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Pairwise orthogonality of n-ary operations *

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Abstract. The notions of hypercube and of the orthogonality of two hypercubes were arised in combinatorial analysis. In [11] a connection between n-dimensional hypercubes and algebraic n-ary operations was established. In this article we use an algebraic approach to the study of orthogonality of two hypercubes (pairwise orthogonality). We give a criterion of orthogonality of two finite k-invertible n-ary operations, which is used by the research of orthogonality and parastrophe-orthogonality of two n-ary T-quasigroups. Some examples are given and connection between admissibility and pairwise orthogonality of n-ary operations is established.

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1 Introduction

It is known that two binary operations A and B, given on a set Q, are called orthogonal if the system of equations $\{A(x,y) = a, B(x,y) = b\}$ has exactly one solution for any $a,b \in Q$ (see [1], where such operations are called compatible). Orthogonal binary operations, in particular, orthogonal quasigroups were considered in different works (see, for example, [1–7]).

In [6] H.B. Mann proved that if A, B, C are quasigroups, given on a set Q and satisfying the equality

$$C(x, B(x,y)) = A(x,y)$$

for all $x, y \in Q$, then the quasigroups A and B are orthogonal.

V.D. Belousov in [3, Lemma 2] gave the following criterion of orthogonality of two binary quasigroups. Let A, B be binary quasigroups on a set Q. Then A and B are orthogonal if and only if the operation $A \cdot B^{-1}$ is a quasigroup, where $(A \cdot B^{-1})(x,y) = A(x, B^{-1}(x,y))$ and B^{-1} is the right inverse quasigroup for B ($B^{-1}(x,z) = y$ if and only if $B(x,y) = z$).

In the case of n-ary operations there exist distinct versions of orthogonality (they are reflected in [11]) which correspond to different types of orthogonality of n-dimensional hypercubes.

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In this article we consider the weakest (for $n > 2$) case of orthogonality of n-ary operations, namely, pairwise orthogonality (see Definition 1). At first orthogonality of two finite k-invertible n-ary operations (pairwise orthogonality) is considered. Then, using the obtained criterion of orthogonality of finite n-ary operations, we give a definition of pairwise orthogonality for arbitrary k-invertible n-operations, in particular, for finite or infinite n-quasigroups. A connection between admissibility and pairwise orthogonality of k-invertible n-ary operations is established. In the last part of the article pairwise orthogonality of n-ary T-quasigroups ($n{-}T$-quasigroups), in particular, $n - T$-quasigroups which are orthogonal to some their parastrophes are studied. Some examples of such quasigroups are given.

2 Necessary notions and results

We recall some notations, concepts and results which are used in the article. At first remember the following designations and notes from [10]. By x_i^j we will denote the sequence $x_i, x_{i+1}, \ldots, x_j$, $i \leq j$. If $j < i$, then x_i^j is the empty sequence, $\overline{1,n} = \{1, 2, \ldots, n\}$. Let Q be a finite or an infinite set, $n \geq 1$ be a positive integer and let Q^n denote the Cartesian power of the set Q.

A n-ary operation A (briefly, an n-operation) on a set Q is a mapping $A : Q^n \to Q$ defined by $A(x_1^n) \to x_{n+1}$, and in this case we write $A(x_1^n) = x_{n+1}$.

A finite n-groupoid (Q, A) *of order* m is a set Q with one n-ary operation A defined on Q, where $|Q| = m$.

A *n-ary quasigroup* is an n-groupoid such that in the equality

$$A(x_1^n) = x_{n+1}$$

each of n elements from x_1^{n+1} uniquely defines the $(n + 1)$-th element. Usually itself quasigroup n-operation A is considered as a n-quasigroup.

The n-operation E_i, $1 \leq i \leq n$, on Q with $E_i(x_1^n) = x_i$ is called *the i-th identity operation (or the i-th selector) of arity n.*

An n-operation A on Q is called *i-invertible* for some $i \in \overline{1,n}$ if the equation

$$A(a_1^{i-1}, x_i, a_{i+1}^n) = a_{n+1}$$

has a unique solution for each fixed n-tuple $(a_1^{i-1}, a_{i+1}^n, a_{n+1}) \in Q^n$.

For an i-invertible n-operation there exists *the i-inverse n-operation* ${}^{(i)}A$ defined in the following way:

$${}^{(i)}A(x_1^{i-1}, x_{n+1}, x_{i+1}^n) = x_i \Leftrightarrow A(x_1^n) = x_{n+1}$$

for all $x_1^{n+1} \in Q^{n+1}$.

It is evident that

$$A(x_1^{i-1}, {}^{(i)}A(x_1^n), x_{i+1}^n) = {}^{(i)}A(x_1^{i-1}, A(x_1^n), x_{i+1}^n) = x_i$$

and ${}^{(i)}[{}^{(i)}A] = A$ for $i \in \overline{1,n}$.

Let Ω_n be the set of all n-ary operations on a finite or an infinite set Q. On Ω_n define a binary operation $\underset{i}{\oplus}$ (the i-multiplication) in the following way:

$$(A \underset{i}{\oplus} B)(x_1^n) = A(x_1^{i-1}, B(x_1^n), x_{i+1}^n),$$

$A, B \in \Omega_n, x_1^n \in Q^n$. Shortly this equality can be written as

$$A \underset{i}{\oplus} B = A(E_1^{i-1}, B, E_{i+1}^n)$$

where E_i is the i-th selector.

In [9] it was proved that $(\Omega_n; \underset{i}{\oplus})$ is a semigroup with the identity E_i. If Λ_i is the set of all i-invertible n-operations from Ω_n for some $i \in \overline{1,n}$, then $(\Lambda_i; \underset{i}{\oplus})$ is a group. In this group E_i is the identity, the inverse element of A is the operation ${}^{(i)}A \in \Lambda_i$, since $A \underset{i}{\oplus} E_i = E_i \underset{i}{\oplus} A$, $A \underset{i}{\oplus} {}^{(i)}A = {}^{(i)}A \underset{i}{\oplus} A = E_i$.

A n-ary quasigroup (Q, A) (or simply A), is an n-groupoid with an i-invertible n-operation for each $i \in \overline{1,n}$ [10].

Let A be an n-quasigroup and $\sigma \in S_{n+1}$, then the n-quasigroup ${}^{\sigma}A$ defined by

$${}^{\sigma}A(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)} \Leftrightarrow A(x_1^n) = x_{n+1}$$

is called the σ-parastrophe (or simple, parastrophe) of A [10].

For any n-operation A there exist the σ-parastrophes ${}^{\sigma}A$, where $\sigma(n+1) = n+1$ (the principal parastrophes). The i-inverse operation ${}^{(i)}A$ for A, $i \in \overline{1,n}$, is the σ-parastrophe defined by the cycle $(i, n+1)$.

Let $(x_1^n)_k$ denote the $(n-1)$-tuple $(x_1^{k-1}, x_{k+1}^n) \in Q^{n-1}$ and let A be an n-operation, then the $(n-1)$-operation A_a:

$$A_a(x_1^n)_k = A(x_1^{k-1}, a, x_{k+1}^n)$$

is called *the* $(n-1)$-*retract* of A, defined by position k, $k \in \overline{1,n}$, with the element a in this position (with $x_k = a$) [10].

An n-ary operation A on Q is called *complete* if there exists a permutation $\overline{\varphi}$ on Q^n such that $A = E_1\overline{\varphi}$ (that is $A(x_1^n) = E_1\overline{\varphi}(x_1^n)$). If a complete n-operation A is finite and has order m, then the equation $A(x_1^n) = a$ has exactly m^{n-1} solutions for any $a \in Q$ [9].

Any i-invertible n-operation A, $i \in \overline{1,n}$, is complete, but there exist complete n-operations, which are not i-invertible for each $i \in \overline{1,n}$ [9].

3 Orthogonality of two n-ary operations

In the case of n-ary operations for $n > 2$ it is possible to consider different versions of orthogonality. The weakest is the notion of the pairwise orthogonality.

Definition 1 [11]. *Two n-ary operations ($n \geq 2$) A and B given on a set Q of order m are called orthogonal (shortly, $A \perp B$) if the system $\{A(x_1^n) = a, B(x_1^n) = b\}$ has exactly m^{n-2} solutions for any $a, b \in Q$.*

This concept corresponds to two orthogonal n-dimensional hypercubes [11, 13]. The following type of orthogonality is strongest.

Definition 2 [8]. *An n-tuple $< A_1, A_2, \dots, A_n >$ of n-operations on a set Q is called orthogonal if the system $\{A_i(x_1^n) = a_i\}_{i=1}^n$ has a unique solution for any $a_1^n \in Q^n$. A set $\Sigma = \{A_1^t\}$, $t \geq n$, of n-operations is called orthogonal if any n-tuple of distinct n-operations from Σ is orthogonal.*

This concept corresponds to an orthogonal n-tuple of n-dimensional hypercubes [11–13]. Orthogonal n-operations and their sets in the sense of Definition 2 were considered in many articles (see, for example, [8, 11–17, 19, 20, 22]).

In [11] intermediate types of orthogonality of n-operations and their sets were studied.

Definition 3 [11]. *A k-tuple $< A_1^k >, 2 \leq k \leq n$, of distinct n-operations on a set Q of order m is called orthogonal if the system $\{A_i(x_1^n) = a_i\}_{i=1}^k$ has exactly m^{n-k} solutions for any $a_1^k \in Q^k$. A set $\Sigma = \{A_1^t\}$, $t \geq k$, of n-operations is called k-wise orthogonal if any k-tuple of distinct n-operations from Σ is orthogonal.*

The following connection exists between different considered types of orthogonality.

Theorem 1 [11]. *If a set $\Sigma = \{A_1^t\}$, $t \geq k$, of finite n-operations is k-wise orthogonal, then Σ is l-wise orthogonal for any l, $2 \leq l \leq k$.*

Thus, every pair of different n-ary operations from an orthogonal n-tuple is orthogonal.

Let $A_1, A_2, \dots, A_n$ be n-operations given on a set Q. In [14] it is proved that a n-tuple $< A_1^n >$ of n-operations is orthogonal if and only if the mapping $\bar{\theta} : x_1^n \to (A_1(x_1^n), A_2(x_1^n), \dots, A_n(x_1^n)) = (A_1, A_2, \dots, A_n)(x_1^n)$ is a permutation on Q^n.

In [1] V.D. Belousov proved that a binary operation A has an operation which is orthogonal to A (an orthogonal mate) if and only if A is a complete operation. This is valid and for finite n-operations.

Proposition 1. *A finite n-operation A has an orthogonal mate if and only if A is complete.*

Proof. By Proposition 5 of [11] A is a complete n-operation if and only if it is a component of some permutation $\overline{\theta} = (A, B_1^{n-1})$ on Q^n, where $< A, B_1^{n-1} >$ is an orthogonal n-tuple. By Theorem 1 $A \perp B_i$ for any $i \in \overline{1, n-1}$.

Conversely, if B is an orthogonal mate for A , that is $A \perp B$, then by Corollary 4 of [11] the pair A, B can be embedded in an orthogonal n-tuple of n-operations and by Proposition 5 of [11] A is a complete n-operation. □

Now we shall consider orthogonality of k-invertible n-operations for some fixed k, $1 \le k \le n$. For them the following criterion is valid.

Theorem 2. *Let k be a fixed number from $\overline{1,n}$. Two finite k-invertible n-operations A and B on a set Q are orthogonal if and only if the $(n-1)$-retract C_a of the n-operation $C = B \underset{k}{\oplus} {}^{(k)}A$, defined by $x_k = a$, is complete for every $a \in Q$.*

Proof. We shall prove this statement when $k = n$ for the sake of simplicity. For the rest $k \in \overline{1, n-1}$ the proof is similar.

Let a be an arbitrary element of Q, $|Q| = m$ and the $(n-1)$-retract C_a by $x_n = a$ of n-operation $C = B \underset{n}{\oplus} {}^{(n)}A$ is complete for any $a \in Q$. Then the equation

$$C_a(x_1^{n-1}) = C(x_1^{n-1}, a) = (B \underset{n}{\oplus} {}^{(n)}A)(x_1^{n-1}, a) = B(x_1^{n-1}, {}^{(n)}A(x_1^{n-1}, a)) = b$$

has $m^{(n-1)-1}$ solutions for any $a, b \in Q$. From the last equation we have ${}^{(n)}B(x_1^{n-1}, b) = {}^{(n)}A(x_1^{n-1}, a) = z$, whence it follows that the system $\{A(x_1^{n-1}, z) = a, B(x_1^{n-1}, z) = b\}$ has m^{n-2} solutions. Thus, $A \perp B$.

Conversely, let $A \perp B$, that is the system $\{A(x_1^n) = a, B(x_1^n) = b\}$ has m^{n-2} solutions for any $a, b \in Q$. From the first equality we have $x_n = {}^{(n)}A(x_1^{n-1}, a)$ and then the equation $B(x_1^{n-1}, {}^{(n)}A(x_1^{n-1}, a)) = b$ or $C_a(x_1^{n-1}) = (B \underset{n}{\oplus} {}^{(n)}A)(x_1^{n-1}, a) = b$ has m^{n-2} solutions for any $a, b \in Q$. Therefore, the $(n-1)$-retract of $B \underset{n}{\oplus} {}^{(n)}A$, defined by any $a \in Q$, is complete. □

For the binary case from Theorem 2 we have the following

Corollary 1. *Two finite invertible from the right (that is 2-invertible) binary operations A, B on Q are orthogonal if and only if the operation $C(x, y) = (A{\cdot}B^{-1})(x, y) = A(x, B^{-1}(x, y))$ is a quasigroup.*

Proof. The operation $C = B \cdot A^{-1} (= B \underset{2}{\oplus} {}^{(2)}A)$ is always invertible from the right. If the operation $C_a x = C(x, a)$ is complete for any $a \in Q$, that is the equation $C(x, a) = b$ has exactly $m^{2-2} = 1$ solutions for any $a, b \in Q$, then the operation C is invertible from the left (that is 1-invertible). Thus, C is a quasigroup.

Conversely, if C is a quasigroup, then any its (unary) retract is complete (that is a permutation). □

From this corollary the criterion of V.D.Belousov [3, Lemma 2] for finite binary quasigroups follows.

Proposition 2. *If A and B are k-invertible n-operations on a set Q for some $k \in \overline{1,n}$, then the following equalities are equivalent: $C = B \underset{k}{\oplus} {}^{(k)}A$, $C \underset{k}{\oplus} A = B$,*

$A = {}^{(k)}C \underset{k}{\oplus} B$, $C \underset{k}{\oplus} A \underset{k}{\oplus} {}^{(k)}B = E_k$, ${}^{(k)}A \underset{k}{\oplus} {}^{(k)}C \underset{k}{\oplus} B = E_k$, $A \underset{k}{\oplus} {}^{(k)}B \underset{k}{\oplus} C = E_k$, ${}^{(k)}C \underset{k}{\oplus} B \underset{k}{\oplus} {}^{(k)}A = E_k$.

Proof. It is easy to see taking into account that all k-invertible n-operations on Q form a group with the identity E_k with the respect to the k-multiplication of n-operations. □

Remark 1. If A and B are n-quasigroups, then they are k-invertible for any $k \in \overline{1,n}$, so $A \perp B$ if and only if for some $k \in \overline{1,n}$ the $(n-1)$-retract C_a of $C = B \underset{k}{\oplus} {}^{(k)}A$, defined by $x_k = a$, is complete for any $a \in Q$. If that holds for some fixed $k \in \overline{1,n}$, then the $(n-1)$-retract of $C_1 = B \underset{l}{\oplus} {}^{(l)}A$, defined by $x_l = a$, is also complete for any $l \in \overline{1,n}$ and any $a \in Q$.

From Proposition 2 and Theorem 2 we have the following

Corollary 2. *If A and B are finite n-quasigroups on Q, $C = B \underset{k}{\oplus} {}^{(k)}A$ and $A \perp B$, then $C \perp {}^{(k)}A$, ${}^{(k)}C \perp {}^{(k)}B$ for any $k \in \overline{1,n}$.*

Proof. $C \perp {}^{(k)}A$ (${}^{(k)}C \perp {}^{(k)}B$) follows from the second (from the third) equality of Proposition 2 and Theorem 2, since A and B are n-quasigroups and so any $(n-1)$-retract of B (A) is an $(n-1)$-quasigroup which is always complete. Further use Remark 1. □

Using the criterion of orthogonality of two finite n-operations from Theorem 2 we can define a pairwise orthogonality of arbitrary k-invertible n-operations (finite or infinite).

Definition 4. *Two k-invertible n-operations A and B , given on an arbitrary set Q, are called orthogonal if the $(n-1)$-retract of the n-operation $B \underset{k}{\oplus} {}^{(k)}A$,defined by $x_k = a$, is complete for each $a \in Q$.*

As it was noted above, an n-operation A on Q is called complete if there exists a permutation (a bijection) $\overline{\varphi}$ on Q^n such that $A = E_1\overline{\varphi}$. In the case of Definition 4 each $(n-1)$-retract

$$C_a(x_1^n)_k = C(x_1^{k-1}, a, x_{k+1}^n) = B(x_1^{k-1}, {}^{(k)}A(x_1^{k-1}, a, x_{k+1}^n), x_{k+1}^n)$$

is complete, that is $C_a = E_1\overline{\psi}$ for some permutation $\overline{\psi}$ of Q^{n-1}.

Remark 2. Note that for binary case (n=2) Definition 4 is equivalent to the usual definition of orthogonality of two 1- or 2-invertible operations.

Indeed, let A, B be 2-invertible binary operations on a set Q and $A \perp B$, that is the system $\{A(x,y) = a, B(x,y) = b\}$ has a unique solution for any $a, b \in Q$. Then $A^{-1}(x,a) = y$ and the equation $B(x, A^{-1}(x,a)) = b$ has a unique solution x for any

$a, b \in Q$, that is $C_a(x) = B(x, R_a x) = E\varphi_a x = \varphi_a x$ where $R_a x = A^{-1}(x, a)$, E is the selector in the 1-ary case ($Ex = \varepsilon x = x$) and so φ_a is a bijection Q on Q. Thus, $C_a = \varphi_a$ is a complete 1-ary (unary) operation for any $a \in Q$.

Conversely, if $C_a = \varphi_a$ is a bijection for any $a \in Q$, then the equation $B(x, A^{-1}(x, a)) = b$ has a unique solution for any $a, b \in Q$ and the system $\{A(x, y) = a, B(x, y) = b\}$ has a unique solution.

For 1-invertible binary operations the proof is similar.

Now we consider a connection between orthogonality of two n-operations and their admissibility.

It is known that a binary quasigroup $(Q, \cdot)$ is called admissible if it has a complete permutation (a bijection) (or a transversal).

A permutation θ on Q is called *complete* for a quasigroup $(Q, \cdot)$ if the mapping θ': $\theta' x = x \cdot \theta x$ is a permutation on Q. All elements $\theta' x$, $x \in Q$, are different and form *a transversal* which is defined by the permutation θ [5].

A binary quasigroup of order m has an orthogonal mate if and only if it has m disjoint transversals $\theta'_1, \theta'_2, \dots, \theta'_m$ (or m disjoint complete permutations $\theta_1, \theta_2, \dots, \theta_m$), that is $\theta'_i x \neq \theta'_j x, i \neq j$, for any $x \in Q$ [5].

Using the criterion of Corollary 4 of orthogonality of binary 2-invertible (or 1-invertible) operations A and B on Q of order m, it is easy to find in this case m disjoint transversals.

Indeed, if $A \perp B$, then the operation $A \cdot B^{-1}$ ($(A \cdot B^{-1})(x, y) = A(x, B^{-1}(x, y))$) is a quasigroup. By $y = a$ we have $A(x, B^{-1}(x, a)) = A(x, R_a x) = C_a x$ and C_a is a permutation where $R_a : R_a x = B^{-1}(x, a)$ is also permutation. Thus, in A there exist m disjoint complete permutations $\{R_a, a \in Q\}$ which define m disjoint transversals $\{C_a, a \in Q\}$.

In [20, 22] the admissibility of n-quasigroups and their connection with orthogonality were considered. By analogue with n-quasigroups (see [21]) the following definition of admissible n-operations was given.

Definition 5. *An n-operation B given on a set Q is called admissible if for some k, $1 \leq k \leq n$, on Q there exists an $(n-1)$-operation A such that the $(n-1)$-operation C:*

$$C(x_1^n)_k = B(x_1^{k-1}, A(x_1^n)_k, x_{k+1}^n)$$

is complete. In this case the $(n-1)$-operation C is called a k-transversal of the n-operation B, defined by the $(n-1)$-operation A.

The n-tuples $(x_1^{k-1}, A(x_1^n)_k, x_{k+1}^n)$ are positions of elements of a k-transversal C. The values $C(x_1^n)_k$, when $(n-1)$-tuples $(x_1^n)_k$ run through Q^{n-1}, are the elements of the k-transversal C.

Two k-transversals of an n-operation B defined by $(n-1)$-operations A_1 and A_2 are called disjoint if $A_1(x_1^n)_k \neq A_2(x_1^n)_k$ for all $(x_1^n)_k \in Q^{n-1}$.

From Theorem 2 it follows

Proposition 3. *Let A, B be finite k-invertible n-operations given on a set Q of order m, $A \perp B$. Then the $(n-1)$-operations ${}^{(k)}A_a(x_1^n)_k = {}^{(k)}A(x_1^{k-1}, a, x_{k+1}^n)$, $a \in Q$, define m pairwise disjoint k-transversals in B.*

Proof. By Theorem 2 $A \perp B$ if and only if the $(n-1)$-operation

$$C_a(x_1^n)_k = B(x_1^{k-1}, {}^{(k)}A(x_1^{k-1}, a, x_{k+1}^n), x_{k+1}^n) = B(x_1^{k-1}, {}^{(k)}A_a(x_1^n)_k, x_{k+1}^n)$$

is complete for any $a \in Q$. Thus, by Definition 5 the operations ${}^{(k)}A_a$, $a \in Q$, define m transversals C_a, $a \in Q$. It is evident that ${}^{(k)}A_a(x_1^n)_k \neq {}^{(k)}A_b(x_1^n)_k$, if $a \neq b$, since A is a k-invertible n-operation. Moreover, in this case we have $C_a(x_1^n)_k \neq C_b(x_1^n)_k$ by virtue of k-invertibility of the n-operation B. $\square$

Let A, B be two n-operations on a set Q. Recall that an n-operation B is called isotopic to an n-operation A if there exists an $(n+1)$-tuple $T = (\alpha_1, \alpha_2, \dots, \alpha_n, \gamma)$ of permutations (bijections) of Q such that $B(x_1^n) = \gamma^{-1}A(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n)$ for all $x_1^n \in Q^n$ (shortly, $B = A^T$)[10].

It is easy to prove that the following statement is valid.

Proposition 4. *Any n-operation B which is isotopic to a complete finite or infinite n-operation A is also complete.*

Proof. Let A be a complete n-operation on a set Q, $B = A^T$, $T = (\alpha_1, \alpha_2, \dots, \alpha_n, \gamma)$, then $A = E_1\overline{\varphi}$ for some permutation $\overline{\varphi} = (C_1, C_2, \dots, C_n)$ (where the n-tuple $< C_1, C_2, \dots, C_n >$ of n-operations is orthogonal) and

$$B(x_1^n) = \gamma^{-1}A(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n) =$$

$$\gamma^{-1}E_1\overline{\varphi}(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n) = \gamma^{-1}E_1(C_1, C_2, \dots, C_n)(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n) =$$

$$E_1(\gamma^{-1}\overline{C}_1, \overline{C}_2, \dots, \overline{C}_n)(x_1^n)$$

where $\overline{C}_i(x_1^n) = C_i(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n)$. It is easy to see that the n-tuple $< \gamma^{-1}\overline{C}_1, \overline{C}_2, \dots, \overline{C}_n >$ is also orthogonal. Thus, $B = E_1\overline{\psi}$, where

$$\overline{\psi} = (\gamma^{-1}\overline{C}_1, \overline{C}_2, \dots, \overline{C}_n). \qquad \square$$

From Proposition 1 and Proposition 3 we obtain the following

Corollary 3. *If a finite n-operation A has an orthogonal mate and $B = A^T, T = (\alpha_1, \alpha_2, \dots, \alpha_n, \gamma)$, then B has an orthogonal mate too.*

4 Pairwise orthogonal n – T-quasigroups

Below we shall consider in more detail orthogonality of two n-ary T-quasigroups (briefly, $n-T$-quasigroups) which are closely connected with finite or infinite abelian groups and generalize the known binary T-quasigroups.

Definition 6 [18]. *An n-quasigroup (Q,A) is called an $n-T$-quasigroup if there exist a binary abelian group $(Q,+)$, its automorphisms $\alpha_1,\alpha_2,\dots,\alpha_n$ and an element $a\in Q$ such that*

$$A(x_1^n)=\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n+a \tag{1}$$

for all $x_1^n\in Q^n$.

Let $k\in\overline{1,n}$, then the k-inverse n-operation ${}^{(k)}A$ for an $n-T$-quasigroup A of (1) has the form

$${}^{(k)}A(x_1^n)=\alpha_k^{-1}(-\alpha_1x_1-\alpha_2x_2-\cdots-\alpha_{k-1}x_{k-1}+x_k-\alpha_{k+1}x_{k+1}-\cdots-\alpha_nx_n-a) \tag{2}$$

and is also $n-T$-quasigroup, since the mapping $I: Ix=-x$ is an automorphism in an abelian group.

Proposition 5. *Let (Q,A) and (Q,B) be two finite $n-T$-quasigroups over a group $(Q,+)$ of odd order,*

$$A(x_1^n)=\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_{k-1}x_{k-1}+x_k+\alpha_{k+1}x_{k+1}+\cdots+\alpha_nx_n,$$

$$B(x_1^n)=\beta_1x_1+\beta_2x_2+\cdots+\beta_{k-1}x_{k-1}+x_k+\beta_{k+1}x_{k+1}+\cdots+\beta_nx_n,$$

where $\beta_i=2\alpha_i$ for each $i\in\overline{1,n}$, $i\neq k$, then $C=B\underset{k}{\oplus}{}^{(k)}A=A$, $B=A\underset{k}{\oplus}A$ and $A\perp(A\underset{k}{\oplus}A)$, $A\perp^{(k)}A$, ${}^{(k)}(A\underset{k}{\oplus}A)\perp^{(k)}A$.

Proof. In this case $\beta_i=2\alpha_i$ is an automorphism for any $i\in\overline{1,n}$, $i\neq k$, since in a group $(Q,+)$ of odd order the mapping $x\to 2x$ is a permutation. Find the form of the n-operation C using (2): $C(x_1^n)=(B\underset{k}{\oplus}{}^{(k)}A)(x_1^n)=B(x_1^{k-1},{}^{(k)}A(x_1^n),x_{k+1}^n)=2\alpha_1x_1+2\alpha_2x_2+\cdots+2\alpha_{k-1}x_{k-1}-\alpha_1x_1-\alpha_2x_2-\cdots-\alpha_{k-1}x_{k-1}+x_k-\alpha_{k+1}x_{k+1}-\cdots-\alpha_nx_n+2\alpha_{k+1}x_{k+1}+\cdots+2\alpha_nx_n=\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_{k-1}x_{k-1}+x_k+\cdots+\alpha_{k+1}x_{k+1}+\cdots+\alpha_nx_n=A(x_1^n)$. Any $(n-1)$-retract of $C=A$ is a $(n-1)$-quasigroup, so is complete and $A\perp B$ by Definition 4 (or by Theorem 2). Since $C=B\underset{k}{\oplus}{}^{(k)}A=A$, then $B=A\underset{k}{\oplus}A$. Orthogonality of the rest n-operations pointed in the proposition follows from Corollary 2. □

The following useful criterion of orthogonality of two $n-T$-quasigroups is valid.

Theorem 3. *Two $n-T$-quasigroups (Q,A) and (Q,B) where*

$$A(x_1^n)=\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n+a,\ B(x_1^n)=\beta_1x_1+\beta_2x_2+\cdots+\beta_nx_n+b$$

are orthogonal if and only if the $(n-1)$*-operation* $\overline{C}$*:*

$$\overline{C}(x_1^n)_k = \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_{k-1}x_{k-1} + \gamma_{k+1}x_{k+1} + \cdots + \gamma_n x_n \qquad (3)$$

is complete, where

$$\gamma_i x_i = \beta_i x_i - \beta_k \alpha_k^{-1} \alpha_i x_i = (\beta_i - \beta_k \alpha_k^{-1}\alpha_i)x_i, \quad i \in \overline{1,n}, i \neq k.$$

Proof. By Remark 1 and Definition 4 we need to prove that $\overline{C}$ is complete if and only if the $(n-1)$-retract C_c of $C = B \underset{k}{\oplus} {}^{(k)}A$ defined by $x_k = c$, for some $k \in \overline{1,n}$ and $c \in Q$, is complete. Using (2) we have $C(x_1^n) = (B \underset{k}{\oplus} {}^{(k)}A)(x_1^n) = B(x_1^{k-1}, {}^{(k)}A(x_1^n), x_{k+1}^n) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{k-1}x_{k-1} + \beta_k \alpha_k^{-1}(-\alpha_1 x_1 - \alpha_2 x_2 - \cdots - \alpha_{k-1}x_{k-1} + x_k - \alpha_{k+1}x_{k+1} - \cdots - \alpha_n x_n - a) + \beta_{k+1}x_{k+1} + \cdots + \beta_n x_n + b = (\beta_1 - \beta_k\alpha_k^{-1}\alpha_1)x_1 + (\beta_2 - \beta_k\alpha_k^{-1}\alpha_2)x_2 + \cdots + (\beta_{k-1} - \beta_k\alpha_k^{-1}\alpha_{k-1})x_{k-1} + \beta_k\alpha_k^{-1}x_k + (\beta_{k+1} - \beta_k\alpha_k^{-1}\alpha_{k+1})x_{k+1} + .. + (\beta_n - \beta_k\alpha_k^{-1}\alpha_n)x_n - \beta_k\alpha_k^{-1}a + b = \overline{C}(x_1^n)_k + \beta_k\alpha_k^{-1}x_k - \beta_k\alpha_k^{-1}a + b$ (see (3)).

Let $x_k = c$ be an arbitrary element of Q, then we have

$$C(x_1^{k-1}, c, x_{k+1}^n) = C_c(x_1^n)_k = \overline{C}(x_1^n)_k + d = R_d\overline{C}(x_1^n)_k,$$

where $d = \beta_k\alpha_k^{-1}c - \beta_k\alpha_k^{-1}a + b$, $R_d x = x + d$. Thus, the $(n-1)$-retract $C_c(x_1^n)_k$ of C, defined by $x_k = c$, is isotopic to the $(n-1)$-ary operation $\overline{C}$: $C_c = \overline{C}^T$, $T = (\varepsilon, \varepsilon, \ldots, \varepsilon, R_d^{-1})$ (ε denotes the identity permutation on Q) and by Proposition 4 C_c is complete if and only if $\overline{C}$ is complete. $\square$

Remark 3. Note that if the conditions of Theorem 3 hold for some $k \in \overline{1,n}$, then they hold for any $k \in \overline{1,n}$ (see Remark 1 for n-quasigroups).

Corollary 4. *If in Theorem 3* $\gamma_{i_0} = \beta_{i_0} - \beta_k\alpha_k^{-1}\alpha_{i_0}$ *is a permutation for some* $i_0 \in \overline{1,n}$*,* $i_0 \neq k$*, then* $A \perp B$*.*

Proof. In this case the $(n-1)$-operation $\overline{C}$ of (3) is i_0-invertible, so it is complete. $\square$

From Theorem 3 and Corollary 4 a number of useful statements follow.

Corollary 5. *Let in Theorem 3* $\alpha_k = \beta_k$ *for some* $k \in \overline{1,n}$*. Then*

(i) if $\beta_{i_0} - \alpha_{i_0}$ *is a permutation for some* $i_0 \in \overline{1,n}$*,* $i_0 \neq k$*, then* $A \perp B$*;*

(ii) if $(Q,+)$ *is an (abelian) group of odd order and* $\beta_{i_0} = 2\alpha_{i_0}$ *for some* $i_0 \in \overline{1,n}$*,* $i \neq k$*, then* $A \perp B$*.*

Proof. By $\alpha_k = \beta_k$ we have $\beta_k\alpha_k^{-1} = \varepsilon$ and $\gamma_i = \beta_i - \alpha_i$ for all $i \in \overline{1,n}$, $i \neq k$. In (i) use Corollary 4. Item (ii) is a particular case of (i), since $\beta_{i_0} = 2\alpha_{i_0}$ is a permutation (and so an automorphism) in a group of odd order. $\square$

Corollary 6. *Let $\Sigma = \{A_1^t\}$ be a set of $n-T$-quasigroups on a set Q over the same group $(Q,+)$:*

$$A_i(x_1^n) = \alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in}x_n, i \in \overline{1,t}, \tag{4}$$

where $\alpha_{1k} = \alpha_{2k} = \cdots = \alpha_{tk}$ for some $k \in \overline{1,n}$. If for all $i,j \in \overline{1,t}$, $i \neq j$ there exists one number $s \in \overline{1,n}$, $s \neq k$ such that $\alpha_{is} - \alpha_{js}$ is a permutation, then the set Σ is pairwise orthogonal.

Proof. In this case $A_i \perp A_j$ for each $i,j \in \overline{1,t}$, $i \neq j$ by virtue of item (i) of Corollary 5 since $\alpha_{ik} = \alpha_{jk}$ for all $i,j \in \overline{1,t}$, $i \neq j$. □

Example 1. Let $\Sigma = \{A_1^{p-1}\}$ be a set of $n-T$-quasigroups over a group $(Q,+)$ (with the identity 0) of a prime order p, where $n-T$-quasigroups of (4) have the form

$$A_1(x_1^n) = a_1x_1 + a_{12}x_2 + \cdots + a_{1,n-1}x_{n-1} + ax_n,$$

$$A_2(x_1^n) = a_2x_1 + a_{22}x_2 + \cdots + a_{2,n-1}x_{n-1} + ax_n,$$

$$\cdots$$

$$A_{p-1}(x_1^n) = a_{p-1}x_1 + a_{p-1,2}x_2 + \cdots + a_{p-1,n-1}x_{n-1} + ax_n,$$

$\alpha_{i1}x = a_ix$, $a_i \neq a_j$, if $i \neq j$, $i,j \in \overline{1,p-1}$, $\alpha_{ik}x = a_{ik}x$, if $k \neq 1$ and $k \neq n$,$a_{in} = a$, $i \in \overline{1,p-1}$, $a, a_i, a_{ik} \in Q \setminus 0$ for all $i \in \overline{1,p-1}$.

By Corollary 6 the set Σ is pairwise orthogonal by $s = 1$ since $a_{i1} - a_{j1} = a_i - a_j \neq 0$, so the mapping $x \to (a_i - a_j)x$ is a permutation by $i \neq j$ and by $\alpha_{1n}x = \alpha_{2n}x = \cdots = \alpha_{p-1,n}x = ax$ (here $k = n$).

Further we shall establish some conditions for orthogonality of an $n-T$-quasigroup to some its parastrophes, using Theorem 3. Parastrophe-orthogonality of binary quasigroups and minimal identities connected with such orthogonality were in detail studied by V.D.Belousov in [4].

At first we recall that an automorphism α of a group $(Q,+)$ is called complete if the mapping $x \to x + \alpha x$ is a permutation of Q, that is if α is a complete permutation [5].

Proposition 6. *If an $n-T$-quasigroup (Q,A), $A(x_1^n) = \alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n + a$ where α_l is a complete automorphism of the group $(Q,+)$ for some $l \in \overline{1,n}$, then $A \perp {}^{(l)}A$.*

Proof. Using expression (2) for ${}^{(l)}A$ and taking in Theorem 3 $k \neq l$, $B = {}^{(l)}A$ we obtain $\beta_l = \alpha_l^{-1}$ and $\beta_k = -\alpha_l^{-1}\alpha_k$. Then $\gamma_l = \beta_l - \beta_k\alpha_k^{-1}\alpha_l = \alpha_l^{-1} + \alpha_l^{-1}\alpha_k\alpha_k^{-1}\alpha_l = \alpha_l^{-1}(\varepsilon + \alpha_l)$ is a permutation and so $A \perp {}^{(l)}A$ by Corollary 4. □

Corollary 7. *An $n-T$-quasigroup (Q,A) over a group $(Q,+)$ with $A(x_1^n) = \alpha x_1 + \alpha x_2 + \cdots + \alpha x_n + a$, where α is a complete automorphism of $(Q,+)$, is orthogonal to ${}^{(l)}A$ for each $l \in \overline{1,n}$. Moreover, if, in addition, $n \geq 3$, then the set $\Sigma = \{A, {}^{(1)}A, \ldots, {}^{(n)}A\}$ is pairwise orthogonal.*

Proof. The first statement follows immediately from Proposition 6. Prove that ${}^{(i)}A \perp {}^{(j)}A$ for each $i, j \in \overline{1,n}$, $i \neq j$. By (2) we have

$$
{}^{(i)}A(x_1^n) = \alpha^{-1}(-\alpha x_1 - \alpha x_2 - \cdots - \alpha x_{i-1} + x_i - \alpha x_{i+1} - \cdots - \alpha x_n - a) =
$$

$$
-x_1 - x_2 - \cdots - x_{i-1} + \alpha^{-1}x_i - x_{i+1} - \cdots - x_n - \alpha^{-1}a =
$$

$$
Ix_1 + Ix_2 + \cdots + Ix_{i-1} + \alpha^{-1}x_i + Ix_{i+1} + \cdots + Ix_n + b =
$$

$$
\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + b, b = -\alpha^{-1}a,
$$

$$
{}^{(j)}A(x_1^n) = Ix_1 + Ix_2 + \cdots + Ix_{j-1} + \alpha^{-1}x_j + Ix_{j+1} + \cdots + Ix_n + b =
$$

$$
\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + b.
$$

Since $i \neq j$ and $n \geq 3$ then there exists $k \in \overline{1,n}$ such that $\alpha_k = \beta_k$ $(k \neq i, j)$. In this case we have $\alpha^{-1}x_j - (Ix_j) = (\alpha^{-1} + \varepsilon)x_j$, so the map $\beta_j - \alpha_j = \alpha^{-1} + \varepsilon$ is a permutation since α is a complete automorphism. By item (i) of Corollary 5 (if $i_0 = j$)) ${}^{(i)}A \perp {}^{(j)}A$. Taking into account that $A \perp {}^{(l)}A$ for any $l \in \overline{1,n}$, we obtain that Σ is a pairwise orthogonal set. $\square$

From Corollary 7, in particular, it follows that if A is an $n-T$-quasigroup $(n \geq 3)$ (Q, A): $A(x_1^n) = x_1 + x_2 + \cdots + x_n + a$ over a group of odd order, then $\Sigma = \{A, {}^{(1)}A, \ldots, {}^{(n)}A\}$ is pairwise orthogonal set, since the identity automorphism ε in such group is complete.

A direct corollary of Theorem 3 for an $n-T$-quasigroup which is orthogonal to some its principal σ-parastrophe is the following

Proposition 7. *Let (Q, A) be an $n-T$-quasigroup over a group $(Q, +)$: $A(x_1^n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + a$, $\sigma(n+1) = n+1$. Then $A \perp {}^{\sigma}A$ if and only if for some $k \in \overline{1,n}$ the $(n-1)$-operation $\overline{C}$:*

$$
\overline{C}(x_1^n)_k = (\alpha_{\sigma 1} - \alpha_{\sigma k}\alpha_k^{-1}\alpha_1)x_1 + (\alpha_{\sigma 2} - \alpha_{\sigma k}\alpha_k^{-1}\alpha_2)x_2 + \cdots + (\alpha_{\sigma(k-1)} - \alpha_{\sigma k}\alpha_k^{-1}\alpha_{k-1})x_{k-1} +
$$

$$
(\alpha_{\sigma(k+1)} - \alpha_{\sigma k}\alpha_k^{-1}\alpha_{k+1})x_{k+1} + \cdots + (\alpha_{\sigma n} - \alpha_{\sigma k}\alpha_k^{-1}\alpha_n)x_n
$$

is complete.

Proof. By the definition of a principal parastrophe ${}^{\sigma}A$ $(\sigma(n+1) = n+1)$ of A

$$
{}^{\sigma}A(x_1^n) = A(x_{\sigma^{-1}1}^{\sigma^{-1}n}) = \alpha_1 x_{\sigma^{-1}1} + \alpha_2 x_{\sigma^{-1}2} + \cdots + \alpha_n x_{\sigma^{-1}n} + a =
$$

$$
\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + a,
$$

where $\beta_i x_i = \alpha_{\sigma i} x_i$, $i \in \overline{1,n}$. Further use Theorem 3 with $\gamma_i = \beta_i - \beta_k \alpha_k^{-1}\alpha_i = \alpha_{\sigma i} - \alpha_{\sigma k}\alpha_k^{-1}\alpha_i$. $\square$

Corollary 8. *If (Q, A) is an $n-T$-quasigroup, $n \geq 3$, $A(x_1^n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + a$, $\sigma(n+1) = n+1$, $\sigma k = k$ for some $k \in \overline{1,n}$ and $\alpha_{\sigma i_0} - \alpha_{i_0}$ is a* *permutation for some* $i_0 \in \overline{1,n}$, $i_0 \neq k$, *then* $A \perp {}^{\sigma}A$. *If, in addition,* $(Q,+)$ *has odd order and* $\alpha_{\sigma i_0} = 2\alpha_{i_0}$, *then* $A \perp {}^{\sigma}A$.

Proof. We have ${}^{\sigma}A(x_1^n) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{k-1}x_{k-1} + \alpha_k x_k + \beta_{k+1}x_{k+1} + \cdots + \beta_n x_n + a$, where $\beta_i = \alpha_{\sigma i}$, so $\beta_k = \alpha_k$, as $\sigma k = k$ and we can use items (i) and (ii) of Corollary 5, respectively. □

Note that for $n = 2$ we have $\sigma = \varepsilon$ (that is ${}^{\sigma}A = A$) by the conditions of this corollary (if $\sigma 3 = 3$, $\sigma 1 = 1$, then $\sigma = (1, \sigma 2, 3) \Rightarrow \sigma 2 = 2$).

Example 2. Let (Q,A) be an $n-T$-quasigroup, $n \geq 3$, over a group of odd order with $A(x_1^n) = \alpha_1 x_1 + 2\alpha_1 x_2 + \cdots + \alpha_n x_n + a$, $i_0 = 1$, $\sigma(n+1) = n+1, \sigma 1 = 2$ and $\sigma k = k$ for some $k \in \overline{1,n}$, $k \neq 1$. Then $\alpha_{\sigma 1} - \alpha_1 = \alpha_2 - \alpha_1 = 2\alpha_1 - \alpha_1 = \alpha_1$. By Corollary 8 $A \perp {}^{\sigma}A$ for any $\alpha_i \neq 0$, $i \in \overline{1,n}$, $i \neq 2$.

Corollary 9. *If* (Q,A) *is an* $n-T$*-quasigroup* , $n \geq 3$, *over a group of a prime order,* $A(x_1^n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + a$, $a_i \neq 0$, $a_i \neq a_j$, *if* $i \neq j$, $\sigma(n+1) = n+1$, $\sigma k = k$ *for some* $k \in \overline{1,n}$ *and* $\sigma i_0 \neq i_0$ *for some* $i_0 \neq k$, *then* $A \perp {}^{\sigma}A$.

Proof. In a group of a prime order all mappings $x \to ax$, where $a \neq 0$ are automorphisms. If $\sigma i_0 \neq i_0$, then the mapping $x \to (a_{\sigma i_0} - a_{i_0})x$ is a permutation (an automorphism), so by Corollary 8 $A \perp {}^{\sigma}A$. □

Example 3. Let $(Q,+) = (Z_p,+)$ be a group of a prime order $p \geq 7$, $Q = \{0,1,2,\ldots,p-1\}$, $A(x_1^5) = 3x_1 + 5x_2 + 4x_3 + 2x_4 + x_5$ and $\sigma = (2,3)$, then $\sigma 3 = 2 \neq 3$, $\sigma 4 = 4$ $(k = 4, i_0 = 3)$, ${}^{\sigma}A(x_1^5) = A(x_{\sigma^{-1}1}^{\sigma^{-1}5}) = 3x_1 + 5x_3 + 4x_2 + 2x_4 + x_5 = 3x_1 + 4x_2 + 5x_3 + 2x_4 + x_5$. By Corollary 9 $A \perp {}^{\sigma}A$.

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