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Some n-ary analogs of the notion of a normalizer of an n-ary subgroup in a group

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Abstract. In this article n-ary analogs of the concept of normalizer of a subgroup of a group are constructed. It is proved that in an n-ary group the role of these n-ary analogs play the concepts of a normalizer and seminormalizer of n-ary subgroup in n-ary group. A connection of these analogs with its binary prototypes is established.

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In this article for any n-ary group $< A, [\ ] >$ we denote by θ_A the introduced by Post [1] equivalence which is defined on a free semigroup F_A by the rule: $(\alpha, \beta) \in \theta_A$ if and only if there exist sequences γ and δ such that $[\gamma\alpha\delta] = [\gamma\beta\delta]$.

For any n-ary group $< A, [\ ] >$ Post defined also the universal enveloping group $A^* = F_A/\theta_A$. In this enveloping group he selected a normal subgroup

$$A_o = \{\theta_A(a_1 \dots a_{n-1}) | a_1, \dots, a_{n-1} \in A\},$$

which is called a corresponding group for the group $< A, [\ ] >$ and he proved that

$$A^*/A_o = \{\theta_A(a)A_o, \theta_A^2(a)A_o, \dots, \theta_A^{n-1}(a)A_o = A_o\}$$

for any $a \in A$, moreover A^*/A_o is a cyclic group of order $n-1$, but generating coset of this cyclic group is an n-ary group that is isomorphic to the n-ary group $< A, [\ ] >$.

We use standard notations. We remark only that one can find definition of n-ary group in the book of V.D. Belousov [2]. In this book the existence of the group A^* is proved and properties of a solution of the equation $[x \underbrace{a \dots a}_{n-1}] = a$ are given. This solution is denoted by the symbol $\bar{a}$ and is called a skew element for the element a.

We recall the normalizer of a subset B in an n-ary group $< A, [\ ] >$ ([3]) is the set

$$N_A(B) = \{x \in A | [x \overset{n-1}{B}] = [\overset{i}{B} x \overset{n-1-i}{B}], \forall i = 1, \dots, n-1\},$$

and a seminormalizer of a subset B in n-ary group $< A, [\ ] >$ ([3]) is the set

$$HN_A(B) = \{x \in A | [x \overset{n-1}{B}] = [\overset{n-1}{B} x]\}.$$

In [3] it is proved that if $< B, [\ ] >$ is an n-ary subgroup of an n-ary group $< A, [\ ] >$, then $< N_A(B), [\ ] >$ and $< HN_A(B), [\ ] >$ are n-ary subgroups in $< A, [\ ] >$ and $N_A(B) \subseteq HN_A(B)$.

For any subset B of an n-ary group $< A, [\ ] >$ it is supposed [4]:

$$B_o(A) = \{\theta_A(\alpha) \in A_o \mid \exists b_1, \dots, b_{n-1} \in B,\ \alpha\theta_A b_1 \dots b_{n-1}\};$$

$$B^*(A) = \{\theta_A(\alpha) \in A^* \mid \exists b_1, \dots, b_i \in B (i \geq 1),\ \alpha\theta_A b_1 \dots b_i\}.$$

It is clear that $B^*(A) \subseteq A^*$, $B_o(A) \subseteq A_o$. In particular, $A^*(A) = A^*$, $A_o(A) = A_o$.

If $< B, [\ ] >$ is an n-ary subgroup of an n-ary group $< A, [\ ] >$, then $B^*(A)$ is a subgroup of the group A^* that is isomorphic to the group B^*, and $B_o(A)$ is a subgroup of the group A_o, which is isomorphic to the group B_o [4].

Lemma 1. *Let $< B, [\ ] >$ be an n-ary subgroup of an n-ary group $< A, [\ ] >$, $b \in B$, $u = \theta_A(x \underbrace{b \dots b}_{n-2}) \in N_{A_o}(B_o(A))$. Then $x \in HN_A(B)$.*

Proof. By condition $u^{-1}vu \in B_o(A)$ for any

$$v = \theta_A(b_o \underbrace{b \dots b}_{n-2}) \in B_o(A), b_o \in B,$$

i.e.

$$\theta_A(\bar{b}\bar{x} \underbrace{x \dots x}_{n-3})\theta_A(b_o \underbrace{b \dots b}_{n-2})\theta_A(x \underbrace{b \dots b}_{n-2}) \in B_o(A),$$

whence

$$\bar{b}\bar{x} \underbrace{x \dots x}_{n-3} b_o \underbrace{b \dots b}_{n-2} x \underbrace{b \dots b}_{n-2} \theta_A b_1 \dots b_{n-1}$$

for some $b_1, \dots, b_{n-1} \in B$. Then

$$[b_o \underbrace{b \dots b}_{n-2} x] = [x \underbrace{b \dots b}_{n-2} b_1 \dots b_{n-1} \bar{b}] \in [x \underbrace{B \dots B}_{n-1}].$$

Since b_o is an arbitrary element of B, then the inclusion is proved

$$[\underbrace{B \dots B}_{n-1} x] \subseteq [x \underbrace{B \dots B}_{n-1}]. \tag{1}$$

If we again apply the condition, then we obtain $uvu^{-1} \in B_o(A)$, i.e.

$$\theta_A(x \underbrace{b \dots b}_{n-2})\theta_A(b_o \underbrace{b \dots b}_{n-2})\theta_A(\bar{b}\bar{x} \underbrace{x \dots x}_{n-3}) \in B_o(A),$$

whence

$$x \underbrace{b \dots b}_{n-2} b_o \underbrace{b \dots b}_{n-2} \bar{b}\bar{x} \underbrace{x \dots x}_{n-3} \theta_A c_1 \dots c_{n-1}$$

for some $c_1, \ldots, c_{n-1} \in B$. Then

$$[x\underbrace{b \ldots b}_{n-2} b_o] = [c_1 \ldots c_{n-1} x] \in [\underbrace{B \ldots B}_{n-1} x],$$

whence

$$[x\underbrace{B \ldots B}_{n-1}] \subseteq [\underbrace{B \ldots B}_{n-1} x]. \tag{2}$$

From (1) and (2) it follows

$$[x\underbrace{B \ldots B}_{n-1}] = [\underbrace{B \ldots B}_{n-1} x].$$

Therefore, $x \in HN_A(B)$. The lemma is proved.

Theorem 1. *If* $< B, [\ ] >$ *is an* n*-ary subgroup in an* n*-ary group* $< A, [\ ] >$, *then*

$$(HN_A(B))_o(A) = N_{A_o}(B_o(A)).$$

Proof. We fix $h \in HN_A(B)$ and choose an arbitrary

$$u = \theta_A(h_o \underbrace{h \ldots h}_{n-2}) \in (HN_A(B))_o(A), h_o \in HN_A(B).$$

If b_o is an arbitrary element, b is a fixed element of the set B, then

$$v = \theta_A(b_o \underbrace{b \ldots b}_{n-2})$$

is an arbitrary element of $B_o(A)$. Since h_o, $\bar{h} \in HN_A(B)$, then

$$u^{-1}vu = \theta_A(\bar{h}\bar{h}_o \underbrace{h_o \ldots h_o}_{n-3})\theta_A(b_o \underbrace{b \ldots b}_{n-2})\theta_A(h_o \underbrace{h \ldots h}_{n-2}) =$$

$$= \theta_A(\bar{h}\bar{h}_o \underbrace{h_o \ldots h_o}_{n-3} b_o \underbrace{b \ldots b}_{n-2} h_o \underbrace{h \ldots h}_{n-2}) = \theta_A(\bar{h}\bar{h}_o \underbrace{h_o \ldots h_o}_{n-3}[b_o \underbrace{b \ldots b}_{n-2} h_o] \underbrace{h \ldots h}_{n-2}) =$$

$$= \theta_A(\bar{h}\bar{h}_o \underbrace{h_o \ldots h_o}_{n-3}[h_o b_1 \ldots b_{n-1}] \underbrace{h \ldots h}_{n-2}) = \theta_A([\bar{h} b_1 \ldots b_{n-1}] \underbrace{h \ldots h}_{n-2}) =$$

$$= \theta_A([b'_1 \ldots b'_{n-1}\bar{h}] \underbrace{h \ldots h}_{n-2}) = \theta_A(b'_1 \ldots b'_{n-1}),$$

where $b_1, \ldots, b_{n-1}, b'_1, \ldots, b'_{n-1} \in B$. Then, $u^{-1}vu \in B_o(A)$, whence $u \in N_{A_o}(B_o(A))$ and the inclusion is proved

$$(HN_A(B))_o(A) \subseteq N_{A_o}(B_o(A)). \tag{3}$$

Since any element $u \in N_{A_o}(B_o(A))$ can be presented in the form

$$u = \theta_A(x\underbrace{b\ldots b}_{n-2}), b \in B,$$

then by Lemma 1 $x \in HN_A(B)$, whence, taking into consideration $B \subseteq HN_A(B)$, we have

$$u = \theta_A(x\underbrace{b\ldots b}_{n-2}) \in (HN_A(B))_o(A).$$

Therefore,

$$N_{A_o}(B_o(A)) \subseteq (HN_A(B))_o(A). \tag{4}$$

From (3) and (4) it follows the needed equality. The theorem is proved.

By remark 2.2.20 [4], corresponding group N_o of n-ary subgroup $< N, [\ ] >$ of n-ary group $< A, [\ ] >$ is isomorphic to subgroup $N_o(A)$ of corresponding group A_o. Therefore from Theorem 1 follows

Corollary 1. *The corresponding Post group of semi-normalizer $< HN_A(B), [\ ] >$ of n-ary subgroup $< B, [\ ] >$ in n-ary group $< A, [\ ] >$ is isomorphic to normalizer of subgroup $B_o(A)$ in corresponding group A_o:*

$$(HN_A(B))_o \simeq N_{A_o}(B_o(A)).$$

Thus Theorem 1 and Corollary 1 establish a correspondence between a semi-normalizer of n-ary subgroup in an n-ary group and its binary prototype in the corresponding Post group.

We notice in [5] a correspondence between a semi-normalizer of an n-ary subgroup in an n-ary group and its binary prototype in the group to which the n-ary group is reducible by Gluskin-Hossu theorem. Namely, the following propositions are proved.

Theorem 2 [5]. *A semi-normalizer of n-ary subgroup $< B, [\ ] >$ in n-ary group $< A, [\ ] >$ coincides with the normalizer of the subgroup $< B_a, ⓐ >$ in the group $< A, ⓐ >$ for any $a \in HN_A(B)$, where $B_a = [\underbrace{B\ldots B}_{n-1}a]$, and the operation ⓐ is defined in the following way*

$$x ⓐ y = [x\bar{a}\underbrace{a\ldots a}_{n-3}y].$$

Corollary 2 [5]. *A semi-normalizer of n-ary subgroup $< B, [\,] >$ in n-ary group $< A, [\,] >$ coincides with the normalizer of the subgroup $< B, ⓐ >$ in group $< A, ⓐ >$ for any $a \in B$.*

We establish now a connection between normalizer of an n-ary subgroup in n-ary group and its binary prototype in enveloping Post group.

Lemma 2 [3]. *If $< B, [\,] >$ is an n-ary subgroup of an n-ary group $< A, [\,] >$, then*

$$N_A(B) = \{x \in A | [xB \underbrace{x \ldots x}_{n-3} \bar{x}] = B\} = \{x \in A | [\bar{x} \underbrace{x \ldots x}_{n-3} Bx] = B\}.$$

Lemma 3. *If $< B, [\,] >$ is an n-ary subgroup of an n-ary group $< A, [\,] >$, $x \in N_A(B)$, then*

$$[\underbrace{x \ldots x}_{i-1} B \underbrace{x \ldots x}_{n-i-1} \bar{x}] = B, \quad [\bar{x} \underbrace{x \ldots x}_{n-i-1} B \underbrace{x \ldots x}_{i-1}] = B$$

for any $i = 1, \ldots, n-1$.

Proof. We prove the second equality. If $i = 1$, then $B = B$. If $i = 2$, then by Lemma 2

$$[\bar{x} \underbrace{x \ldots x}_{n-3} Bx] = B.$$

From the last equality we have

$$[\bar{x} \underbrace{x \ldots x}_{n-3} [\bar{x} \underbrace{x \ldots x}_{n-3} Bx] x] = [\bar{x} \underbrace{x \ldots x}_{n-3} Bx], \quad [\bar{x} \underbrace{x \ldots x}_{n-4} Bxx] = B,$$

whence

$$[\bar{x} \underbrace{x \ldots x}_{n-3} [\bar{x} \underbrace{x \ldots x}_{n-4} Bxx] x] = [\bar{x} \underbrace{x \ldots x}_{n-3} Bx], \quad [\bar{x} \underbrace{x \ldots x}_{n-5} Bxxx] = B.$$

Further we have

$$[\bar{x} x B \underbrace{x \ldots x}_{n-3}] = B, \quad [\bar{x} B \underbrace{x \ldots x}_{n-2}] = B.$$

Therefore, the second equality is true for any $i = 1, \ldots, n-1$.

The first equality is proved similarly. The lemma is proved.

Theorem 3. *If $< B, [\,] >$ is an n-ary subgroup of an n-ary group $< A, [\,] >$, then*

$$(N_A(B))^*(A) = N_{A_*}(B^*(A)).$$

Proof. We fix an element $h \in N_A(B)$ and take any element

$$u = \theta_A(h_o \underbrace{h \dots h}_{i-1}) \in (N_A(B))^*(A),\ h_o \in N_A(B).$$

If b_o is any element and b is a fixed element from B, then

$$v = \theta_A(b_o \underbrace{b \dots b}_{j-1})$$

is any element from $B^*(A)$. Since $h_o, h \in N_A(B)$, then, if we apply Lemma 2, after that Lemma 3, then we obtain

$$u^{-1}vu = \theta_A(\bar{h} \underbrace{h \dots h}_{n-i-1} \bar{h}_o \underbrace{h_o \dots h_o}_{n-3})\theta_A(b_o \underbrace{b \dots b}_{j-1})\theta_A(h_o \underbrace{h \dots h}_{i-1}) =$$

$$= \theta_A(\bar{h} \underbrace{h \dots h}_{n-i-1} \bar{h}_o \underbrace{h_o \dots h_o}_{n-3} b_o \underbrace{b \dots b}_{j-1} h_o \underbrace{h \dots h}_{i-1}) =$$

$$= \theta_A(\bar{h} \underbrace{h \dots h}_{n-i-1} [\bar{h}_o \underbrace{h_o \dots h_o}_{n-3} b_o h_o] \underbrace{[\bar{h}_o \underbrace{h_o \dots h_o}_{n-3} b h_o] \dots [\bar{h}_o \underbrace{h_o \dots h_o}_{n-3} b h_o]}_{j-1} \underbrace{h \dots h}_{i-1}) =$$

$$= \theta_A(\bar{h} \underbrace{h \dots h}_{n-i-1} b'_o \underbrace{b' \dots b'}_{j-1} \underbrace{h \dots h}_{i-1}) =$$

$$= \theta_A([\bar{h} \underbrace{h \dots h}_{n-i-1} b'_o \underbrace{h \dots h}_{i-1}] \underbrace{[\bar{h} \underbrace{h \dots h}_{n-i-1} b' \underbrace{h \dots h}_{i-1}] \dots [\bar{h} \underbrace{h \dots h}_{n-i-1} b' \underbrace{h \dots h}_{i-1}]}_{j-1}) = \theta_A(b''_o \underbrace{b'' \dots b''}_{j-1}),$$

where $b'_o, b', b''_o, b'' \in B$. Therefore, $u^{-1}vu \in B^*(A)$, $u \in N_{A^*}(B^*(A))$ and the following inclusion is proved

$$(N_A(B))^*(A) \subseteq N_{A^*}(B^*(A)). \tag{5}$$

Let $c \in B$ and

$$u = \theta_A(x \underbrace{c \dots c}_{i-1}) = \theta_A(x)\theta_A(\underbrace{c \dots c}_{i-1})$$

be an element of $N_{A^*}(B^*(A))$. Since

$$\theta_A(\underbrace{c \dots c}_{i-1}) \in B^*(A) \subseteq N_{A^*}(B^*(A)),$$

then from the last equality it follows

$$\theta_A(x) \in N_{A^*}(B^*(A)). \tag{6}$$

Thus $\theta_A^{-1}(x)\theta_A(b)\theta_A(x) \in B^*(A)$ for any $b \in B$, whence

$$\theta_A(\bar{x}\underbrace{x\ldots x}_{n-3}bx) \in B^*(A),$$

i.e.

$$[\bar{x}\underbrace{x\ldots x}_{n-3}bx] = b'$$

for some $b' \in B$. Since the element b was an arbitrary element of B, then

$$[\bar{x}\underbrace{x\ldots x}_{n-3}Bx] \subseteq B. \tag{7}$$

From (6) it follows also that $\theta_A(x)\theta_A(b)\theta_A^{-1}(x) \in B^*(A)$ for any $b \in B$, whence

$$[xB\underbrace{x\ldots x}_{n-3}\bar{x}] \subseteq B.$$

From the last inclusion it follows that

$$B \subseteq [\bar{x}\underbrace{x\ldots x}_{n-3}Bx]. \tag{8}$$

Then from (7) and (8) it follows

$$[\bar{x}\underbrace{x\ldots x}_{n-3}Bx] = B,$$

whence, taking in consideration Lemma 2, $x \in N_A(B)$. Then

$$u = \theta_A(x\underbrace{c\ldots c}_{i-1}) \in (N_A(B))^*(A),$$

whence

$$N_{A^*}(B^*(A)) \subseteq (N_A(B))^*(A). \tag{9}$$

From (5) and (9) the required equality follows. The theorem is proved.

By Theorem 2.2.19 [4] universal enveloping Post group N^* of an n-ary subgroup $< N, [\,] >$ of an n-ary group $< A, [\,] >$ is isomorphic to a subgroup $N^*(A)$ of universal enveloping Post group A^*. Therefore from Theorem 3 it follows

Corollary 3. *An universal enveloping Post group of a normalizer $< N_A(B), [\,] >$ of n-ary subgroup $< B, [\,] >$ in n-ary group $< A, [\,] >$ is isomorphic to the normalizer of subgroup $B^*(A)$ in universal enveloping Post group A^*:*

$$(N_A(B))^* \simeq N_{A^*}(B^*(A)).$$

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