*y), *or*

G is commutative, $(xx)^* = xx$ for every $x \in G$ and (β,γ,δ) is among
(xy, xy, g_0xy^*), $(x^*y, xy, g_0x^*y^*)$, *or*

$G/Z(G)$ is an elementary abelian 2-group, $(xx)^* = xx$ for every $x \in G$ and (β,γ,δ) is among
(xy, xy, g_0xy^*), $(x^*y, yx, g_0y^*x^*)$, (yx, xy, g_0y^*x), $(yx^*, yx, g_0x^*y^*)$.

Theorem 8.7. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is an LC-loop iff it is a C-loop or if the following conditions are satisfied:*

$G/Z(G)$ is an elementary abelian 2-group and (β,γ,δ) is among
(xy, yx, g_0yx), (x^*y, xy, g_0x^*y), (yx, xy, g_0xy), (yx^*, yx, g_0yx^*), *or*

G is commutative, $(xx)^* = xx$ for every $x \in G$ and (β,γ,δ) is among
(xy, xy, g_0x^*y), (x^*y, xy, g_0xy), *or*

$G/Z(G)$ is an elementary abelian 2-group, $(xx)^* = xx$ for every $x \in G$ and (β,γ,δ) is among
(xy, xy, g_0x^*y), (xy, yx, g_0yx^*), (x^*y, xy, g_0xy), (x^*y, yx, g_0yx),
(yx, xy, g_0x^*y), (yx, yx, g_0yx^*), (yx^*, xy, g_0xy), (yx^*, yx, g_0yx).

Theorem 8.8. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is an RC-loop iff it is a C-loop or if the following conditions are satisfied:*

$G/Z(G)$ is an elementary abelian 2-group and (β,γ,δ) is among
(xy, yx, g_0xy), (x^*y, xy, g_0yx^*), (yx, xy, g_0yx), (yx^*, yx, g_0x^*y), *or*

G is commutative, $(xx)^* = xx$ for every $x \in G$ and (β,γ,δ) is among
(xy, xy, g_0xy^*), $(x^*y, xy, g_0x^*y^*)$, *or*

$G/Z(G)$ is an elementary abelian 2-group, $(xx)^* = xx$ for every $x \in G$ and (β,γ,δ) is among
(xy, xy, g_0xy^*), (xy, yx, g_0xy^*), $(x^*y, xy, g_0y^*x^*)$, $(x^*y, yx, g_0y^*x^*)$,
(yx, xy, g_0y^*x), (yx, yx, g_0y^*x), $(yx^*, xy, g_0x^*y^*)$, $(yx^*, yx, g_0x^*y^*)$.

Theorem 8.9. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is flexible iff it is Moufang or if the following conditions are satisfied:*

(β, γ, δ) is among
$(xy, xy, g_0y^*x^*)$, (x^*y, yx, g_0xy^*), (x^*y, yx, g_0x^*y), (x^*y, yx, g_0y^*x),
$(yx, yx, g_0x^*y^*)$, (yx, yx, g_0yx), (yx^*, xy, g_0xy^*), (yx^*, xy, g_0x^*y),
(yx^*, xy, g_0yx^*), (yx^*, xy, g_0y^*x), *or*

$G/Z(G)$ is an elementary abelian 2-group and (β, γ, δ) is among
$(xy, xy, g_0x^*y^*)$, (xy, xy, g_0yx), (yx, yx, g_0xy), $(yx, yx, g_0y^*x^*)$.

Theorem 8.10. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is left alternative iff it is left Bol or an LC-loop or if the following conditions are satisfied:*

(β, γ, δ) is among
(xy, xy, g_0x^*y), (xy, yx, g_0yx^*), (x^*y, xy, g_0x^*y), (yx, xy, g_0x^*y),
(yx, yx, g_0yx^*), (yx^*, xy, g_0x^*y), (yx^*, yx, g_0yx^*), *or*

$(xx)^* = xx$ for every $x \in G$ and (β, γ, δ) is equal to
(x^*y, xy, g_0xy).

Theorem 8.11. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is right alternative iff it is right Bol or an RC-loop or if the following conditions are satisfied:*

(β, γ, δ) is among
(xy, xy, g_0xy^*), (xy, yx, g_0xy^*), (x^*y, xy, g_0yx^*), (yx, xy, g_0y^*x),
(yx, yx, g_0y^*x), (yx^*, xy, g_0x^*y), (yx^*, yx, g_0x^*y), *or*

$(xx)^* = xx$ for every $x \in G$ and (β, γ, δ) is equal to
$(yx^*, yx, g_0x^*y^*)$.

Theorem 8.12. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is a left nuclear square loop iff it is an LC-loop or if the following conditions are satisfied:*

(β, γ, δ) is among
(xy, xy, g_0xy^*), (yx^*, yx, g_0x^*y), $(yx^*, yx, g_0x^*y^*)$, *or*

$G/Z(G)$ is an elementary abelian 2-group and (β, γ, δ) is among
(xy, xy, g_0yx), (xy, xy, g_0y^*x), (xy, yx, g_0xy), (xy, yx, g_0xy^*),
(xy, yx, g_0y^*x), $(x^*y, xy, g_0x^*y^*)$, (x^*y, xy, g_0yx^*), $(x^*y, xy, g_0y^*x^*)$,
(x^*y, yx, g_0x^*y), $(x^*y, yx, g_0x^*y^*)$, $(x^*y, yx, g_0y^*x^*)$, (yx, xy, g_0xy^*),
(yx, xy, g_0yx), (yx, xy, g_0y^*x), (yx, yx, g_0xy), (yx, yx, g_0xy^*),
(yx, yx, g_0y^*x), $(yx^*, xy, g_0x^*y^*)$, (yx^*, xy, g_0yx^*), $(yx^*, xy, g_0y^*x^*)$,
$(yx^*, yx, g_0y^*x^*)$, *or*

$G/Z(G)$ is an elementary abelian 2-group, $(xx)^* = xx$ for every $x \in G$ and (β, γ, δ) is among
$(xy, xy, g_0x^*y^*)$, (xy, xy, g_0yx^*), $(xy, xy, g_0y^*x^*)$, (xy, yx, g_0x^*y),
$(xy, yx, g_0x^*y^*)$, $(xy, yx, g_0y^*x^*)$, (x^*y, xy, g_0xy^*), (x^*y, xy, g_0yx),
(x^*y, xy, g_0y^*x), (x^*y, yx, g_0xy), (x^*y, yx, g_0xy^*), (x^*y, yx, g_0y^*x),

$(yx, xy, g_0x^*y^*)$, (yx, xy, g_0yx^*), $(yx, xy, g_0y^*x^*)$, (yx, yx, g_0x^*y),
$(yx, yx, g_0x^*y^*)$, $(yx, yx, g_0y^*x^*)$, (yx^*, xy, g_0xy^*), (yx^*, xy, g_0yx),
(yx^*, xy, g_0y^*x), (yx^*, yx, g_0xy), (yx^*, yx, g_0xy^*), (yx^*, yx, g_0y^*x).

Theorem 8.13. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is a middle nuclear square loop iff it is an LC-loop or an RC-loop or if the following conditions are satisfied:*

(β, γ, δ) is among
$(xy, xy, g_0y^*x^*)$, (yx^*, xy, g_0xy^*), (yx^*, xy, g_0yx^*), *or*

$G/Z(G)$ is an elementary abelian 2-group and (β, γ, δ) is among
$(xy, xy, g_0x^*y^*)$, (xy, xy, g_0yx), $(xy, yx, g_0x^*y^*)$, $(xy, yx, g_0y^*x^*)$,
(x^*y, xy, g_0xy^*), (x^*y, xy, g_0y^*x), (x^*y, yx, g_0xy^*), (x^*y, yx, g_0x^*y),
(x^*y, yx, g_0y^*x), $(yx, xy, g_0x^*y^*)$, $(yx, xy, g_0y^*x^*)$, (yx, yx, g_0xy),
$(yx, yx, g_0x^*y^*)$, $(yx, yx, g_0y^*x^*)$, (yx^*, xy, g_0y^*x), (yx^*, yx, g_0xy^*),
(yx^*, yx, g_0y^*x), *or*

$G/Z(G)$ is an elementary abelian 2-group, $(xx)^* = xx$ for every $x \in G$ and (β, γ, δ) is among
(xy, xy, g_0yx^*), (xy, xy, g_0y^*x), (xy, yx, g_0x^*y), (xy, yx, g_0y^*x),
$(x^*y, xy, g_0x^*y^*)$, (x^*y, xy, g_0yx), (x^*y, yx, g_0xy), $(x^*y, yx, g_0x^*y^*)$,
(yx, xy, g_0xy^*), (yx, xy, g_0yx^*), (yx, yx, g_0xy^*), (yx, yx, g_0x^*y),
(yx^*, xy, g_0yx), $(yx^*, xy, g_0y^*x^*)$, (yx^*, yx, g_0xy), $(yx^*, yx, g_0y^*x^*)$.

Theorem 8.14. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is a right nuclear square loop iff it is an RC-loop or if the following conditions are satisfied:*

(β, γ, δ) is among
(xy, xy, g_0x^*y), (x^*y, xy, g_0xy), (x^*y, xy, g_0x^*y), *or*

$G/Z(G)$ is an elementary abelian 2-group and (β, γ, δ) is among
(xy, xy, g_0yx), (xy, xy, g_0yx^*), (xy, yx, g_0x^*y), (xy, yx, g_0yx),
(xy, yx, g_0yx^*), (x^*y, xy, g_0yx), (x^*y, yx, g_0xy), (x^*y, yx, g_0x^*y),
(x^*y, yx, g_0yx), (yx, xy, g_0xy), (yx, xy, g_0x^*y), (yx, xy, g_0yx^*),
(yx, yx, g_0xy), (yx, yx, g_0x^*y), (yx, yx, g_0yx^*), (yx^*, xy, g_0xy),
(yx^*, xy, g_0yx), (yx^*, xy, g_0yx^*), (yx^*, yx, g_0xy), (yx^*, yx, g_0yx),
(yx^*, yx, g_0yx^*), *or*

$G/Z(G)$ is an elementary abelian 2-group, $(xx)^* = xx$ for every $x \in G$ and (β, γ, δ) is among
$(xy, xy, g_0x^*y^*)$, (xy, xy, g_0y^*x), $(xy, xy, g_0y^*x^*)$, $(xy, yx, g_0x^*y^*)$,
(xy, yx, g_0y^*x), $(xy, yx, g_0y^*x^*)$, (x^*y, xy, g_0xy^*), $(x^*y, xy, g_0x^*y^*)$,
(x^*y, xy, g_0y^*x), (x^*y, yx, g_0xy^*), $(x^*y, yx, g_0x^*y^*)$, (x^*y, yx, g_0y^*x),
(yx, xy, g_0xy^*), $(yx, xy, g_0x^*y^*)$, $(yx, xy, g_0y^*x^*)$, (yx, yx, g_0xy^*),
$(yx, yx, g_0x^*y^*)$, $(yx, yx, g_0y^*x^*)$, (yx^*, xy, g_0xy^*), (yx^*, xy, g_0y^*x),
$(yx^*, xy, g_0y^*x^*)$, (yx^*, yx, g_0xy^*), (yx^*, yx, g_0y^*x), $(yx^*, yx, g_0y^*x^*)$.

9 Concluding remarks

(I) Figure 1 and Theorems 8.1–8.14 taken together tell us more than if we consider them separately. For instance, Theorem 8.1 and Theorem 8.3 plus the fact that every group is a Moufang loop imply that the construction of Theorem 8.3 yields a nonassociative loop if and only if the group G is not commutative. In other words, the two theorems encompass Theorem 1.1, and, in addition, show that Chein's construction is unique for Moufang loops.

(II) Note that we have also recovered (an isomorphic copy of) the construction (4) of de Barros and Juriaans. Our results on Bol loops agree with those of [10], obtained by hand.

(III) To illustrate how the algorithm works for loops that are not of Bol-Moufang type, we show the output for nonassociative RIF loops. A loop is an *RIF loop* if it satisfies $(xy)(z(xy)) = ((x(yz))x)y$.

Theorem 9.1. *The loop $Q(G, *, g_0, \theta_{xy}, \beta, \gamma, \delta)$ is RIF iff it is associative or if the following conditions are satisfied:*

(β, γ, δ) is among
(x^*y, yx, g_0yx^*), (yx^*, xy, g_0x^*y), or

(β, γ, δ) and G are as in the following list:
(yx, yx, g_0yx) and $xyzxy = yxzyx$.

Note that the algorithm did not manage to decipher the meaning of the group identity $xyzxy = yxzyx$, so it simply listed it.

(IV) We conclude the paper with the following observation:

Lemma 9.2. *Let $L = Q(G, *, g_0, \alpha, \beta, \gamma, \delta)$ be a loop. Then L has two-sided inverses.*

Proof. Let $g \in G$. Since $g^*(g^{-1})^* = (g^{-1}g)^* = 1^* = 1$, we have $(g^*)^{-1} = (g^{-1})^*$, and the antiautomorphisms $^{-1}$ and $*$ commute. Let us denote $(g^{-1})^* = (g^*)^{-1}$ by g^{-*}.

We show that for every $\alpha \in \Theta_0$ and $g \in G$, there is $h \in G$ such that $\Delta\alpha(g, h) = g \circ h = 1 = h \circ g = \Delta\alpha(h, g)$. The proof for $gu \in Gu$ is similar.

Assume that $\alpha \in \{\theta_{xy}, \theta_{xy^*}, \theta_{x^*y}, \theta_{x^*y^*}\}$. Then

$$\begin{aligned}
\Delta\theta_{xy}(g, g^{-1}) &= gg^{-1} = 1 = g^{-1}g = \Delta\theta_{xy}(g^{-1}, g),\\
\Delta\theta_{xy^*}(g, g^{-*}) &= g(g^{-*})^* = 1 = g^{-*}g^* = \Delta\theta_{xy^*}(g^{-*}, g),\\
\Delta\theta_{x^*y}(g, g^{-*}) &= g^*g^{-*} = 1 = (g^{-*})^*g = \Delta\theta_{x^*y}(g^{-*}, g),\\
\Delta\theta_{x^*y^*}(g, g^{-1}) &= g^*g^{-*} = 1 = g^{-*}g^* = \Delta\theta_{x^*y^*}(g^{-1}, g)
\end{aligned}$$

show that the two-sided inverse h exists. The case $\alpha \in \{\theta_{yx}, \theta_{y^*x}, \theta_{yx^*}, \theta_{y^*x^*}\}$ is similar. The general case $\alpha \in \Theta_0$ then follows thanks to $g_0 = g_0^* \in Z(G)$. □

References

[1] Bruck R.H. *A Survey of Binary Systems, third printing, corrected.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge **20**, Springer-Verlag, 1971.

[2] De Barros L.G.X., Juriaans S.O. *Some loops whose loop algebras are flexible II.* International Journal of Mathematics, Game Theory and Algebra **8**, N 1, p. 73–80.

[3] Chein O. *Moufang loops of small order.* Mem. Amer. Math. Soc., 1978, **13**, N 197.

[4] Fenyves F. *Extra loops II, On loops with identities of Bol-Moufang type.* Publ. Math. Debrecen, 1969, **16**, p. 187–192.

[5] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.3; Aachen, St Andrews (1999). (Visit http://www-gap.dcs.st-and.ac.uk/~gap).

[6] Pflugfelder H.O. *Quasigroups and Loops: Introduction.* Sigma series in pure mathematics, **7**, Heldermann Verlag Berlin, 1990.

[7] Phillips J.D., Vojtěchovský P. *On C-loops* (to appear in Publ. Math. Debrecen).

[8] Phillips J.D., Vojtěchovský P. *The varieties of loops of Bol-Moufang type* (to appear in Algebra Universalis).

[9] Vojtěchovský P. *On the uniqueness of loops $M(G,2)$.* Comment. Math. Univ. Carolin., 2003, **44**, N 4, p. 629–635.

[10] Vojtěchovský P. *A class of Bol loops with a subgroup of index two.* Comment. Math. Univ. Carolin., 2004, **45**, N 2, p. 371–381.

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