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On the Riemann extension of the Gödel space-time metric

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Abstract. Some properties of the Gödel space-metric and its Riemann extension are studied. The spectrum of de Rham operator acting on 1-forms is studied. The examples of translation surfaces of the Gödel space-metric are constructed.

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1 Introduction

The notion of the Riemann extension of nonriemannian spaces was introduced first in [1]. Main idea of this theory is application of the methods of Riemann geometry for studying properties of nonriemaniann spaces.

For example the system of differential equations of the form

$$\frac{d^2x^k}{ds^2} + \Pi^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{1}$$

with arbitrary coefficients $\Pi^k_{ij}(x^l)$ can be considered as a system of geodesic equations of affinely connected space with local coordinates x^k.

For n-dimensional Riemannian spaces with the metrics

$${}^n ds^2 = g_{ij}dx^i dx^j$$

the system of geodesic equations looks similarly but the coefficients $\Pi^k_{ij}(x^l)$ now have very special form and depend on the choice of the metric g_{ij};

$$\Pi^i_{kl} = \Gamma^i_{kl} = \frac{1}{2}g^{im}(g_{mk,l} + g_{ml,k} - g_{kl,m})$$

In order that methods of Riemann geometry can be applied for studying properties of spaces with equations (1) the construction of $2n$-dimensional extension of the space with local coordinates x^i was introduced .

The metric of extended space is constructed with the help of coefficients of equation (1) and looks as follows

$${}^{2n}ds^2 = -2\Pi^k_{ij}(x^l)\Psi_k dx^i dx^j + 2d\Psi_k dx^k \tag{2}$$

where Ψ_k are the coordinates of additional space.

An important property of such type metric is that the geodesic equations of metric (2) decompos into two parts

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0, \tag{3}$$

and

$$\frac{\delta^2\Psi_k}{ds^2} + R^l_{kji}\dot{x}^j\dot{x}^i\Psi_l = 0, \tag{4}$$

where

$$\frac{\delta\Psi_k}{ds} = \frac{d\Psi_k}{ds} - \Gamma^l_{jk}\Psi_l\frac{dx^j}{ds}.$$

The first part (3) of the full system is the system of equations for geodesics of basic space with local coordinates x^i and it does not contain coordinates Ψ_k.

The second part (4) of system of geodesic equations has the form of linear 4×4 matrix system of second order ODE's for coordinates Ψ_k

$$\frac{d^2\vec{\Psi}}{ds^2} + A(s)\frac{d\vec{\Psi}}{ds} + B(s)\vec{\Psi} = 0 \tag{5}$$

with the matrix

$$A = A(x^i(s), \dot{x}^i(s)), \quad B = B(x^i(s), \dot{x}^i(s)).$$

From this point of view we have the case of geodesic extension of the basic space (x^i). It is important to note that the geometry of extended space is connected with geometry of basic space.

For example the property of the space to be a Ricci-flat

$$R_{ij} = 0, \quad R_{ij;k} + R_{ki;j} + R_{jk;i} = 0,$$

or symmetrical

$$R_{ijkl;m} = 0$$

keeps also for the extended space.

This fact givs us the possibility to use the linear system of equation (5) for studying properties of basic space.

In particular the invariants of 4×4 matrix-function

$$E = B - \frac{1}{2}\frac{dA}{ds} - \frac{1}{4}A^2$$

under change of coordinates Ψ_k can be used for that.

For example the condition

$$E = B - \frac{1}{2}\frac{dA}{ds} - \frac{1}{4}A^2 = 0$$

for a given system means that it is equivalent to the simplest system

$$\frac{d^2\vec{\Phi}}{ds^2} = 0$$

and corresponding extended space is a flat space.

Other cases of integrability of the system (5) are connected with non-flat spaces having special form of the curvature tensor.

Remark that for extended spaces all scalar invariants constructed with the help of curvature tensor and its covariant derivatives are vanishing.

The first applications of the notion of extended spaces to the studying of non-linear second order differential equations and the Einstein spaces were done in the works of author [2–11]

Here we consider properties of the Gödel space-time and its Riemann extension.

2 The Gödel space-time metric

The line element of the metric of the Gödel space-time in coordinates x, y, z, t has the form

$$ds^2 = -dt^2 + dx^2 - 2\,e^{\frac{x}{a}} dt dy - 1/2\,e^{2\,\frac{x}{a}} dy^2 + dz^2. \tag{6}$$

Here the parameter a is the velocity of rotation [12].

The geodesic equations of the metric (6) are given by

$$2\left(\frac{d^2}{ds^2}x\,(s)\right)a + \left(e^{\frac{x(s)}{a}}\right)^2\left(\frac{d}{ds}y\,(s)\right)^2 + 2\,e^{\frac{x(s)}{a}}\left(\frac{d}{ds}t\,(s)\right)\frac{d}{ds}y\,(s) = 0, \tag{7}$$

$$\left(\frac{d^2}{ds^2}y\,(s)\right)e^{\frac{x(s)}{a}}a - 2\left(\frac{d}{ds}t\,(s)\right)\frac{d}{ds}x\,(s) = 0, \tag{8}$$

$$\frac{d^2}{ds^2}z(s) = 0, \tag{9}$$

$$\left(\frac{d^2}{ds^2}t\,(s)\right)a + e^{\frac{x(s)}{a}}\left(\frac{d}{ds}y\,(s)\right)\frac{d}{ds}x\,(s) + 2\left(\frac{d}{ds}t\,(s)\right)\frac{d}{ds}x\,(s) = 0. \tag{10}$$

The first integral of geodesics satisfies the condition

$$-\left(\frac{d}{ds}t\,(s)\right)^2 + \left(\frac{d}{ds}x\,(s)\right)^2 - 2\,e^{\frac{x(s)}{a}}\left(\frac{d}{ds}t\,(s)\right)\frac{d}{ds}y\,(s) - 1/2\,e^{2\,\frac{x(s)}{a}}\left(\frac{d}{ds}y\,(s)\right)^2 +$$

$$+\left(\frac{d}{ds}z\,(s)\right)^2 - \mu = 0.$$

The symbols of Christoffel of the metric (6) are

$$\Gamma^4_{12} = \frac{\exp(x/a)}{2a}, \quad \Gamma^2_{14} = -\frac{\exp(-x/a)}{a}, \quad \Gamma^4_{14} = \frac{1}{a}, \quad \Gamma^1_{22} = \frac{\exp(2x/a)}{2a},$$

$$\Gamma^1_{24} = \frac{\exp(x/a)}{2a}.$$

To find solutions of the equations of geodesics (7)–(10) we present the metric (6) in equivalent form [13]

$$ds^2 = -(dt + \frac{a\sqrt{2}}{y}dx)^2 + \frac{a^2}{y^2}(dx^2 + dy^2) + dz^2. \tag{11}$$

The correspondence between the both forms of the metrics is given by the relations

$$y = a\sqrt{2}\exp(-x/a), \quad x = y.$$

The equations of geodesics of the metric (11) are defined by

$$\left(\frac{d^2}{ds^2}x(s)\right)a + \sqrt{2}\left(\frac{d}{ds}t(s)\right)\frac{d}{ds}y(s) = 0, \tag{12}$$

$$\left(\frac{d^2}{ds^2}y(s)\right)y(s)a - \left(\frac{d}{ds}x(s)\right)^2 a - \sqrt{2}\left(\frac{d}{ds}t(s)\right)\left(\frac{d}{ds}x(s)\right)y(s) - \left(\frac{d}{ds}y(s)\right)^2 a = 0, \tag{13}$$

$$\left(\frac{d^2}{ds^2}t(s)\right)(y(s))^2 - \sqrt{2}a\left(\frac{d}{ds}y(s)\right)\frac{d}{ds}x(s) - 2\left(\frac{d}{ds}t(s)\right)\left(\frac{d}{ds}y(s)\right)y(s) = 0, \tag{14}$$

$$\frac{d^2}{ds^2}z(s) = 0. \tag{15}$$

The geodesic equations admit the first integral

$$\frac{dt}{ds} = \frac{(-c_2/\sqrt{2} + \sqrt{2}y)}{c_0}, \qquad \frac{dx}{ds} = \frac{y(c_2 - y)}{ac_0},$$

$$\frac{dy}{ds} = \frac{y(x - c_1)}{ac_0}, \qquad \frac{dz}{ds} = \frac{c_3}{c_0}, \tag{16}$$

where c_i, a_0 are parameters.

Remark 1. In the theory of varieties the Chern-Simons characteristic class is constructed from a matrix gauge connection A^i_{jk} as

$$W(A) = \frac{1}{4\pi^2}\int d^3x\epsilon^{ijk}tr\left(\frac{1}{2}A_i\partial_j A_k + \frac{1}{3}A_iA_jA_k\right).$$

This term can be translated into a three-dimensional geometric quantity by replacing the matrix connection A^i_{jk} with the Christoffel connection Γ^i_{jk}.

For the density of Chern-Simons invariant the expression can be obtained [14]

$$CS(\Gamma) = \epsilon^{ijk}(\Gamma^p_{iq}\Gamma^q_{kp;j} + \frac{2}{3}\Gamma^p_{iq}\Gamma^q_{jr}\Gamma^r_{kp}).$$

For the metric (11) by the condition $z = const$

$$ds^2 = -a^2/y^2dx^2 - 2\sqrt{2}a/ydxdt + a^2/y^2dy^2 - dt^2$$

we find the quantity

$$CS(\Gamma) = -\frac{\sqrt{2}}{ay^2}.$$

For the spatial metric

$${}^3ds^2 = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}$$

of the metric (11)

$$-ds^2 = \frac{a^2}{y^2}(dx^2 + dy^2) + dz^2$$

the quantity $CS(\Gamma) = 0$.

3 The Riemann extension of the Gödel metric

The Christofell symbols of the metric (11) are

$$\Gamma^2_{11} = -\frac{1}{y}, \quad \Gamma^2_{22} = -\frac{1}{y}, \quad \Gamma^2_{14} = -\frac{\sqrt{2}}{2a}, \quad \Gamma^1_{24} = \frac{\sqrt{2}}{2a}, \quad \Gamma^4_{12} = -\frac{a\sqrt{2}}{2y^2}, \quad \Gamma^4_{14} = -\frac{1}{y}.$$

Now with the help of the formulae (2) we construct eight-dimensional extension of the metric (11).

It has the form

$${}^8ds^2 = \frac{2}{y}Qdx^2 + \frac{2a\sqrt{2}}{y^2}Vdxdy + \frac{2\sqrt{2}}{a}Qdxdt + \frac{2}{y}Qdy^2 + (\frac{4}{y}V - 2\frac{\sqrt{2}}{a}P)dydt +$$

$$+2dxdP + 2dydQ + 2dzdU + 2dtdV. \tag{17}$$

where (P, Q, U, V) are additional coordinates.

The Ricci tensor of the four-dimensional Gödel space with the metric (11) or (6) satisfies the condition

$${}^4R_{ik;l} + {}^4R_{li;k} + {}^4R_{kl;i} = 0.$$

This property is valid for the eight-dimensional space in local coordinates (x, y, z, t, P, Q, U, V) with the metric (11)

$${}^8R_{ik;l} + {}^8R_{li;k} + {}^8R_{kl;i} = 0.$$

The full system of geodesic equations for the metric (7) decomposes into two parts.

The first part coincides with the equations (12)–(15) on the coordinates (x, y, z, t) and second part forms the linear system of equations for coordinates P, Q, U, V.

They are defined as

$$\frac{d^2}{ds^2}P(s) = -\frac{\left(\sqrt{2}a^2\left(\frac{d}{ds}x\right)^2 + 2\left(\frac{d}{ds}t\right)\left(\frac{d}{ds}x\right)ya - \sqrt{2}a^2\left(\frac{d}{ds}y\right)^2\right)V(s)}{(y)^3\,a} -$$

-
$$\frac{\left(2\,\left(\frac{d}{ds}x\right)(y)^2 a+\sqrt{2}\left(\frac{d}{ds}t\right)(y)^3\right)\frac{d}{ds}Q(s)}{(y)^3 a}-\frac{\sqrt{2}a\left(\frac{d}{ds}V(s)\right)\frac{d}{ds}y}{(y)^2}, \tag{18}$$

$$\frac{d^2}{ds^2}Q(s)=\frac{\left(-3\,\left(\frac{d}{ds}x\right)^2 ya-2\sqrt{2}\left(\frac{d}{ds}t\right)\left(\frac{d}{ds}x\right)(y)^2-\left(\frac{d}{ds}y\right)^2 ya\right)Q(s)}{(y)^3 a}+$$
$$+\frac{\left(2\,a\left(\frac{d}{ds}y\right)\left(\frac{d}{ds}x\right)y+2\,\left(\frac{d}{ds}t\right)\left(\frac{d}{ds}y\right)(y)^2\sqrt{2}\right)P(s)}{(y)^3 a}+$$
$$+\frac{\left(-2\,\left(\frac{d}{ds}t\right)\left(\frac{d}{ds}y\right)ya-2\sqrt{2}a^2\left(\frac{d}{ds}y\right)\frac{d}{ds}x\right)V(s)}{(y)^3 a}+\frac{\sqrt{2}\left(\frac{d}{ds}t\right)\frac{d}{ds}P(s)}{a}-$$
$$-2\,\frac{\left(\frac{d}{ds}y\right)\frac{d}{ds}Q(s)}{y}+\frac{\left(-\sqrt{2}a^2\left(\frac{d}{ds}x\right)y-2\,\left(\frac{d}{ds}t\right)(y)^2 a\right)\frac{d}{ds}V(s)}{(y)^3 a}, \tag{19}$$

$$\frac{d^2}{ds^2}U(s)=0, \tag{20}$$

$$\frac{d^2}{ds^2}V(s)=\frac{\left(\left(\frac{d}{ds}x\right)^2 a\sqrt{2}y+2\,\left(\frac{d}{ds}t\right)\left(\frac{d}{ds}x\right)(y)^2+\sqrt{2}\left(\frac{d}{ds}y\right)^2 ay\right)P(s)}{a^2\,(y)^2}+$$
$$+\frac{\left(-2\sqrt{2}\left(\frac{d}{ds}t\right)\left(\frac{d}{ds}x\right)y(s)a-2\,a^2\left(\frac{d}{ds}x\right)^2\right)V(s)}{a^2\,(y)^2}+\frac{\sqrt{2}\left(\frac{d}{ds}y\right)\frac{d}{ds}P(s)}{a}-$$
$$-\frac{\sqrt{2}\left(\frac{d}{ds}x\right)\frac{d}{ds}Q(s)}{a}-2\,\frac{\left(\frac{d}{ds}y\right)\frac{d}{ds}V(s)}{y}+2\,\frac{Q(s)\left(\frac{d}{ds}t\right)\frac{d}{ds}y}{a^2}. \tag{21}$$

In result we have got a linear matrix-second order ODE for the coordinates U, V, P, Q

$$\frac{d^2\Psi}{ds^2}=A(x,\phi,z,t)\frac{d\Psi}{ds}+B(x,\phi,z,t)\Psi, \tag{22}$$

where

$$\Psi(s)=\begin{pmatrix} P(s)\\ Q(s)\\ U(s)\\ V(s)\end{pmatrix}$$

and A, B are some 4×4 matrix-functions depending on the coordinates $x(s)$, $y(s)$, $z(s)$, $t(s)$ and their derivatives.

Now we shall investigate properties of the matrix system of equations (18)–(21).

To integrate this system we use the relation

$$\dot{x}(s)P(s) + \dot{y}(s)Q(s) + \dot{z}(s)U(s) + \dot{t}(s)V(s) - \frac{s}{2} - \mu = 0, \tag{23}$$

which is valid for every Riemann extensions of affinely connected space and which is a consequence of the well known first integral of geodesic equations $g_{ik}\dot{x}^i\dot{x}^k = \nu$ of arbitrary Riemann space.

Using the expressions for the first integrals of geodesic (16) and $U(s) = \alpha s + \beta$ from the equation (20) the system of equations (18)–(21) may be simplified.

In result we get the system of equations for additional coordinates

$$\frac{d^2}{ds^2}P(s) = \frac{\left(\sqrt{2}c_0\,ac_1 - \sqrt{2}c_0\,ax\right)\frac{d}{ds}V(s)}{yc_0{}^2a} +$$

$$+\frac{\left(-\sqrt{2}c_2\,y - 2\sqrt{2}xc1 + \sqrt{2}c_1{}^2 + \sqrt{2}\,(y)^2 + \sqrt{2}\,(x)^2\right)V(s)}{yc_0{}^2a} - \frac{\left(\frac{d}{ds}Q(s)\right)c_2}{c_0\,a}, \tag{24}$$

$$\frac{d^2}{ds^2}Q(s) = -\frac{\left(2\,(y)^2\,c_1 - 2\,(y)^2\,x\right)P(s)}{ya^2c_0{}^2} -$$

$$-\frac{\left(y\,(x)^2 - (y)^3 + yc_1{}^2 + yc_2{}^2 - 2\,yxc_1\right)Q(s)}{ya^2c_0{}^2} -$$

$$-\frac{\left(2\,ac_0\,yx - 2\,ac_0\,yc_1\right)\frac{d}{ds}Q(s)}{ya^2c_0{}^2} - \frac{\left(ac_0\,yc_2 - 2\,ac_0\,(y)^2\right)\frac{d}{ds}P(s)}{ya^2c_0{}^2} - \frac{\sqrt{2}\frac{d}{ds}V(s)}{c_0} -$$

$$-\frac{\left(\sqrt{2}c_2\,ax - \sqrt{2}c_2\,ac_1\right)V(s)}{ya^2c_0{}^2}, \tag{25}$$

$$\frac{d^2}{ds^2}V(s) = \frac{\left((y)^2\sqrt{2}c_2 - (y)^3\sqrt{2} + y\sqrt{2}\,(x)^2 - 2\,y\sqrt{2}xc_1 + y\sqrt{2}c_1{}^2\right)P(s)}{c_0{}^2a^3} +$$

$$+\frac{\left(-\sqrt{2}yxc_2 + \sqrt{2}yc_1\,c_2 + 2\sqrt{2}\,(y)^2\,x - 2\sqrt{2}\,(y)^2\,c_1\right)Q(s)}{c_0{}^2a^3} +$$

$$+\frac{\left(-\sqrt{2}yac_0\,c_2 + \sqrt{2}\,(y)^2\,ac_0\right)\frac{d}{ds}Q(s)}{c_0{}^2a^3} + \frac{\left(\sqrt{2}yac_0\,x - \sqrt{2}yac_0\,c_1\right)\frac{d}{ds}P(s)}{c_0{}^2a^3} +$$

$$+\frac{\left(-2\,a^2c_0\,x + 2\,a^2c_0\,c_1\right)\frac{d}{ds}V(s)}{c_0{}^2a^3} + \frac{\left(-2\,ac_2\,y + 2\,a\,(y)^2\right)V(s)}{c_0{}^2a^3}. \tag{26}$$

The relation (23) in this case takes the form

$$-1/2\,\frac{(-2\,\alpha\,c3\,a+c_0\,a)\,s}{c_0\,a}-1/2\,\frac{\left(c_2\,\sqrt{2}a-2\,y\sqrt{2}a\right)V\,(s)}{c_0\,a}-$$

$$-1/2\,\frac{(2\,c_1\,y-2\,xy)\,Q\,(s)}{c_0\,a}-1/2\,\frac{\left(2\,(y)^2-2\,c_2\,y\right)P\,(s)}{c_0\,a}-\nu=0 \qquad (27)$$

and the system from three equations (24)–(26) can be reduced to the system of two coupled equations.

As example the substitution of the expression

$$Q\,(s)=1/2\,\frac{(-2\,\alpha\,c_3\,a+c_0\,a)\,s}{y\,(x-c_1)}+1/2\,\frac{\left(c_2\,\sqrt{2}a-2\,y\sqrt{2}a\right)V\,(s)}{y\,(x-c_1)}+$$

$$+1/2\,\frac{\left(2\,(y)^2-2\,c_2\,y\right)P\,(s)}{y\,(x\,(s)-c_1)}+\frac{\nu\,c_0\,a}{y\,(x-c_1)}$$

into the equation for $Q(s)$ give us the identity and in result our system takes the form

$$\frac{d^2}{ds^2}P\,(s)=-\frac{(-c_2+y)\,c_2\,\frac{d}{ds}P\,(s)}{c_0\,a\,(x-c_1)}-$$

$$-1/2\,\frac{\left(2\,(x)^2-4\,xc_1-2\,c_2\,y+{c_2}^2+2\,{c_1}^2\right)\sqrt{2}\frac{d}{ds}V\,(s)}{c_0\,(x-c_1)\,y}-$$

$$-\frac{c_2\,y\left((y)^2+{c_2}^2-2\,c_2\,y+{c_1}^2-2\,xc_1+(x)^2\right)P\,(s)}{a^2{c_0}^2\left((x)^2-2\,xc_1+{c_1}^2\right)}+L(s)V\,(s)-$$

$$-1/2\,\frac{(-2\,\alpha\,c_3\,ac_2+c_0\,ac_2)\,sy\,(s)}{a^2{c_0}^2\left((x)^2-2\,xc_1+{c_1}^2\right)}-1/2\,\frac{\left(-c_0\,a{c_2}^2+2\,\alpha\,c_3\,a{c_2}^2\right)s}{a^2{c_0}^2\left((x)^2-2\,xc_1+{c_1}^2\right)}-$$

$$-1/2\,\frac{\left((2\,\alpha\,c_3\,ac_2-c_0\,ac_2)\,(x)^2+(2\,c_1\,c_0\,ac_2-4\,\alpha\,c_3\,c_1\,ac_2)\,x\right)s}{a^2{c_0}^2y\left((x)^2-2\,xc_1+{c_1}^2\right)}-$$

$$-1/2\,\frac{\left(2\,\alpha\,c_3\,{c_1}^2ac_2-{c_1}^2c_0\,ac_2\right)s}{a^2{c_0}^2y\left((x)^2-2\,xc_1+{c_1}^2\right)}, \qquad (28)$$

where

$$L\,(s)=\frac{(y)^2\,\sqrt{2}c_2}{a{c_0}^2\left((x)^2-2\,xc_1+{c_1}^2\right)}+1/2\,\frac{\sqrt{2}\left(2\,(x)^2-3\,{c_2}^2-4\,xc_1+2\,{c_1}^2\right)y}{a{c_0}^2\left((x)^2-2\,xc_1+{c_1}^2\right)}+$$

$$+1/2\,\frac{\sqrt{2}\left(-2\,c_2\,(x)^2-2\,c_1{}^2c_2+c_2{}^3+4\,c_2\,c_1\,x\right)}{a c_0{}^2\left((x)^2-2\,x c_1+c_1{}^2\right)}+$$

$$+1/2\,\frac{\sqrt{2}\left(2\,(x)^4-8\,(x)^3\,c_1+\left(c_2{}^2+12\,c_1{}^2\right)(x)^2+\left(-2\,c_2{}^2c_1-8\,c_1{}^3\right)x\right)}{a c_0{}^2 y\left((x(s))^2-2\,x c_1+c_1{}^2\right)}+$$

$$+1/2\,\frac{\sqrt{2}\left(c_1{}^2c_2{}^2+2\,c_1{}^4\right)}{a c_0{}^2 y\left((x(s))^2-2\,x c_1+c_1{}^2\right)},$$

and

$$\frac{d^2}{ds^2}V(s)=-\frac{\left(2\,(x)^2-4\,x c_1-3\,c_2\,y+2\,(y)^2+2\,c_1{}^2+c_2{}^2\right)\frac{d}{ds}V(s)}{a c_0\,(x-c_1)}+$$

$$+\frac{\left((y)^2+c_2{}^2-2\,c_2\,y+c_1{}^2-2\,x c_1+(x)^2\right)\sqrt{2}y\frac{d}{ds}P(s)}{c_0\,a^2\,(x-c_1)}+$$

$$+M(s)V(s)+N(s)P(s)+1/2\,\frac{\sqrt{2}\,(c_0-2\,\alpha\,c_3)\,s\,(y)^3}{a^2c_0{}^2\,(x-c_1{}^2)}+$$

$$+1/2\,\frac{\sqrt{2}\,(-2\,c_0\,c_2+4\,\alpha\,c_3\,c_2)\,s\,(y)^2}{a^2c_0{}^2\,(x-c_1{}^2)}+$$

$$+1/2\,\frac{\sqrt{2}\left(-2\,\alpha\,c_3\,c_2{}^2+c_2{}^2c_0+c_1{}^2c_0+x^2(c_0-2\,\alpha\,c_3)-2\,\alpha\,c_3\,c_1{}^2\right)sy}{a^2c_0{}^2\,(x-c_1{}^2)}+$$

$$+1/2\,\frac{\sqrt{2}\,((4\,\alpha\,c_3c_1-2c_0\,c_1)\,x)\,sy}{a^2c_0{}^2\,(x-c_1{}^2)},\tag{29}$$

where

$$M(s)=-2\,\frac{(y)^4}{a^2c_0{}^2\left((x)^2-2\,x(s)\,c_1+c_1{}^2\right)}+5\,\frac{(y)^3\,c_2}{a^2c_0{}^2\left((x)^2-2\,x(s)\,c_1+c_1{}^2\right)}-$$

$$-\frac{\left(-4\,x c_1+2\,(x)^2+4\,c_2{}^2+2\,c_1{}^2\right)(y)^2}{a^2c_0{}^2\left((x)^2-2\,x c_1+c_1{}^2\right)}-\frac{\left(-c_2\,(x)^2-c_1{}^2c_2+2\,c_2\,c_1\,x-c_2{}^3\right)y}{a^2c_0{}^2\left((x)^2-2\,x c_1+c_1{}^2\right)},$$

$$N(s)=\frac{\sqrt{2}\,(y)^5}{a^3c_0{}^2\left((x)^2-2\,x c_1+c_1{}^2\right)}-3\,\frac{\sqrt{2}\,(y)^4\,c_2}{a^3c_0{}^2\left((x)^2-2\,x c_1+c_1{}^2\right)}+$$

$$+\frac{\sqrt{2}\left(2\,(x)^2+3\,c_2{}^2+2\,c_1{}^2-4\,x c_1\right)(y)^3}{a^3c_0{}^2\left((x)^2-2\,x c_1+c_1{}^2\right)}+$$

$$+\frac{\sqrt{2}\left(-3\,c_2\,(x)^2-c_2{}^3-3\,c_1{}^2c_2+6\,c_2\,c_1\,x\right)(y)^2}{a^3c_0{}^2\left((x)^2-2\,x\,c_1+c_1{}^2\right)}+$$

$$+\frac{\sqrt{2}\left((x)^4-4\,(x)^3\,c1+\left(6\,c1^2+c2^2\right)(x)^2+\left(-2\,c2^2c1-4\,c1^3\right)x\right)}{a^3c0^2\left((x)^2-2\,x\,c1+c1^2\right)}+$$

$$+\frac{\sqrt{2}\left(c1^4+c1^2c2^2\right)y}{a^3c0^2\left((x)^2-2\,x\,c1+c1^2\right)}.$$

The expressions for functions $x(s)$ and $y(s)$ are dependent on the choice of parameters and can be defined from the equations (16).

The integration of the equations (28)–(29) for the additional coordinates $P(s)$, $Q(s)$ is reduced to investigation of a 2×2 system of second order ODE's with variable coefficients.

Remark that the matrix E and its properties play important role in the analysis of such type of system of equations.

In result we get the correspondence between the geodesic in the x, y, x, t-space and the geodesic in the space with local coordinates P, Q, U, V (partner space).

The studying of such type of correspondence may be useful from various points of view.

4 Translation surfaces of the Gödel spaces

Now we discuss some properties of translation surfaces of the Gödel spaces.

According with definition ([15]) translation surfaces in arbitrary Riemannian space are defined by the systems of equations for local coordinates $x^i(u,v)$ of the space

$$\frac{\partial x^i(u,v)}{\partial u\partial v}+\Gamma^i_{jk}(x^m)\frac{\partial x^j(u,v)}{\partial u}\frac{\partial x^k(u,v)}{\partial v}=0, \tag{30}$$

where Γ^i_{jk} are the Christoffel coefficients.

In the case of the Gödel metric (6) we get the equations

$$\frac{\partial^2}{\partial u\partial v}x\,(u,v)+\frac{\sqrt{2}}{2a}\frac{\partial y(u,v)}{\partial u}\frac{\partial t(u,v)}{\partial v}+\frac{\sqrt{2}}{2a}\frac{\partial t(u,v)}{\partial u}\frac{\partial y(u,v)}{\partial v}=0, \tag{31}$$

$$\frac{\partial^2}{\partial u\partial v}y\,(u,v)-\frac{\sqrt{2}}{2a}\frac{\partial x(u,v)}{\partial v}\frac{\partial t(u,v)}{\partial u}-\frac{\sqrt{2}}{2a}\frac{\partial t(u,v)}{\partial v}\frac{\partial x(u,v)}{\partial u}-$$
$$-\frac{1}{y(u,v)}\frac{\partial x(u,v)}{\partial u}\frac{\partial x(u,v)}{\partial v}-\frac{1}{y(u,v)}\frac{\partial y(u,v)}{\partial u}\frac{\partial y(u,v)}{\partial v}=0, \tag{32}$$

$$\frac{\partial^2}{\partial u \partial v} t\left(u, v\right) - \frac{1}{y(u,v)} \frac{\partial y(u,v)}{\partial u} \frac{\partial t(u,v)}{\partial v} - \frac{1}{y(u,v)} \frac{\partial t(u,v)}{\partial u} \frac{\partial y(u,v)}{\partial v} -$$
$$- \frac{\sqrt{2a}}{2y(u,v)^2} \frac{\partial x(u,v)}{\partial u} \frac{\partial y(u,v)}{\partial v} - \frac{\sqrt{2a}}{2y(u,v)^2} \frac{\partial x(u,v)}{\partial v} \frac{\partial y(u,v)}{\partial u} = 0, \tag{33}$$

$$\frac{\partial^2}{\partial u \partial v} z\left(u, v\right) = 0. \tag{34}$$

Full integration of this nonlinear system of equations is a difficult problem. Give one example.

With this aim we present our system of equations in new coordinates $u = r + s, v = r - s$.

It takes the form

$$2 \left(1/4 \frac{\partial^2}{\partial r^2} x(r,s) - 1/4 \frac{\partial^2}{\partial s^2} x(r,s)\right) a +$$
$$+ \sqrt{2} \left(1/2 \frac{\partial}{\partial r} y(r,s) + 1/2 \frac{\partial}{\partial s} y(r,s)\right) \left(1/2 \frac{\partial}{\partial r} t(r,s) - 1/2 \frac{\partial}{\partial s} t(r,s)\right) +$$
$$+ \sqrt{2} \left(1/2 \frac{\partial}{\partial r} t(r,s) + 1/2 \frac{\partial}{\partial s} t(r,s)\right) \left(1/2 \frac{\partial}{\partial r} y(r,s) - 1/2 \frac{\partial}{\partial s} y(r,s)\right) = 0,$$

$$-2 \left(1/4 \frac{\partial^2}{\partial r^2} y(r,s) - 1/4 \frac{\partial^2}{\partial s^2} y(r,s)\right) y(r,s) a +$$
$$+2 \left(1/2 \frac{\partial}{\partial r} x(r,s) + 1/2 \frac{\partial}{\partial s} x(r,s)\right) \left(1/2 \frac{\partial}{\partial r} x(r,s) - 1/2 \frac{\partial}{\partial s} x(r,s)\right) a +$$
$$+ \sqrt{2} \left(1/2 \frac{\partial}{\partial r} x(r,s) + 1/2 \frac{\partial}{\partial s} x(r,s)\right) \left(1/2 \frac{\partial}{\partial r} t(r,s) - 1/2 \frac{\partial}{\partial s} t(r,s)\right) y(r,s) +$$
$$+2 \left(1/2 \frac{\partial}{\partial r} y(r,s) + 1/2 \frac{\partial}{\partial s} y(r,s)\right) \left(1/2 \frac{\partial}{\partial r} y(r,s) - 1/2 \frac{\partial}{\partial s} y(r,s)\right) a +$$
$$+ \sqrt{2} \left(1/2 \frac{\partial}{\partial r} t(r,s) + 1/2 \frac{\partial}{\partial s} t(r,s)\right) \left(1/2 \frac{\partial}{\partial r} x(r,s) - 1/2 \frac{\partial}{\partial s} x(r,s)\right) y(r,s) = 0,$$

$$-2 \left(1/4 \frac{\partial^2}{\partial r^2} t(r,s) - 1/4 \frac{\partial^2}{\partial s^2} t(r,s)\right) \left(y(r,s)\right)^2 +$$
$$+ \sqrt{2} a \left(1/2 \frac{\partial}{\partial r} x(r,s) + 1/2 \frac{\partial}{\partial s} x(r,s)\right) \left(1/2 \frac{\partial}{\partial r} y(r,s) - 1/2 \frac{\partial}{\partial s} y(r,s)\right) +$$
$$+ \sqrt{2} a \left(1/2 \frac{\partial}{\partial r} y(r,s) + 1/2 \frac{\partial}{\partial s} y(r,s)\right) \left(1/2 \frac{\partial}{\partial r} x(r,s) - 1/2 \frac{\partial}{\partial s} x(r,s)\right) +$$

$$+2\left(1/2\,\frac{\partial}{\partial r}y(r,s)+1/2\,\frac{\partial}{\partial s}y(r,s)\right)\left(1/2\,\frac{\partial}{\partial r}t(r,s)-1/2\,\frac{\partial}{\partial s}t(r,s)\right)y(r,s)+$$

$$+2\left(1/2\,\frac{\partial}{\partial r}t(r,s)+1/2\,\frac{\partial}{\partial s}t(r,s)\right)\left(1/2\,\frac{\partial}{\partial r}y(r,s)-1/2\,\frac{\partial}{\partial s}y(r,s)\right)y(r,s)=0.$$

A solution of this system of equations we shall seek in the form

$$y(r,s)=B(r),\quad t(r,s)=C(r)-s,\quad x(r,s)=s+A(r),$$

where $B(r),C(r),A(r)$ are some unknown functions.

In result our system takes the form

$$\left(\frac{d^2}{dr^2}C(r)\right)(B(r))^2-\sqrt{2}a\left(\frac{d}{dr}B(r)\right)\frac{d}{dr}A(r)-2\left(\frac{d}{dr}B(r)\right)B(r)\frac{d}{dr}C(r)=0,$$

$$\left(\frac{d^2}{dr^2}B(r)\right)B(r)a-a\left(\frac{d}{dr}A(r)\right)^2+a-\sqrt{2}B(r)\left(\frac{d}{dr}A(r)\right)\frac{d}{dr}C(r)-\sqrt{2}B(r)-$$

$$-\left(\frac{d}{dr}B(r)\right)^2a=0,$$

$$\left(\frac{d^2}{dr^2}A(r)\right)a+\sqrt{2}\left(\frac{d}{dr}B(r)\right)\frac{d}{dr}C(r)=0.$$

Using the first integral

$$\frac{d}{dr}C(r)=-\frac{\sqrt{2}a\frac{d}{dr}A(r)}{B(r)}+\alpha,$$

the system can be written in the form

$$\left(\frac{d^2}{dr^2}B(r)\right)B(r)a+a\left(\frac{d}{dr}A(r)\right)^2+a-\sqrt{2}\left(\frac{d}{dr}A(r)\right)\alpha\,B(r)-\sqrt{2}B(r)-$$

$$-\left(\frac{d}{dr}B(r)\right)^2a=0,$$

$$\left(\frac{d^2}{dr^2}A(r)\right)aB(r)-2\left(\frac{d}{dr}B(r)\right)a\frac{d}{dr}A(r)+\sqrt{2}\left(\frac{d}{dr}B(r)\right)\alpha\,B(r)=0.$$

Integration of the last equation gives us the expression for the function $A(r)$

$$A(r)=\int\frac{B(r)\left(\sqrt{2}\alpha+B(r)C_1\,a\right)}{a}dr+C_2$$

with parameters C_1,C_2 and α.

After substitution of the expression for $A(r)$ in the first equation we get the equation

$$\left(\frac{d^2}{dr^2}B(r)\right)B(r)a+(B(r))^3\sqrt{2}\alpha\,C_1+(B(r))^4\,{C_1}^2a+a-\sqrt{2}B(r)-$$

$$-\left(\frac{d}{dr}B(r)\right)^2 a = 0$$

for the function $B(r)$.

Remark that this equation can be written in the form

$$\frac{d^2}{dr^2}E(r) = \frac{\sqrt{2}e^{-E(r)}}{a} - {C_1}^2 e^{2E(r)} - \frac{\sqrt{2}\alpha\, C_1\, e^{E(r)}}{a} - e^{-2E(r)}$$

after the change of variable $B(r) = \exp(E(r))$.

With the help of its solutions the examples of the translation surfaces of the Gödel space (6) can be constructed.

They can be presented in form

$$t + x = A(r) + C(r), \quad y = B(r),$$

or $t(x, y) = x + \phi(y)$ with some function $\phi(y)$.

Detailed consideration of properties of this type of translation surfaces, their intrinsic geometry and characteristic lines will be done in following publications of author.

Remark that with the help of the translation surfaces the properties of closed trajectories of the Gödel space can be investigated.

Let us consider the eight-dimensional extension of the Gödel space with the metric (17).

Translation surfaces in this case are determined by the equations (28)–(31) for coordinates x, y, z, t and by the linear system of equations in coordinates P, Q, U, V

$$\frac{\partial^2}{\partial u \partial v}P(u, v) -$$
$$-\frac{1}{2ay^3}\left(-2a^2\sqrt{2}\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} - 2ay\frac{\partial x}{\partial u}\frac{\partial t}{\partial v} + 2a^2\sqrt{2}\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} - 2ay\frac{\partial t}{\partial u}\frac{\partial x}{\partial v}\right)V(u, v)+$$
$$+\frac{1}{2ay^3}\left(2a^2\frac{\partial x}{\partial u} + \sqrt{2}y^3\frac{\partial t}{\partial u}\right)\frac{\partial Q(u, v)}{\partial v} + \frac{1}{2ay^3}\left(2a^2\frac{\partial x}{\partial v} + \sqrt{2}y^3\frac{\partial t}{\partial v}\right)\frac{\partial Q(u, v)}{\partial u}+$$
$$+\frac{a\sqrt{2}}{2y^2}\frac{\partial y}{\partial u}\frac{\partial V(u, v)}{\partial v} + \frac{a\sqrt{2}}{2y^2}\frac{\partial y}{\partial v}\frac{\partial V(u, v)}{\partial u} = 0, \quad (35)$$

$$\frac{\partial^2}{\partial u \partial v}Q(u, v) +$$
$$+\frac{1}{2ay^3}\left(6ay\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + 2\sqrt{2}y^2\frac{\partial x}{\partial u}\frac{\partial t}{\partial v} + 2\sqrt{2}y^2\frac{\partial x}{\partial v}\frac{\partial t}{\partial u} + 2ay\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\right)Q(u, v)+$$
$$+\frac{1}{2ay^3}\left(-2ay\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - 2\sqrt{2}y^2\frac{\partial y}{\partial u}\frac{\partial t}{\partial v} - 2\sqrt{2}y^2\frac{\partial y}{\partial v}\frac{\partial t}{\partial u} - 2ay\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)P(u, v)+$$

$$+\frac{1}{2ay^3}\left(2ay\frac{\partial y}{\partial u}\frac{\partial t}{\partial v}+2\sqrt{2}y^2\frac{\partial y}{\partial v}\frac{\partial x}{\partial u}+2\sqrt{2}y^2\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}+2ay\frac{\partial y}{\partial v}\frac{\partial t}{\partial u}\right)V(u,v)+$$
$$+\frac{1}{y}\frac{\partial y}{\partial u}\frac{\partial Q(u,v)}{\partial v}+\frac{1}{y}\frac{\partial y}{\partial v}\frac{\partial Q(u,v)}{\partial u}-\frac{\sqrt{2}}{2a}\frac{\partial t}{\partial u}\frac{\partial P(u,v)}{\partial v}-\frac{\sqrt{2}}{2a}\frac{\partial t}{\partial v}\frac{\partial P(u,v)}{\partial u}+$$
$$+\frac{1}{2ay^3}\left(2ay^2\frac{\partial t}{\partial u}+\sqrt{2}a^2y\frac{\partial x}{\partial u}\right)\frac{\partial V(u,v)}{\partial v}+$$
$$+\frac{1}{2ay^3}\left(2ay^2\frac{\partial t}{\partial v}+\sqrt{2}a^2y\frac{\partial x}{\partial v}\right)\frac{\partial V(u,v)}{\partial u}=0, \tag{36}$$

$$\frac{\partial^2}{\partial u\partial v}V(u,v)+\;-\frac{1}{a^2}\left(\frac{\partial y}{\partial u}\frac{\partial t}{\partial v}+\frac{\partial y}{\partial v}\frac{\partial t}{\partial u}\right)Q(u,v)-$$
$$-\frac{1}{a^2y}\left(y\frac{\partial x}{\partial u}\frac{\partial t}{\partial v}+a\sqrt{2}\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+a\sqrt{2}\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+y\frac{\partial t}{\partial u}\frac{\partial x}{\partial v}\right)P(u,v)+$$
$$+\frac{1}{ay^2}\left(2a\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+y\sqrt{2}\frac{\partial x}{\partial u}\frac{\partial t}{\partial v}+y\sqrt{2}\frac{\partial t}{\partial u}\frac{\partial x}{\partial v}\right)V(u,v)+$$
$$+\frac{\sqrt{2}}{2a}\frac{\partial x}{\partial u}\frac{\partial Q}{\partial v}+\frac{\sqrt{2}}{2a}\frac{\partial x}{\partial v}\frac{\partial Q}{\partial u}+\frac{1}{y}\frac{\partial y}{\partial u}\frac{\partial V}{\partial v}+\frac{1}{y}\frac{\partial y}{\partial v}\frac{\partial V}{\partial u}-$$
$$-\frac{\sqrt{2}}{2a}\frac{\partial y}{\partial v}\frac{\partial P}{\partial u}-\frac{\sqrt{2}}{2a}\frac{\partial y}{\partial u}\frac{\partial P}{\partial v}=0, \tag{37}$$

$$\frac{\partial^2}{\partial u\partial v}U(u,v)=0. \tag{38}$$

The system of linear equations (32)–(35) is the matrix analog of the Laplace equation

$$\frac{\partial^2\Psi(u,v)}{\partial u\partial v}+A(u,v)\frac{\partial\Psi(u,v)}{\partial u}+B(u,v)\frac{\partial\Psi(u,v)}{\partial v}+C(u,v)\Psi(u,v)=0, \tag{39}$$

where

$$\Psi(u,v)=\begin{pmatrix}P(u,v)\\Q(u,v)\\U(u,v)\\V(u,v)\end{pmatrix}$$

is a vector-function, and $A(u,v), B(u,v), C(u,v)$ are matrices depending on the variables (u,v).

For integration of such type of equation the matrix generalization of the Darboux Invariants [16] can be used.

Remark 2. We remind basic facts on the integration of the matrix Laplace-equation.

The system (39) can be presented in the form

$$(\partial_u + B)(\partial_v + A)\Psi - H\Psi = 0,$$

or

$$(\partial_v + A)(\partial_u + B)\Psi - K\Psi = 0,$$

where

$$H = \frac{\partial A}{\partial u} + BA - C, \quad K = \frac{\partial B}{\partial v} + AB - C,$$

are the Darboux invariants of the system.

In the case $K = 0$ or $H = 0$ the system can be integrated.

If $K \neq 0$ and $H \neq 0$ the system may be presented in a similar form for the functions

$$\Psi_1 = \frac{\partial \Psi}{\partial y} + A\Psi.$$

or

$$\Psi_{-1} = \frac{\partial \Psi}{\partial x} + B\Psi.$$

In the first case one gets

$$\frac{\partial^2 \Psi_1(u,v)}{\partial u \partial v} + A_1(u,v)\frac{\partial \Psi_1(u,v)}{\partial u} + B_1(u,v)\frac{\partial \Psi_1(u,v)}{\partial v} + C_1(u,v)\Psi_1(u,v) = 0,$$

where

$$A_1 = HAH^{-1} - H_v H^{-1}, \quad B_1 = B, \quad C_1 = B_v - H + (HAH^{-1} - H_v H^{-1})B.$$

The invariants H_1 and K_1 for this equation are

$$H_1 = H - B_v + (HAH^{-1} - H_y H^{-1})_u + B(HAH^{-1} - H_v H^{-1}) - (HAH^{-1} - H_v H^{-1})B,$$

$$K_1 = H.$$

In the case $H_1 = 0$ the system can be integrated.

In the second case we get the equation for the function Ψ_{-1}

$$\frac{\partial^2 \Psi_{-1}(u,v)}{\partial u \partial v} + A_{-1}(u,v)\frac{\partial \Psi_{-1}(u,v)}{\partial u} + B_{-1}(u,v)\frac{\partial \Psi_{-1}(u,v)}{\partial v} + C_{-1}(u,v)\Psi_{-1}(u,v) = 0,$$

where

$$B_{-1} = KBK^{-1} - K_u K^{-1}, \quad A_{-1} = A, \quad C_{-1} = A_u - K + (KBK^{-1} - K_u K^{-1})A.$$

The invariants H_{-1} and K_{-1} for this equation are

$$K_{-1} = K - A_u + (KBK^{-1} - K_u K^{-1})_v + A(KBK^{-1} - K_u K^{-1}) - (KBK^{-1} - K_u K^{-1})A,$$

$$H_{-1} = K$$

and by the condition $K_{-1} = 0$ the system is also integrable.

To integrate the system of equations (39) in explicit form it is necessary to use the expressions for coordinates $x(u,v), y(u,v), z(u,v), t(u,v)$ of translation surfaces of the basic space.

The properties of the invariants H and K also may be important for classifications of translation surfaces of the basic and extended Gödel space.

5 On the spectrum of the Gödel space-time metric

In this section the spectrum λ of de Rham operator

$$\Delta = g^{ij}\nabla_i\nabla_j - Ricci,$$

defined on a four-dimensional Riemannian manifold and acting on 1-forms

$$\omega = A_i(x,y,z)dx^i = u(x,y,z,t)dx + v(x,y,z,t)dy + p(x,y,z,t)dz + q(x,y,z,t)dt$$

will be calculated.

The problem is reduced to the solution of the system of equations

$$g^{ij}\nabla_i\nabla_j A_k - R^l_k A_l - \mu^2 A_k = 0, \tag{40}$$

where ∇_k is a symbol of covariant derivative and R^i_j is the Ricci tensor of the metric g^{ij} of the Gödel space-time.

We use the Gödel space-time metric in form (11) and for simplicity sake the components of the 1-form ω will be presented as

$$A_k = [0, v(y,t), 0, q(y,t)].$$

As this takes place the system (40) looks as

$$-\frac{\partial}{\partial t}v(y,t) + \frac{\partial}{\partial y}q(y,t) = 0,$$

$$2\left(\frac{\partial}{\partial y}v(y,t)\right)y + \left(\frac{\partial^2}{\partial y^2}v(y,t)\right)y^2 + a^2\frac{\partial^2}{\partial t^2}v(y,t) - \mu^2 v(y,t)a^2 = 0,$$

$$2\left(\frac{\partial}{\partial y}q(y,t)\right)y - 2\left(\frac{\partial}{\partial t}v(y,t)\right)y + \left(\frac{\partial^2}{\partial y^2}q(y,t)\right)y^2 + \left(\frac{\partial^2}{\partial t^2}q(y,t)\right)a^2 -$$
$$-\mu^2 q(y,t)a^2 = 0.$$

It is equivalent to the following non-homogeneous equation

$$-\left(\frac{\partial^2}{\partial y^2}\Phi(y,t)\right)y^2 - \left(\frac{\partial^2}{\partial t^2}\Phi(y,t)\right)a^2 + \mu^2\Phi(y,t)a^2 + \epsilon = 0, \tag{41}$$

where

$$q(y,t) = \frac{\partial\Phi(y,t)}{\partial t}, v(y,t) = \frac{\partial\Phi(y,t)}{\partial y}$$

and ϵ is a parameter.

The simplest solution of homogeneous equation can be presented in the form

$$\Phi(y,t) = F_1(y)F_2(t), \tag{42}$$

where

$$\frac{d^2}{dt^2}F_2(t) = -\frac{c_1 F_2(t)}{a^2} + \mu^2 F_2(t),$$
$$\frac{d^2}{dy^2}F_1(y) = \frac{c_1 F_1(y)}{y^2}, \tag{43}$$

and c_1 is a parameter.

The second equation from (43) has the form

$$\frac{d^2}{dy^2}F_1(y) - \frac{c_1 F_1(y)}{y^2} = 0$$

and its solutions are defined by the relations

$$\frac{F_1}{\sqrt{y}} = C_1 \cos(b\ln(y)) + C_2 \sin(b\ln(y)), \quad b^2 = -c_1 - \frac{1}{4} > 0,$$

$$\frac{F_1}{\sqrt{y}} = C_1 x^b + C_2 x^{-b}, \quad b^2 = \frac{1}{4} + c_1 > 0,$$
$$\frac{F_1}{\sqrt{y}} = C_1 + C_2 \ln(y), \quad c_1 = -\frac{1}{4},$$

which depend on the parameter c_1.

The solutions of the first equation of the system (43) are

$$F_2(t) = C_3 \ \sin(1/4\,\frac{\sqrt{2}\sqrt{c_1 - 8\,\mu^2 a^2}t}{a}) + C_4 \ \cos(1/4\,\frac{\sqrt{2}\sqrt{c_1 - 8\,\mu^2 a^2}t}{a}).$$

In result the general solution of the equation (41) can be constructed with the help of solutions $F_1(y)$ and $F_2(t)$.

So in dependence depend upon the choice of c_1 the spectrum of manifold and the solutions of the equation (41) will be various.

The problem of solutions of the system (40) in more general case of the 1-form $\omega = A_i(x,y,z)dx^i$ requires more detailed consideration.

Remark 3. For determination of the spectrum λ of Laplace operator

$$\Delta = g^{ij}\nabla_i\nabla_j$$

acting on the $0-form$- function $\psi(x,y,z,t)$ defined on the manifold with the metric g_{ij}, it is necessary to solve the equation

$$g^{ij}\nabla_i\nabla_j\psi = \lambda\psi.$$

In the case of Gödel space-time metric (6) we get

$$a\left(\frac{\partial^2}{\partial x^2}\psi(x,y,z,t)\right)e^{\frac{2x}{a}} + 2a\left(\frac{\partial^2}{\partial y^2}\psi(x,y,z,t)\right) + \left(\frac{\partial}{\partial x}\psi(x,y,z,t)\right)e^{\frac{2x}{a}} -$$

$$-4a\,e^{\frac{x}{a}}\left(\frac{\partial^2}{\partial t\partial y}\psi(x,y,z,t)\right)+a\left(\frac{\partial^2}{\partial z^2}\psi(x,y,z,t)\right)e^{\frac{2x}{a}}+a\left(\frac{\partial^2}{\partial t^2}\psi(x,y,z,t)\right)e^{\frac{2x}{a}}-$$

$$-a\lambda\,\psi(x,y,z,t)e^{\frac{2x}{a}}=0.$$

The substitution here the function $\psi(x,y,z,t)$ in form

$$\psi(x,y,z,t)=V(x)\exp(-x/(2a))\exp(my+nz+kt),$$

where m,n,k are the parameters lead to the equation

$$\frac{d^2}{dx^2}V(x)-\left(-2m^2\,e^{-\frac{2x}{a}}+4km\,e^{-\frac{x}{a}}-n^2-k^2+1/4\,a^{-2}+\lambda\right)V(x)=0.$$

having the form of one dimensional Schrödinger equation for the spectrum of the particle in the field with the Morse potential.

In such type of potential a finite number of stationary states λ_n at the some relations between the parameters m,n,k,a may be existed.

This fact is important for understanding of the properties of the Gödel space-time metric.

6 Spatial metric of the four-dimensional Gödel space-time

The spatial metric of any four-dimensional metric

$${}^4ds^2=g_{\alpha\beta}dx^\alpha dx^\beta+2g_{0\alpha}dx^0dx^\alpha+g_{00}dx^0dx^0$$

has the form

$${}^3dl^2=\gamma_{\alpha\beta}dx^\alpha dx^\beta,$$

where

$$\gamma_{\alpha\beta}=-g_{\alpha\beta}+\frac{g_{0\alpha}g_{0\beta}}{g_{00}}$$

is a three-dimensional tensor determining the properties of the space.

In the case of the Gödel space-time the spatial three-dimensional metric has the form

$$-{}^3dl^2=\frac{a^2}{y^2}(dx^2+dy^2)+dz^2. \tag{44}$$

Three-dimensional space with the metric (44) belongs to one of the eight types of W.Thurston geometries and has diverse global properties.

In particular it admits the surfaces bundle.

As example we consider the translation surfaces of the space (44).

They defined by the system of equations for coordinates $x(u,v),y(u,v)$ and $z(u,v)$

$$\left(\frac{\partial^2}{\partial u\partial v}x(u,v)\right)y(u,v)-\left(\frac{\partial}{\partial u}x(u,v)\right)\frac{\partial}{\partial v}y(u,v)-\left(\frac{\partial}{\partial u}y(u,v)\right)\frac{\partial}{\partial v}x(u,v)=0,$$

$$\left(\frac{\partial^2}{\partial u \partial v} y(u,v)\right) y(u,v) + \left(\frac{\partial}{\partial u} x(u,v)\right) \frac{\partial}{\partial v} x(u,v) - \left(\frac{\partial}{\partial u} y(u,v)\right) \frac{\partial}{\partial v} y(u,v) = 0, \tag{45}$$

$$\frac{\partial^2}{\partial u \partial v} z(u,v) = 0.$$

The simplest solutions of these equations are of the form

$$y(u,v) = 1/2 \left(1 + \left(\left(\frac{u}{v}\right)^{C_1}\right)^2 \left(e^{C_2\, C_1}\right)^{-2}\right) e^{C_2\, C_1} \left(\left(\frac{u}{v}\right)^{C_1}\right)^{-1} C_1{}^{-1},$$

$$x(u,v) = \ln(u) + \ln(v),$$

and

$$z(u,v) = A(u) + B(v),$$

where $A(v)$ and $B(v)$ are arbitrary functions and C_1, C_2 are parameters.

In particular case $C_1 = 1$, $C_2 = 0$ one get

$$y(u,v) = 1/2\, \frac{u^2+v^2}{uv}, \qquad x(u,v) = \ln(uv).$$

From here we find

$$v = \sqrt{ye^x + e^x\sqrt{y^2-1}}, \qquad u = \frac{e^x}{\sqrt{ye^x + e^x\sqrt{y^2-1}}}$$

and

$$z(x,y) = A(\sqrt{ye^x + e^x\sqrt{y^2-1}}) + B(\frac{e^x}{\sqrt{ye^x + e^x\sqrt{y^2-1}}})$$

with arbitrary functions $A(u), B(v)$.

The properties of surfaces are dependent on the choice of the functions A and B.

Remark 4. From the system (45) we find the relations

$$\left(\frac{\partial}{\partial v} x(u,v)\right)^2 - \frac{e^{2\,z(u,v)}}{v^2} + \left(\frac{\partial}{\partial v} z(u,v)\right)^2 e^{2\,z(u,v)} = 0,$$

$$\left(\frac{\partial}{\partial u} x(u,v)\right)^2 - \frac{e^{2\,z(u,v)}}{u^2} + \left(\frac{\partial}{\partial u} z(u,v)\right)^2 e^{2\,z(u,v)} = 0,$$

where $z(u,v) = \ln(y(u,v))$.

This fact allows us to get one equation in variable $y(u,v)$ only.

$$\left(-\sqrt{(y)^2 - \left(\frac{\partial}{\partial v} y\right)^2 v^2}\, v \left(\frac{\partial}{\partial u} y\right) u^2 + \sqrt{(y)^2 - \left(\frac{\partial}{\partial u} y\right)^2 u^2}\, u \left(\frac{\partial}{\partial v} y\right) v^2\right) \frac{\partial^2}{\partial u \partial v} y +$$

$$+\sqrt{(y(u,v))^2 - \left(\frac{\partial}{\partial v} y\right)^2 v^2}\, v \left(\frac{\partial}{\partial v} y\right) y - \sqrt{(y)^2 - \left(\frac{\partial}{\partial u} y\right)^2 u^2}\, u \left(\frac{\partial}{\partial u} y\right) y = 0.$$

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