Pairwise orthogonality of n-ary operations

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Abstract. The notions of hypercube and of the orthogonality of two hypercubes were arisen in combinatorial analysis. In [11] a connection between n-dimensional hypercubes and algebraic n-ary operations was established. In this article we use an algebraic approach to the study of orthogonality of two hypercubes (pairwise orthogonality). We give a criterion of orthogonality of two finite $k$-invertible n-ary operations, which is used by the research of orthogonality and parastrophe-orthogonality of two n-ary $T$-quasigroups. Some examples are given and connection between admissibility and pairwise orthogonality of n-ary operations is established.

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1 Introduction

It is known that two binary operations $A$ and $B$, given on a set $Q$, are called orthogonal if the system of equations \( \{A(x, y) = a, B(x, y) = b\} \) has exactly one solution for any $a, b \in Q$ (see [1], where such operations are called compatible). Orthogonal binary operations, in particular, orthogonal quasigroups were considered in different works (see, for example, [1–7]).

In [6] H.B. Mann proved that if $A, B, C$ are quasigroups, given on a set $Q$ and satisfying the equality

\[
C(x, B(x, y)) = A(x, y)
\]

for all $x, y \in Q$, then the quasigroups $A$ and $B$ are orthogonal.

V.D. Belousov in [3, Lemma 2] gave the following criterion of orthogonality of two binary quasigroups. Let $A, B$ be binary quasigroups on a set $Q$. Then $A$ and $B$ are orthogonal if and only if the operation $A \cdot B^{-1}$ is a quasigroup, where $(A \cdot B^{-1})(x, y) = A(x, B^{-1}(x, y))$ and $B^{-1}$ is the right inverse quasigroup for $B$ $(B^{-1}(x, z) = y$ if and only if $B(x, y) = z)$.

In the case of n-ary operations there exist distinct versions of orthogonality (they are reflected in [11]) which correspond to different types of orthogonality of $n$-dimensional hypercubes.

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In this article we consider the weakest (for \( n > 2 \)) case of orthogonality of \( n \)-ary operations, namely, pairwise orthogonality (see Definition 1). At first orthogonality of two finite \( k \)-invertible \( n \)-ary operations (pairwise orthogonality) is considered. Then, using the obtained criterion of orthogonality of finite \( n \)-ary operations, we give a definition of pairwise orthogonality for arbitrary \( k \)-invertible \( n \)-operations, in particular, for finite or infinite \( n \)-quasigroups. A connection between admissibility and pairwise orthogonality of \( k \)-invertible \( n \)-ary operations is established. In the last part of the article pairwise orthogonality of \( n \)-ary \( T \)-quasigroups (\( n-T \)-quasigroups), in particular, \( n-T \)-quasigroups which are orthogonal to some their parastrophes are studied. Some examples of such quasigroups are given.

2 Necessary notions and results

We recall some notations, concepts and results which are used in the article. At first remember the following designations and notes from [10]. By \( x^i_1 \) we will denote the sequence \( x_1, x_{i+1}, \ldots, x_j, i \leq j \). If \( j < i \), then \( x^i_1 \) is the empty sequence, \( \overline{1,n} = \{1,2,\ldots,n\} \). Let \( Q \) be a finite or an infinite set, \( n \geq 1 \) be a positive integer and let \( Q^n \) denote the Cartesian power of the set \( Q \).

A \( n \)-ary operation \( A \) (briefly, an \( n \)-operation) on a set \( Q \) is a mapping \( A : Q^n \rightarrow Q \) defined by \( A(x^i_1) = x_{n+1} \), and in this case we write \( A(x^i_1) = x_{n+1} \).

A finite \( n \)-groupoid \((Q,A)\) of order \( m \) is a set \( Q \) with one \( n \)-ary operation \( A \) defined on \( Q \), where \( |Q| = m \).

A \( n \)-ary quasigroup is an \( n \)-groupoid such that in the equality

\[
A(x^i_1) = x_{n+1}
\]

each of \( n \) elements from \( x^{n+1}_1 \) uniquely defines the \((n+1)\)-th element. Usually itself quasigroup \( n \)-operation \( A \) is considered as a \( n \)-quasigroup.

The \( n \)-operation \( E_i, 1 \leq i \leq n \), on \( Q \) with \( E_i(x^i_1) = x_i \) is called the \( i \)-th identity operation (or the \( i \)-th selector) of arity \( n \).

An \( n \)-ary operation \( A \) on \( Q \) is called \( i \)-invertible for some \( i \in \overline{1,n} \) if the equation

\[
A(a^i_1, x_i, a^n_{i+1}) = a_{n+1}
\]

has a unique solution for each fixed \( n \)-tuple \((a^{i-1}_1, a^i_{i+1}, a_{n+1}) \in Q^n \).

For an \( i \)-invertible \( n \)-operation there exists the \( i \)-inverse \( n \)-operation \((^i)A\) defined in the following way:

\[
(^i)A(x^i_1, x_{n+1}, x^{n+1}_{i+1}) = x_i \Leftrightarrow A(x^i_1) = x_{n+1}
\]

for all \( x^{n+1}_1 \in Q^{n+1} \).

It is evident that

\[
A(x^i_1, (^i)A(x^i_1), x^{n+1}_{i+1}) = (^i)A(x^{i-1}_1, A(x^i_1), x^{n+1}_{i+1}) = x_i
\]

and \((^i)[(^i)A] = A\) for \( i \in \overline{1,n} \).
Let $\Omega_n$ be the set of all $n$-ary operations on a finite or an infinite set $Q$. On $\Omega_n$ define a binary operation $\oplus_i$ (the $i$-multiplication) in the following way:

$$(A \oplus_i B)(x^n) = A(x^{i-1}_1, B(x^n_i), x^n_{i+1}),$$

$A, B \in \Omega_n, x^n_i \in Q^n$. Shortly this equality can be written as

$$A \oplus_i B = A(E^i_1, B, E^n_{i+1})$$

where $E_i$ is the $i$-th selector.

In [9] it was proved that $(\Omega_n; \oplus)$ is a semigroup with the identity $E_i$. If $\Lambda_i$ is the set of all $i$-invertible $n$-operations from $\Omega_n$ for some $i \in \overline{1,n}$, then $(\Lambda_i; \oplus)$ is a group.

In this group $E_i$ is the identity, the inverse element of $A$ is the operation $\bigl(i\bigr) A$ for each $i \in \overline{1,n}$.

A $n$-ary quasigroup $(Q, A)$ (or simply $A$), is an $n$-groupoid with an $i$-invertible $n$-operation for each $i \in \overline{1,n}$ [10].

Let $A$ be an $n$-quasigroup and $\sigma \in S_{n+1}$, then the $n$-quasigroup $\sigma A$ defined by

$$\sigma A(x^n_{\sigma 1}) = x_{\sigma(n+1)} \Leftrightarrow A(x^n_1) = x_{n+1}$$

is called the $\sigma$-parastrophe (or simple, parastrophe) of $A$ [10].

For any $n$-operation $A$ there exist the $\sigma$-parastrophes of $A$, where $\sigma(n+1) = n + 1$ (the principal parastrophes). The $i$-inverse operation $\bigl(i\bigr) A$ for $A, i \in \overline{1,n}$, is the $\sigma$-parastrophe defined by the cycle $(i, n+1)$.

Let $(x^n_1)_k$ denote the $(n-1)$-tuple $(x^n_{k-1}, x^n_{k+1}) \in Q^{n-1}$ and let $A$ be an $n$-operation, then the $(n-1)$-operation $A_a$:

$$A_a(x^n_1)_k = A(x^n_{k-1}, a, x^n_{k+1})$$

is called the $(n-1)$-retract of $A$, defined by position $k, k \in \overline{1,n}$, with the element $a$ in this position (with $x_k = a$) [10].

An $n$-ary operation $A$ on $Q$ is called complete if there exists a permutation $\sigma$ on $Q^n$ such that $A = E_1 \sigma$ (that is $A(x^n_1) = E^n_1 \sigma(x^n_1))$. If a complete $n$-operation $A$ is finite and has order $m$, then the equation $A(x^n_1) = a$ has exactly $m^{n-1}$ solutions for any $a \in Q$ [9].

Any $i$-invertible $n$-operation $A, i \in \overline{1,n}$, is complete, but there exist complete $n$-operations, which are not $i$-invertible for each $i \in \overline{1,n}$ [9].

3 Orthogonality of two $n$-ary operations

In the case of $n$-ary operations for $n > 2$ it is possible to consider different versions of orthogonality. The weakest is the notion of the pairwise orthogonality.
Definition 1 [11]. Two $n$-ary operations ($n \geq 2$) $A$ and $B$ given on a set $Q$ of order $m$ are called orthogonal (shortly, $A \perp B$) if the system \{\(A(x^n_i) = a, B(x^n_i) = b\)\} has exactly $m^{n-2}$ solutions for any $a, b \in Q$.

This concept corresponds to two orthogonal $n$-dimensional hypercubes [11, 13]. The following type of orthogonality is strongest.

Definition 2 [8]. An $n$-tuple $<A_1, A_2, \ldots, A_n>$ of $n$-operations on a set $Q$ is called orthogonal if the system \{\(A_i(x^n_i) = a_i\)\} has a unique solution for any $a^n_i \in Q^n$. A set $\Sigma = \{A^t_i\}, t \geq n,$ of $n$-operations is called orthogonal if any $n$-tuple of distinct $n$-operations from $\Sigma$ is orthogonal.

This concept corresponds to an orthogonal $n$-tuple of $n$-dimensional hypercubes [11–13]. Orthogonal $n$-operations and their sets in the sense of Definition 2 were considered in many articles (see, for example, [8, 11–17, 19, 20, 22]).

In [11] intermediate types of orthogonality of $n$-operations and their sets were studied.

Definition 3 [11]. A $k$-tuple $<A^k_1>$, $2 \leq k \leq n$, of distinct $n$-operations on a set $Q$ of order $m$ is called orthogonal if the system \{\(A_i(x^n_i) = a_i\)\} has exactly $m^{n-k}$ solutions for any $a^n_i \in Q^k$. A set $\Sigma = \{A^t_i\}, t \geq k,$ of $n$-operations is called $k$-wise orthogonal if any $k$-tuple of distinct $n$-operations from $\Sigma$ is orthogonal.

The following connection exists between different considered types of orthogonality.

Theorem 1 [11]. If a set $\Sigma = \{A^t_i\}, t \geq k,$ of finite $n$-operations is $k$-wise orthogonal, then $\Sigma$ is $l$-wise orthogonal for any $l, 2 \leq l \leq k$.

Thus, every pair of different $n$-ary operations from an orthogonal $n$-tuple is orthogonal.

Let $A_1, A_2, \ldots, A_n$ be $n$-operations given on a set $Q$. In [14] it is proved that a $n$-tuple $<A^t_i>$ of $n$-operations is orthogonal if and only if the mapping $\bar{\theta} : x^n_1 \rightarrow (A_1(x^n_1), A_2(x^n_1), \ldots, A_n(x^n_1)) = (A_1, A_2, \ldots, A_n)(x^n_1)$ is a permutation on $Q^n$.

In [1] V.D. Belousov proved that a binary operation $A$ has an operation which is orthogonal to $A$ (an orthogonal mate) if and only if $A$ is a complete operation. This is valid and for finite $n$-operations.

Proposition 1. A finite $n$-operation $A$ has an orthogonal mate if and only if $A$ is complete.

Proof. By Proposition 5 of [11] $A$ is a complete $n$-operation if and only if it is a component of some permutation $\bar{\mathcal{F}} = (A, B^n_1)$ on $Q^n$, where $<A, B^n_1>$ is an orthogonal $n$-tuple. By Theorem 1 $A \perp B_i$ for any $i \in \Gamma, n-1$. 
Conversely, if $B$ is an orthogonal mate for $A$, that is $A \perp B$, then by Corollary 4 of [11] the pair $A,B$ can be embedded in an orthogonal $n$-tuple of $n$-operations and by Proposition 5 of [11] $A$ is a complete $n$-operation. \hfill $\square$

Now we shall consider orthogonality of $k$-invertible $n$-operations for some fixed $k, 1 \leq k \leq n$. For them the following criterion is valid.

**Theorem 2.** Let $k$ be a fixed number from $\overline{1,n}$. Two finite $k$-invertible $n$-operations $A$ and $B$ on a set $Q$ are orthogonal if and only if the $(n-1)$-retract $C_a$ of the $n$-operation $C = B \oplus \langle k \rangle A$, defined by $x_k = a$, is complete for every $a \in Q$.

**Proof.** We shall prove this statement when $k = n$ for the sake of simplicity. For the rest $k \in \overline{1,n-1}$ the proof is similar.

Let $a$ be an arbitrary element of $Q$, $|Q| = m$ and the $(n-1)$-retract $C_a$ by $x_n = a$ of $n$-operation $C = B \oplus \langle n \rangle A$ is complete for any $a \in Q$. Then the equation 

\[
C_a(x_1^{n-1}) = C(x_1^{n-1}, a) = (B \oplus \langle n \rangle A)(x_1^{n-1}, a) = B(x_1^{n-1}, (n)A(x_1^{n-1}, a)) = b
\]

has $m^{(n-1)-1}$ solutions for any $a,b \in Q$. From the last equation we have 

\[
(B(x_1^{n-1}, b) = (n)A(x_1^{n-1}, a) = z,
\]

whence it follows that the system \{\(A(x_1^{n-1}, z) = a, B(x_1^{n-1}, z) = b\)\} has $m^{n-2}$ solutions. Thus, $A \perp B$.

Conversely, let $A \perp B$, that is the system \{\(A(x_1^n) = a, B(x_1^n) = b\)\} has $m^{n-2}$ solutions for any $a,b \in Q$. From the first equality we have $x_n = (n)A(x_1^{n-1}, a)$ and then the equation $B(x_1^{n-1}, (n)A(x_1^{n-1}, a)) = b$ or $C_a(x_1^{n-1}) = (B \oplus \langle n \rangle A)(x_1^{n-1}, a) = b$ has $m^{n-2}$ solutions for any $a,b \in Q$. Therefore, the $(n-1)$-retract of $B \oplus \langle n \rangle A$, defined by any $a \in Q$, is complete. \hfill $\square$

For the binary case from Theorem 2 we have the following

**Corollary 1.** Two finite invertible from the right (that is 2-invertible) binary operations $A,B$ on $Q$ are orthogonal if and only if the operation $C(x,y) = (A \cdot B^{-1})(x,y) = A(x,B^{-1}(x,y))$ is a quasigroup.

**Proof.** The operation $C = B \cdot A^{-1}(= B \oplus \langle 2 \rangle A)$ is always invertible from the right.

If the operation $C_a x = C(x,a)$ is complete for any $a \in Q$, that is the equation \(C(x,a) = b\) has exactly $m^{2-2} = 1$ solutions for any $a,b \in Q$, then the operation $C$ is invertible from the left (that is 1-invertible). Thus, $C$ is a quasigroup.

Conversely, if $C$ is a quasigroup, then any its (unary) retract is complete (that is a permutation). \hfill $\square$

From this corollary the criterion of V.D.Belousov [3, Lemma 2] for finite binary quasigroups follows.

**Proposition 2.** If $A$ and $B$ are $k$-invertible $n$-operations on a set $Q$ for some $k \in \overline{1,n}$, then the following equalities are equivalent: $C = B \oplus \langle k \rangle A$, $C \oplus A = B$, $C = B \oplus \langle k \rangle A$, $C \oplus A = B$, $C = B \oplus \langle k \rangle A$, $C \oplus A = B$, $C = B \oplus \langle k \rangle A$, $C \oplus A = B$, $C = B \oplus \langle k \rangle A$, $C \oplus A = B$, $C = B \oplus \langle k \rangle A$, $C \oplus A = B
\[ A = (k) C \oplus B, \quad C \oplus A \oplus (k) B = E_k, \quad (k) A \oplus (k) C \oplus B = E_k, \quad A \oplus (k) B \oplus C = E_k, \]
\[ (k) C \oplus B \oplus (k) A = E_k. \]

**Proof.** It is easy to see taking into account that all \( k \)-invertible \( n \)-operations on \( Q \) form a group with the identity \( E_k \) with the respect to the \( k \)-multiplication of \( n \)-operations.

**Remark 1.** If \( A \) and \( B \) are \( n \)-quasigroups, then they are \( k \)-invertible for any \( k \in \mathbb{N} \), so \( A \perp B \) if and only if for some \( k \in \mathbb{N} \), the \((n-1)\)-retract \( C_a \) of \( C = B \oplus (k) A \), defined by \( x_k = a \), is complete for any \( a \in Q \). If that holds for some fixed \( k \in \mathbb{N} \), then the \((n-1)\)-retract of \( C_1 = B \oplus (l) A \), defined by \( x_l = a \), is also complete for any \( l \in \mathbb{N} \) and any \( a \in Q \).

From Proposition 2 and Theorem 2 we have the following

**Corollary 2.** If \( A \) and \( B \) are finite \( n \)-quasigroups on \( Q \), \( C = B \oplus (k) A \) and \( A \perp B \), then \( C \perp (k) A \), \((k) C \perp (k) B \) for any \( k \in \mathbb{N} \).

**Proof.** \( C \perp (k) A \) \((k) C \perp (k) B \) follows from the second (from the third) equality of Proposition 2 and Theorem 2, since \( A \) and \( B \) are \( n \)-quasigroups and so any \((n-1)\)-retract of \( B \) \((A) \) is an \((n-1)\)-quasigroup which is always complete. Further use Remark 1.

Using the criterion of orthogonality of two finite \( n \)-operations from Theorem 2 we can define a pairwise orthogonality of arbitrary \( k \)-invertible \( n \)-operations (finite or infinite).

**Definition 4.** Two \( k \)-invertible \( n \)-operations \( A \) and \( B \), given on an arbitrary set \( Q \), are called orthogonal if the \((n-1)\)-retract of the \( n \)-operation \( B \oplus (k) A \), defined by \( x_k = a \), is complete for each \( a \in Q \).

As it was noted above, an \( n \)-operation \( A \) on \( Q \) is called complete if there exists a permutation (a bijection) \( \varphi \) on \( Q^n \) such that \( A = E_1 \varphi \). In the case of Definition 4 each \((n-1)\)-retract

\[ C_a(x^n_1) = C(x^{n-1}_1, a, x^n_{k+1}) = B(x^{n-1}_1, (k) A(x^{n-1}_1, a, x^n_{k+1}), x^n_{k+1}) \]

is complete, that is \( C_a = E_1 \varphi \) for some permutation \( \varphi \) of \( Q^{n-1} \).

**Remark 2.** Note that for binary case \((n=2)\) Definition 4 is equivalent to the usual definition of orthogonality of two 1- or 2-invertible operations.

Indeed, let \( A, B \) be 2-invertible binary operations on a set \( Q \) and \( A \perp B \), that is the system \( \{ A(x, y) = a, B(x, y) = b \} \) has a unique solution for any \( a, b \in Q \). Then \( A^{-1}(x, a) = y \) and the equation \( B(x, A^{-1}(x, a)) = b \) has a unique solution \( x \) for any
a, b ∈ Q, that is \( C_a(x) = B(x, R_a x) = E \varphi_a x = \varphi_a x \) where \( R_a x = A^{-1}(x, a) \), \( E \) is the selector in the 1-ary case \( (E x = x x = x) \) and so \( \varphi_a \) is a bijection \( Q \) on \( Q \). Thus, \( C_a = \varphi_a \) is a complete 1-ary (unary) operation for any \( a \in Q \).

Conversely, if \( C_a = \varphi_a \) is a bijection for any \( a \in Q \), then the equation \( B(x, A^{-1}(x, a)) = b \) has a unique solution for any \( a, b \in Q \) and the system \( \{ A(x, y) = a, B(x, y) = b \} \) has a unique solution.

For 1-invertible binary operations the proof is similar.

Now we consider a connection between orthogonality of two \( n \)-operations and their admissibility.

It is known that a binary quasigroup \( (Q, \cdot) \) is called admissible if it has a complete permutation (a bijection) (or a transversal).

A permutation \( \theta \) on \( Q \) is called complete for a quasigroup \( (Q, \cdot) \) if the mapping \( \theta': \theta' x = x \cdot \theta x \) is a permutation on \( Q \). All elements \( \theta' x, x \in Q \), are different and form a transversal which is defined by the permutation \( \theta \) [5].

A binary quasigroup of order \( m \) has an orthogonal mate if and only if it has \( m \) disjoint transversals \( \theta'_1, \theta'_2, \ldots, \theta'_m \) (or \( m \) disjoint complete permutations \( \theta_1, \theta_2, \ldots, \theta_m \), that is \( \theta'_i x \neq \theta'_j x, i \neq j \), for any \( x \in Q \) [5].

Using the criterion of Corollary 4 of orthogonality of binary 2-invertible (or 1-invertible) operations \( A \) and \( B \) on \( Q \) of order \( m \), it is easy to find in this case \( m \) disjoint transversals.

Indeed, if \( A \perp B \), then the operation \( A \cdot B^{-1} ((A \cdot B^{-1})(x, y) = A(x, B^{-1}(x, y))) \) is a quasigroup. By \( y = a \) we have \( A(x, B^{-1}(x, a)) = A(x, R_a x) = C_a x \) and \( C_a \) is a permutation where \( R_a : R_a x = B^{-1}(x, a) \) is also permutation. Thus, in \( A \) there exist \( m \) disjoint complete permutations \( \{ R_a, a \in Q \} \) which define \( m \) disjoint transversals \( \{ C_a, a \in Q \} \).

In [20, 22] the admissibility of \( n \)-quasigroups and their connection with orthogonality were considered. By analogue with \( n \)-quasigroups (see [21]) the following definition of admissible \( n \)-operations was given.

**Definition 5.** An \( n \)-operation \( B \) given on a set \( Q \) is called admissible if for some \( k \), \( 1 \leq k \leq n \), on \( Q \) there exists an \((n - 1)\)-operation \( A \) such that the \((n - 1)\)-operation \( C \):

\[
C(x_1^n)_k = B(x_1^{k-1}, A(x_1^n)_k, x_{k+1}^n)
\]

is complete. In this case the \((n - 1)\)-operation \( C \) is called a \( k \)-transversal of the \( n \)-operation \( B \), defined by the \((n - 1)\)-operation \( A \).

The \( n \)-tuples \((x_1^{k-1}, A(x_1^n)_k, x_{k+1}^n)\) are positions of elements of a \( k \)-transversal \( C \). The values \( C(x_1^n)_k \), when \((n - 1)\)-tuples \((x_1^n)_k \) run through \( Q^{n-1} \), are the elements of the \( k \)-transversal \( C \).

Two \( k \)-transversals of an \( n \)-operation \( B \) defined by \((n - 1)\)-operations \( A_1 \) and \( A_2 \) are called disjoint if \( A_1(x_1^n)_k \neq A_2(x_1^n)_k \) for all \((x_1^n)_k \in Q^{n-1} \).

From Theorem 2 it follows
Proposition 3. Let $A, B$ be finite $k$-invertible $n$-operations given on a set $Q$ of order $m$, $A \perp B$. Then the $(n-1)$-operations $(k)A_a(x_1^n) = (k)A(x_1^{k-1}, a, x_n^{k+1}), a \in Q$, define $m$ pairwise disjoint $k$-transversals in $B$.

Proof. By Theorem 2 $A \perp B$ if and only if the $(n-1)$-operation

$$C_a(x_1^n)_k = B(x_1^{k-1}, (k)A(x_1^{k-1}, a, x_n^{k+1}), x_n^{k+1}) = B(x_1^{k-1}, (k)A_a(x_1^n)_k, x_n^{k+1})$$

is complete for any $a \in Q$. Thus, by Definition 5 the operations $(k)A_a, a \in Q$, define $m$ transversals $C_a, a \in Q$. It is evident that $(k)A_a(x_1^n)_k \neq (k)A_b(x_1^n)_k$, if $a \neq b$, since $A$ is a $k$-invertible $n$-operation. Moreover, in this case we have $C_a(x_1^n)_k \neq C_b(x_1^n)_k$ by virtue of $k$-invertibility of the $n$-operation $B$. 

Let $A, B$ be two $n$-operations on a set $Q$. Recall that an $n$-operation $B$ is called isotopic to an $n$-operation $A$ if there exists an $(n+1)$-tuple $T = (\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma)$ of permutations (bijects) of $Q$ such that $B(x_1^n) = \gamma^{-1}A(\alpha_1, x_1, \alpha_2x_2, \ldots, \alpha_n x_n)$ for all $x_1^n \in Q^n$ (shortly, $B = A^T[10]$).

It is easy to prove that the following statement is valid.

Proposition 4. Any $n$-operation $B$ which is isotopic to a complete finite or infinite $n$-operation $A$ is also complete.

Proof. Let $A$ be a complete $n$-operation on a set $Q$, $B = A^T, T = (\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma)$, then $A = E_1:\psi$ for some permutation $\psi = (C_1, C_2, \ldots, C_n)$ (where the $n$-tuple $< C_1, C_2, \ldots, C_n >$ of $n$-operations is orthogonal) and

$$B(x_1^n) = \gamma^{-1}A(\alpha_1, \alpha_2x_2, \ldots, \alpha_n x_n) =$$

$$\gamma^{-1}E_1\psi(\alpha_1, \alpha_2x_2, \ldots, \alpha_n x_n) = \gamma^{-1}E_1(C_1, C_2, \ldots, C_n)(\alpha_1, \alpha_2x_2, \ldots, \alpha_n x_n) =$$

$$E_1(\gamma^{-1}\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_n)(x_1^n)$$

where $\bar{C}_i(x_1^n) = C_i(\alpha_1, \alpha_2x_2, \ldots, \alpha_n x_n)$. It is easy to see that the $n$-tuple $< \gamma^{-1}\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_n >$ is also orthogonal. Thus, $B = E_1:\bar{\psi}$, where

$$\bar{\psi} = (\gamma^{-1}\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_n).$$

From Proposition 1 and Proposition 3 we obtain the following

Corollary 3. If a finite $n$-operation $A$ has an orthogonal mate and $B = A^T, T = (\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma)$, then $B$ has an orthogonal mate too.
4 Pairwise orthogonal n – T-quasigroups

Below we shall consider in more detail orthogonality of two \( n \)-ary \( T \)-quasigroups (briefly, \( n - T \)-quasigroups) which are closely connected with finite or infinite abelian groups and generalize the known binary \( T \)-quasigroups.

**Definition 6 [18].** An \( n \)-quasigroup \( (Q, A) \) is called an \( n - T \)-quasigroup if there exist a binary abelian group \( (Q, +) \), its automorphisms \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and an element \( a \in Q \) such that

\[
A(x^n_1) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + a
\]

for all \( x^n_1 \in Q^n \).

Let \( k \in \mathbb{I}, n \), then the \( k \)-inverse \( n \)-operation \( (k) A \) for an \( n - T \)-quasigroup \( A \) of (1) has the form

\[
(k) A(x^n_i) = \alpha_k^{-1} (-\alpha_1 x_1 - \alpha_2 x_2 - \cdots - \alpha_{k-1} x_{k-1} + x_k - \alpha_{k+1} x_{k+1} - \cdots - \alpha_n x_n - a)
\]

and is also \( n - T \)-quasigroup, since the mapping \( I : Ix = -x \) is an automorphism in an abelian group.

**Proposition 5.** Let \( (Q, A) \) and \( (Q, B) \) be two finite \( n - T \)-quasigroups over a group \( (Q, +) \) of odd order, \( A(x^n_1) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k + \alpha_{k+1} x_{k+1} + \cdots + \alpha_n x_n, \\ B(x^n_1) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \beta_{k+1} x_{k+1} + \cdots + \beta_n x_n, \)

where \( \beta_i = 2\alpha_i \) for each \( i \in \mathbb{I}, n \), \( i \neq k \), then \( C = B \oplus (k) A = A, B = A \oplus A \) and \( A \perp (A \oplus A), A \perp (k) A, (k) (A \oplus A) \perp (k) A \).

**Proof.** In this case \( \beta_i = 2\alpha_i \) is an automorphism for any \( i \in \mathbb{I}, n \), \( i \neq k \), since in a group \( (Q, +) \) of odd order the mapping \( x \rightarrow 2x \) is a permutation. Find the form of the \( n \)-operation \( C \) using (2): \( C(x^n_1) = (B \oplus (k) A)(x^n_1) = B(x^{k-1}_1 \oplus (k) A(x^n_1), x^n_{k+1} = 2\alpha_1 x_1 + 2\alpha_2 x_2 + \cdots + 2\alpha_k x_k - \alpha_1 x_1 - \alpha_2 x_2 - \cdots - \alpha_{k-1} x_k - x_k - \alpha_{k+1} x_{k+1} - \cdots - \alpha_n x_n + 2\alpha_{k+1} x_{k+1} + \cdots + 2\alpha_n x_n = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{k-1} x_{k-1} + x_k + \cdots + \alpha_{k+1} x_{k+1} + \cdots + \alpha_n x_n = A(x^n_k) \). Any \( (n - 1) \)-retract of \( C = A \) is a \( (n - 1) \)-quasigroup, so is complete and \( A \perp B \) by Definition 4 (or by Theorem 2). Since \( C = B \oplus (k) A = A \), then \( B = A \oplus A \). Orthogonality of the rest \( n \)-operations pointed in the proposition follows from Corollary 2.

The following useful criterion of orthogonality of two \( n - T \)-quasigroups is valid.

**Theorem 3.** Two \( n - T \)-quasigroups \( (Q, A) \) and \( (Q, B) \) where

\[
A(x^n_1) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + a, B(x^n_1) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + b
\]
are orthogonal if and only if the \((n-1)\)-operation \(\overline{C}\):
\[
\overline{C}(x^n_i)_k = \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_{k-1} x_{k-1} + \gamma_{k+1} x_{k+1} + \cdots + \gamma_n x_n
\]
is complete, where
\[
\gamma_i x_i = \beta_i x_i - \beta_k \alpha_k^{-1} \alpha_i x_i = (\beta_i - \beta_k \alpha_k^{-1} \alpha_i) x_i, \quad i \in \overline{1,n}, i \neq k.
\]

**Proof.** By Remark 1 and Definition 4 we need to prove that \(\overline{C}\) is complete if and only if the \((n-1)\)-retract \(C\) of \(C = B \oplus (k)A\) defined by \(x_k = c\), for some \(k \in \overline{1,n}\) and \(c \in Q\), is complete. Using (2) we have \(C(x^n_i) = (B \oplus (k)A)(x^n_i) = B(x_1^{k-1}, (k)A(x^n_i), x^n_{k+1}) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{k-1} x_{k-1} + \beta_k \alpha_k^{-1}(-\alpha_1 x_1 - \alpha_2 x_2 - \cdots - \alpha_k x_k + \cdots - \alpha_{k+1} x_{k+1} - \cdots - \alpha_n x_n) + \beta_{k+1} x_{k+1} + \cdots + \beta_n x_n + b = (\beta_1 - \beta_k \alpha_k^{-1} \alpha_1)x_1 + (\beta_2 - \beta_k \alpha_k^{-1} \alpha_2)x_2 + \cdots + (\beta_k - \beta_k \alpha_k^{-1} \alpha_k) x_k + \cdots + (\beta_n - \beta_k \alpha_k^{-1} \alpha_n)x_n + \beta_k \alpha_k^{-1} a + b = (\overline{C}(x^n_i)_k + \beta_k \alpha_k^{-1} x_k - \beta_k \alpha_k^{-1} a + b) \ (\text{see (3)}).

Let \(x_k = c\) be an arbitrary element of \(Q\), then we have
\[
C(x_1^{k-1}, c, x^n_{k+1}) = C_c(x^n_i)_k = \overline{C}(x^n_i)_k + d = R_d \overline{C}(x^n_i)_k,
\]
where \(d = \beta_k \alpha_k^{-1} c - \beta_k \alpha_k^{-1} a + b\), \(R_d x = x + d\). Thus, the \((n-1)\)-retract \(C(x^n_i)_k\) of \(C\), defined by \(x_k = c\), is isotopic to the \((n-1)\)-ary operation \(\overline{C}\): \(C_c = \overline{C}^T\), \(T = (\varepsilon, \varepsilon, \ldots, \varepsilon, R_d^{-1})\) (\(\varepsilon\) denotes the identity permutation on \(Q\)) and by Proposition 4 \(C_c\) is complete if and only if \(\overline{C}\) is complete. \(\Box\)

**Remark 3.** Note that if the conditions of Theorem 3 hold for some \(k \in \overline{1,n}\), then they hold for any \(k \in \overline{1,n}\) (see Remark 1 for \(n\)-quasigroups).

**Corollary 4.** If in Theorem 3 \(\gamma_i = \beta_i - \beta_k \alpha_k^{-1} \alpha_i\) is a permutation for some \(i_0 \in \overline{1,n}\), \(i_0 \neq k\), then \(A \perp B\).

**Proof.** In this case the \((n-1)\)-operation \(\overline{C}\) of (3) is \(i_0\)-invertible, so it is complete. \(\square\)

From Theorem 3 and Corollary 4 a number of useful statements follow.

**Corollary 5.** Let in Theorem 3 \(\alpha_k = \beta_k\) for some \(k \in \overline{1,n}\). Then
\[
\begin{align*}
\text{(i)} \ &\text{if } \beta_i - \alpha_i \text{ is a permutation for some } i_0 \in \overline{1,n}, \ i_0 \neq k, \text{ then } A \perp B; \\
\text{(ii)} \ &\text{if } (Q, +) \text{ is an (abelian) group of odd order and } \beta_i = 2\alpha_i \text{ for some } i_0 \in \overline{1,n}, \ i_0 \neq k, \text{ then } A \perp B.
\end{align*}
\]

**Proof.** By \(\alpha_k = \beta_k\) we have \(\beta_k \alpha_k^{-1} = \varepsilon\) and \(\gamma_i = \beta_i - \alpha_i\) for all \(i \in \overline{1,n}\), \(i \neq k\). In (i) use Corollary 4. Item (ii) is a particular case of (i), since \(\beta_i = 2\alpha_i\) is a permutation (and so an automorphism) in a group of odd order. \(\square\)
Corollary 6. Let $\Sigma = \{A_1^1\}$ be a set of $n-T$-quasigroups on a set $Q$ over the same group $(Q, +)$:

$$A_i(x_i^a) = \alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in}x_n, i \in \overline{1, l},$$

(4)

where $\alpha_{ik} = \alpha_{2k} = \cdots = \alpha_{lk}$ for some $k \in \overline{1, n}$. If for all $i, j \in \overline{1, l}$, $i \neq j$ there exists one number $s \in \overline{1, n}$, $s \neq k$ such that $\alpha_{is} - \alpha_{js}$ is a permutation, then the set $\Sigma$ is pairwise orthogonal.

**Proof.** In this case $A_i \perp A_j$ for each $i, j \in \overline{1, l}$, $i \neq j$ by virtue of item (i) of Corollary 5 since $\alpha_{ik} = \alpha_{jk}$ for all $i, j \in \overline{1, l}$, $i \neq j$. □

**Example 1.** Let $\Sigma = \{A_p^{p-1}\}$ be a set of $n-T$-quasigroups over a group $(Q, +)$ (with the identity 0) of a prime order $p$, where $n-T$-quasigroups of (4) have the form

$$A_1(x_1^a) = a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} + ax_n,$$

$$A_2(x_1^a) = a_2x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} + ax_n,$$

$$\ldots$$

$$A_{p-1}(x_1^a) = a_{p-1}x_1 + a_{p-2}x_2 + \cdots + a_{n-1}x_n + ax_n,$$

$\alpha_{i1}x = a_1x, \alpha_{ij} \neq a_j$ if $i \neq j, i, j \in \overline{1, p-1}, \alpha_{ik}x = a_{ik}x$, if $k \neq 1$ and $k \neq n, \alpha_{in} = a, i \in \overline{1, p-1}, a, a_1, a_{ik} \in Q \setminus 0$ for all $i \in \overline{1, p-1}$.

By Corollary 6 the set $\Sigma$ is pairwise orthogonal by $s = 1$ since $a_1 - a_{j_1} = a_j$, $a_j \neq 0$, so the mapping $x \rightarrow (a_i - a_j)x$ is a permutation by $i \neq j$ and by $\alpha_{1n}x = \alpha_{2n}x = \cdots = a_{p-1,n}x = ax$ (here $k = n$).

Further we shall establish some conditions for orthogonality of an $n-T$-quasigroup to some its parastrophes, using Theorem 3. Parastrophe-orthogonality of binary quasigroups and minimal identities connected with such orthogonality were in detail studied by V.D. Belousov in [4].

At first we recall that an automorphism $\alpha$ of a group $(Q, +)$ is called complete if the mapping $x \rightarrow x + \alpha x$ is a permutation of $Q$, that is if $\alpha$ is a complete permutation [5].

**Proposition 6.** If an $n-T$-quasigroup $(Q, A)$, $A(x_i^a) = \alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n + a$ where $\alpha_l$ is a complete automorphism of the group $(Q, +)$ for some $l \in \overline{1, n}$, then $A \perp (l)A$.

**Proof.** Using expression (2) for $(l)A$ and taking in Theorem 3 $k \neq l$, $B = (l)A$ we obtain $\beta_l = \alpha_l^{-1}$ and $\beta_k = -\alpha_l^{-1} \alpha_k$. Then $\gamma_l = \beta_l - \beta_k \alpha^{-1}_k \alpha_l = \alpha_l^{-1} + \alpha_l^{-1} \alpha_k \alpha^{-1}_k \alpha_l = \alpha_l^{-1}(\varepsilon + \alpha_l)$ is a permutation and so $A \perp (l)A$ by Corollary 4. □

**Corollary 7.** An $n-T$-quasigroup $(Q, A)$ over a group $(Q, +)$ with $A(x_i^a) = ax_1 + \alpha x_2 + \cdots + \alpha x_n + a$, where $\alpha$ is a complete automorphism of $(Q, +)$, is orthogonal to $(l)A$ for each $l \in \overline{1, n}$. Moreover, if, in addition, $n \geq 3$, then the set $\Sigma = \{A, (1)A, \ldots, (n)A\}$ is pairwise orthogonal.
Proof. The first statement follows immediately from Proposition 6. Prove that 

\( (i) A \perp (j) A \) for each \( i, j \in \overline{1, n}, i \neq j \). By (2) we have

\[
(i) A(x^n_i) = \alpha^{-1}(-\alpha x_1 - \alpha x_2 - \cdots - \alpha x_{i-1} + x_i - \alpha x_{i+1} - \cdots - \alpha x_n - a) = \\
-x_1 - x_2 - \cdots - x_{i-1} + \alpha^{-1}x_i - x_{i+1} - \cdots - x_n - \alpha^{-1}a = \\
x_1 + x_2 + \cdots + nx_{i-1} + \alpha^{-1}x_i + x_{i+1} + \cdots + nx_n + b = \\
\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n + b, b = -\alpha^{-1}a,
\]

\[
(j) A(x^n_i) = Ix_1 + Ix_2 + \cdots + Ix_{j-1} + \alpha^{-1}x_j + Ix_{j+1} + \cdots + Ix_n + b = \\
\beta_1x_1 + \beta_2x_2 + \cdots + \beta_nx_n + b.
\]

Since \( i \neq j \) and \( n \geq 3 \) then there exists \( k \in \overline{1, n} \) such that \( \alpha_k = \beta_k \) (\( k \neq i, j \)). In this case we have \( \alpha^{-1}x_j - (Ix_j) = (\alpha^{-1} + \varepsilon)x_j \), so the map \( \beta_j - \alpha_j = \alpha^{-1} + \varepsilon \) is a permutation since \( \alpha \) is a complete automorphism. By item (i) of Corollary 5 (if \( i_0 = j \)) \( (i) A \perp (j) A \). Taking into account that \( A \perp (i) A \) for any \( l \in \overline{1, n} \), we obtain that \( \Sigma \) is a pairwise orthogonal set.

From Corollary 7, in particular, it follows that if \( A \) is an \( n - T \)-quasigroup \( (n \geq 3) \) \((Q, A)\): \( A(x^n_1) = x_1 + x_2 + \cdots + x_n + a \) over a group of odd order, then \( \Sigma = \{A, (1)A, \ldots, (n)A\} \) is pairwise orthogonal set, since the identity automorphism \( \varepsilon \) in such group is complete.

A direct corollary of Theorem 3 for an \( n - T \)-quasigroup which is orthogonal to some its principal \( \sigma \)-parastrophe is the following

Proposition 7. Let \( (Q, A) \) be an \( n - T \)-quasigroup over a group \( (Q, +) \): \( A(x^n_1) = \alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n + a, \sigma(n + 1) = n + 1 \). Then \( A \perp \sigma A \) if and only if for some \( k \in \overline{1, n} \) the \((n-1)\)-operation \( \overline{C} \):

\[
\overline{C}(x^n_1)_k = (\alpha_1 - \alpha_2\alpha_k^{-1}\alpha_1)x_1 + (\alpha_2 - \alpha_3\alpha_k^{-1}\alpha_2)x_2 + \cdots + (\alpha_{\sigma(k-1)} - \alpha_{\sigma(k-1)}\alpha_k^{-1}\alpha_{k-1})x_{k-1} + \\
(\alpha_{\sigma(k+1)} - \alpha_{\sigma(k+1)}\alpha_k^{-1}\alpha_{k+1})x_{k+1} + \cdots + (\alpha_{\sigma n} - \alpha_{\sigma n}\alpha_k^{-1}\alpha_n)x_n
\]

is complete.

Proof. By the definition of a principal parastrophe \( \sigma A \) (\( \sigma(n + 1) = n + 1 \)) of \( A \)

\[
\sigma A(x^n_1) = A(x^{n-1}_{\sigma^{-1}1}) = \alpha_1x_{\sigma^{-1}1} + \alpha_2x_{\sigma^{-1}2} + \cdots + \alpha_nx_{\sigma^{-1}n} + a = \\
\beta_1x_1 + \beta_2x_2 + \cdots + \beta_nx_n + a,
\]

where \( \beta_ix_i = \alpha_{\sigma i}x_i, i \in \overline{1, n} \). Further use Theorem 3 with \( \gamma_i = \beta_i - \beta_k\alpha_k^{-1}\alpha_i = \alpha_{\sigma i} - \alpha_{\sigma k}\alpha_k^{-1}\alpha_i \).

Corollary 8. If \( (Q, A) \) is an \( n - T \)-quasigroup, \( n \geq 3 \), \( A(x^n_1) = \alpha_1x_1 + \alpha_2x_2 + \\
\cdots + \alpha_nx_n + a, \sigma(n + 1) = n + 1, \sigma k = k \) for some \( k \in \overline{1, n} \) and \( \alpha_{\sigma i_0} - \alpha_{i_0} \) is a
permutation for some \( i_0 \in \overline{1,n}, i_0 \neq k \), then \( A \perp \sigma A \). If, in addition, \((Q,+)\) has odd order and \( \alpha_{i_0} = 2\alpha_{i_0} \), then \( A \perp \sigma A \).

**Proof.** We have \( \sigma A(x^n) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{k-1}x_{k-1} + \alpha_k x_k + \beta_{k+1} x_{k+1} + \cdots + \beta_n x_n + a \), where \( \beta_i = \alpha_{\sigma i} \), so \( \beta_k = \alpha_k \), as \( \sigma k = k \) and we can use items (i) and (ii) of Corollary 5, respectively.

Note that for \( n = 2 \) we have \( \sigma = \varepsilon \) (that is \( \sigma A = A \)) by the conditions of this corollary (if \( \sigma = 3, \sigma 1 = 1 \), then \( \sigma = 1, \sigma 2, 3 \Rightarrow \sigma 2 = 2 \)).

**Example 2.** Let \((Q,A)\) be an \( n-T \)-quasigroup, \( n \geq 3 \), over a group of odd order with \( A(x^n) = \alpha_1 x_1 + 2\alpha_1 x_2 + \cdots + \alpha_n x_n + a \), \( i_0 = 1 \), \( \sigma(n + 1) = n + 1, \sigma 1 = 2 \) and \( \sigma k = k \) for some \( k \in \overline{1,n}, k \neq 1 \). Then \( \alpha_{i_1} - \alpha_1 = \alpha_2 - \alpha_1 = 2\alpha_1 - \alpha_1 = \alpha_1 \). By Corollary 8 \( A \perp \sigma A \) for any \( \alpha_i \neq 0, i \in \overline{1,n}, i \neq 2 \).

**Corollary 9.** If \((Q,A)\) is an \( n-T \)-quasigroup, \( n \geq 3 \), over a group of a prime order, \( A(x^n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + a \), \( a \neq 0 \), \( a_i \neq a_j \), \( i \neq j \), \( \sigma(n + 1) = n + 1, \sigma k = k \) for some \( k \in \overline{1,n} \), \( n \neq 0 \neq i_0 \) for some \( i_0 \neq k \), then \( A \perp \sigma A \).

**Proof.** In a group of a prime order all mappings \( x \rightarrow ax \), where \( a \neq 0 \) are automorphisms. If \( \sigma a_i \neq a_i \), then the mapping \( x \rightarrow (a_{i_0} - a_i)x \) is a permutation (an automorphism), so by Corollary 8 \( A \perp \sigma A \). 

**Example 3.** Let \((Q,+) = (Z_p,+)\) be a group of a prime order \( p \geq 7, Q = \{0,1,2,\ldots,p-1\} \), \( A(x^3) = 3x_1 + 5x_2 + x_3 + 2x_4 + x_5 \) and \( \sigma = (2,3) \), then \( \sigma 3 = 2 \neq 3, \sigma 4 = 4 (k = 4, i_0 = 3) \), \( \sigma A(x^5) = A(x^{\sigma -1}) = 3x_1 + 5x_3 + 4x_2 + 2x_4 + x_5 = 3x_1 + 4x_2 + 5x_3 + 2x_4 + x_5 \). By Corollary 9 \( A \perp \sigma A \).

**References**


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