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On a small quasi-compactness

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Abstract. The notion of small quasi-compactness is introduced and studied. Let P be a small quasi-compactness. We prove that the classes of equivalence of P-compactifications of a given space X form a lattice with maximal and minimal elements. Some properties of maximal elements are studied.

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1 Introduction

Compactness is one of the most important notions.

A quasi-compactness is a class of spaces which is multiplicative, hereditary with respect to closed subspaces and contains an infinite T_0-space.

A g-extension of a space X is a pair (Y, f), where Y is a T_0-space, $f : X \to Y$ is a continuous mapping and the set $f(X)$ is dense in Y. If f is an embedding of X into Y, then (Y, f) is an extension of the space X.

Denote by $E(X)$ the class of all extensions of a space X and by $GE(X)$ the class of all g-extensions of the space X. If $e_X(x) = x$ for every $x \in X$, then $(X, e_X) \in E(X)$. Thus $\Phi \neq E(X) \subseteq GE(X)$.

In the family $GE(X)$ there exists a binary relation $\leq$: $(Y_1, f_1) \leq (Y_2, f_2)$ if there exists a continuous mapping $\varphi : Y_2 \to Y_1$ such that $f_1 = \varphi \circ f_2$, i. e. $f_1(x) = \varphi(f_2(x))$ for each $x \in X$.

If $(Y_1, f_1) \leq (Y_2, f_2)$ and $(Y_2, f_2) \leq (Y_1, f_1)$ then we say that the g-extensions (Y_1, f_1) and (Y_2, f_2) are equivalent and we denote $(Y_1, f_1) \approx (Y_2, f_2)$.

We say that (Y, f) is a g-extension with a T_1-remainder if for every point $x \in Y \backslash f(X)$ the set $\{x\}$ is closed in Y.

1.1. Example. Let X be an infinite T_0-space, A and B be two non-empty sets, $Y_1 = X \cup A$, $Y_2 = X \cup B$, $f_1(x) = f_2(x) = x$ for every $x \in X$, X is an open subspace of Y_1 and Y_2, the neighborhoods of the point $x \in A$ are of the form $Y_1 \backslash (F \cup \Phi)$, where F is a closed compact subset of X and Φ is a finite subset of A, the neighborhoods of the points $x \in B$ are of the form $Y_2 \backslash (F \cup \Phi)$, where F is a closed compact subset of X and Φ is a finite subset of B. The pairs (Y_1, f_1) and (Y_2, f_2) are equivalent compactifications of the space X. Let (Y, f) be a g-compactification of the space X with a T_1-remainder and $Y \backslash f(X) \neq \emptyset$. Fix a non-empty set A. We put $Z = Y \cup A$. Consider some mapping $\varphi : A \to Y \backslash f(X)$. On Z we consider the topology with the base $\{U \subseteq Y : U$ is open in $Y\} \cup \{\varphi^{-1}(U) \backslash F) \cup (U \backslash \Phi) : U$ is an open subset of Y, F is a finite subset of A and Φ is a finite subset of $Y \backslash f(X)\}$. Then (Z, f) is a g-compactification of the space X with a T_1-remainder and the g-compactifications (Y, f) and (Z, f) are equivalent. Thus the class of equivalence of some g-compactification of X is not a set.

For every g-extension (Y, f) of a space X by $e(Y, f, X)$ we denote the class of all g-extensions of X equivalent to the g-extension (Y, f) .

1.2. Definition. *A class L of g-extensions of a space X is a lattice of g-extensions if the following conditions are fulfilled:*

– there exists a set $e(L) \subseteq L$ such that $L \subseteq \cup\{e(Y, f, X) : (Y, f) \in e(L)\}$;

– there exists a g-extension $(m_L X, m_L) \in L$ such that $(m_L X, m_L) \leq (Y, f)$ for every $(Y, f) \in L$;

– for every non-empty set $A \subseteq L$ there exists a g-extension $(Z, g) \in L$ such that $(Z, g) = \vee A$ and $(Y, f) \leq (Z, g)$ for every $(Y, f) \in L$.

If L is a lattice of g-extensions of a space X, then by $(\beta_L X, \beta_L)$ we denote some maximal element of the class L.

1.3. Example. M. Hušec [6, 7] constructed an infinite non-compact T_1- space X such that the class of all $T_1 - g$-compactifications of X is not a lattice.

Let P be a quasi-compactness.

A g-extension (Y, f) of a space X is called a $g - P$-extension of X if $Y \in P$. Let $PGE(X) = \{(Y, f) \in GE(X) : Y \in P\}$ be the class of all $g - P$-extensions and $PE(X) = E(X) \cap PGE(X)$ be the class of all P-extensions of the space X.

If $PGE(X)$ is a lattice of g-extensions of the space X, then $(\beta_P X, \beta_P)$ is one of the maximal elements of the class $PGE(X)$.

First General Problem. To find the methods of construction and of study of the P-extensions and of special P-extensions of a given space X.

Second General Problem. Under which conditions the class $PGE(X)$ is a lattice?

Third General Problem. Let P be a compactness and K be a class of spaces. Under which conditions there exists a set valued functor $F : K \to P$ such that:

– $F(X)$ is a non-empty lattice of $g - P$-extensions of the space X for every space X;

– $F(X) \cap PE(X) \neq \emptyset$ for every $X \in K_B$ and the maximal element $(\beta_F X, \beta_F)$ of the lattice $F(X)$ is an extension of X;

– for every closed continuous mapping $f : X \to Y$ of a space $X \in K$ onto a space $Y \in K$ there exists a continuous extension $g = \beta f : \beta_F X \to \beta_F Y$ such that $f = g|X$?

The functor F with these properties is called a functor of the Wallman type.

Fourth General Problem. Let P be a compactness and K be a class of spaces. Under which conditions there exists a set-valued functor $F : K \to P$ such that:

– $F(X) \cap PE(X) \neq \emptyset$ and $F(X)$ is a lattice of $g - P$-extensions of the space X for every space $X \in K$;

– for every continuous mapping $f : X \to Y$ of a space $X \in K$ into a space $Y \in K$ there exists a continuous extension $g = \beta f : \beta_F X \to \beta_F Y$ of the mapping f onto the maximal extensions?

A functor with these properties is called a functor of the Stone-Čech type. Every functor of the Stone-Čech type is a functor of the Wallman type.

1.4. Example. If P is a compactness, i.e. P is a quasi-compactness and every space $X \in P$ is a Hausdorff space, then for every space X the class $PGE(X)$ is a set.

If $K = \{X : E(X) \cap PGE(X) \neq \emptyset\}$, then $F : K \to PGE(X)$, where $F(X) = PG(X)$, is a functor of the Stone-Čech type.

1.5. Example. Let K be the class of all T_0-spaces, ωX be the Wallman extension of the T_0-space X (see [1]). A g-compactification (Y, f) of a space X is called a regular g-compactification of X if $\{cl_Y A : A \subseteq f(X)\}$ is a closed base of Y and there exists a continuous mapping $g : \omega X \to Y$ such that $g(x) = f(x)$ for every $x \in X$. If the mapping g is closed, then the g-compactification (Y, f) is called a $g - \omega\alpha$-compactification of X (see [9]). Let $F(X) = \{(Y, f) : (Y, f)$ is a regular g-compactification of X $\}$ and $\Phi(X) = \{(Y, f) : (Y, f)$ is a $g - \omega\alpha$-compactification of X $\}$. Then F and Φ are functors of the Wallman type.

1.6. Example. Let K be the class of all T_0-spaces and $PGE(X)$ be the set of all spectral g-compactifications of the T_0-space X. Then the correspondence $X \to PGE(X)$ is a functor of the Stone-Čech type (see [1]).

1.7. Example. Let K be the class of all completely regular spaces and $PGE(X)$ be the set of all Hausdorff g-compactifications of the space X. Then the correspondence $X \to PGE(X)$ is a functor of the Stone-Čech type.

The purpose of the present paper is to investigate the class of P-extensions of topological spaces.

In this article we shall use the following notations:

– we denote by $cl_X A$ or clA the closure of a set A in a space X;
– we denote by $|A|$ the cardinality of a set A;
– we denote by $w(X)$ the weight of a space X.
– R is the space of reals, $N = \{1, 2, ...\}$, $I = [0, 1]$;
– every space is considered to be a T_0-space.

We use the terminology from [3, 1].

2 Small quasi-compactness

Let K be a class of T_0-spaces and $2 \leq |X|$ for some $X \in K$. Then there exists a minimal quasi-compactness $P(K)$ such that $K \subseteq P(K)$. We put $KGE(X) = P(K)GE(X)$ for every non-empty T_0-space X.

2.1. Definition. *A quasi-compactness P is called a small quasi-compactness if there exists a set K of spaces such that $P = P(K)$.*

2.2. Proposition. *Let P be a small quasi-compactness and for every space $X \in P$ there exists a point $b_X \in X$ such that the set $\{a_X\}$is closed in X. Then $P = P(\{E\})$ for some space $E \in P$.*

Proof. There exists a non-empty set $K \subseteq P$ of spaces such that $P = P(K)$. Suppose that $K = \{X_\alpha : \alpha \in A\}$. For every $\alpha \in A$ there exists a point $b_\alpha \in X_\alpha$ such that the set $\{b_\alpha\}$ is closed in X_α. We put $E = \Pi\{X_\alpha : \alpha \in A\}$ and $E_\beta = \{(x_\alpha : \alpha \in A) \in E : x_\alpha = b_\alpha$ for every $\alpha \in A\backslash\{\beta\}$ and $x_\beta \in X_\beta\}$. Then E_β is a closed subspace of the space E. For every $\beta \in A$ the spaces E_α and X_α are homeomorphic. Thus $P(K) = P(\{E\})$.

2.3. Lemma. *Let F be a non-empty compact subset of T_0-space X. Then there exists a point $b \in F$ such that the set $\{b\}$ is closed in X.*

Proof. Let ξ be a maximal family of closed subsets of the space X such that:

1. $\emptyset \notin \xi$ and $H \subseteq F$ for every $H \in \xi$;
2. If $H, M \in \xi$, then $H \cap M \in \xi$.

We put $\Phi = \cap\xi$. The set Φ is non-empty and closed in X. There exists a unique point $b \in F$ such that $\Phi = \{b\}$. Really, if $x_1, x_2 \in \Phi$ and $x_1 \neq x_2$, then there exists an open subset U of X such that $U \cap \{x_1, x_2\}$ is a singleton set. Then $\Phi\backslash U \in \xi$ and $\{x_1, x_2\}\backslash\Phi \neq \emptyset$, a contradiction. The proof is complete.

2.4. Corollary. *Let P be a small quasi-compactness and every space $X \in P$ be a compact T_0-space. Then $P = (\{E\})$ for some space $E \in P$.*

Fix a quasi-compactness P and a non-empty space X. For every $Y \in P$ denote by $C(X, Y)$ the set of all continuous mappings of the space X into the space Y. Let $\Phi = \{f_\alpha : X \to Y_\alpha : \alpha \in A\} \subseteq \cup\{C(X, Y) : Y \in P\}$ be a set of mappings. We put $f_\Phi(x) = \{f_\alpha(x) : \alpha \in A\} \in \Pi\{Y_\alpha : \alpha \in A\}$ for every $x \in X$. Denote by $e_\Phi X$ the closure of the set $f_\Phi(X)$ in the space $\Pi\{Y_\alpha : \alpha \in A\}$. Then $(e_\Phi X, f_\Phi) \in PGE(X)$.

2.5. Theorem. *Let K be a class of T_0-spaces and $P = P(K)$. For every T_0-space X and every $g - P$-extension $(Y, f) \in PGE(X)$ there exists a set $\Phi = \{f_\alpha : X \to Y_\alpha : \alpha \in A\} \subseteq \cup\{C(X, Z) : Z \in K\}$ such that the $g - P$-extensions (Y, f) and $(e_\Phi X, f_\Phi)$ are equivalent. Moreover, $f_\alpha \neq f_\beta$ for every $\alpha, \beta \in A$ and $\alpha \neq \beta$.*

Proof. There exists a set $\{E_\beta : \beta \in B\} \subseteq K$ such that Y is homeomorphic to a closed subset of the space $E = \Pi\{E_\beta : \beta \in B\}$. We assume that Y is a closed subspace of the space E. Let $p_\beta : E \to E_\beta$ be the continuous projections of E onto $E_\beta : p_\beta((x_\mu : \mu \in B)) = x_\beta$ for every point $(x_\mu \in E_\mu : \mu \in B) \in E$. For every $\beta \in B$ we put $g_\beta = p_\beta \circ f$. Then $g_\beta : X \to E_\beta$ is a continuous mapping. There exists a minimal set of mappings $\Phi = \{f_\alpha : X \to Y_\alpha : \alpha \in A\}$ such that:

1. For every $\alpha \in A$ there exists a $\beta \in B$ such that $f_\alpha = g_\beta$.
2. For every $\beta \in B$ there exists a unique $\alpha = i(\beta) \in A$ such that $f_\alpha = g_\beta$.

Thus there exists a mapping $i : B \to A$ of B onto A such that $f_\alpha = g_\beta$ for all $\alpha \in A$ and $\beta \in i^{-1}(\alpha)$. If $\alpha, \lambda \in A$ and $\alpha \neq \lambda$, then $f_\alpha \neq f_\lambda$.

We put $B_\alpha = i^{-1}(\alpha)$ for each $\alpha \in A$. By construction, $E_\beta = E_\mu$ for all $\alpha \in A$ and $\beta, \mu \in B_\alpha$. We may consider that $E_\beta = X_\alpha$ for all $\alpha \in A$ and $\beta \in B_\alpha$. For every $\beta \in B_\alpha$ we consider the mapping $\delta_{\alpha\beta} : X_\alpha \to E_\beta$ such that $\delta_{\alpha\beta}(x) = x$ for every $x \in X_\alpha$. We put $\delta_\alpha(x) = (\delta_{\alpha\beta}(x) : \beta \in B_\alpha) \in \Pi\{E_\beta : \beta \in B_\alpha\}$ for every $\alpha \in A$ and every $x \in X_\alpha$. Then $\delta_\alpha : X_\alpha \to \Pi\{E_\beta : \beta \in B_\alpha\}$ is an embedding. The set $\Delta_\alpha(X_\alpha) = \delta_\alpha(X_\alpha)$ is the diagonal of the space X_α in $\Pi\{E_\beta : \beta \in B_\alpha\}$. Fix $\beta(\alpha) \in B_\alpha$. Let $h_\alpha : \Pi\{E_\beta : \beta \in B_\alpha\} \to X_\alpha$ be the projection $h_\alpha(x_\beta : \beta \in B_\alpha) = x_{\beta(\alpha)}$ for every point $(x_\beta : \beta \in B_\alpha) \in \Pi\{E_\beta : \beta \in B_\alpha\}$.

Now we consider the continuous mapping $h : \Pi\{E_\beta : \beta \in B\} = \Pi\{\Pi\{E_\beta : \beta \in B_\alpha\} : \alpha \in A\} \to \Pi\{X_\alpha : \alpha \in A\}$, where $h(x_\beta : \beta \in B) = (h_\alpha(x_\beta : \beta \in B_\alpha) : \alpha \in A)$ for every point $(x_\beta : \beta \in B) \in \Pi\{E_\beta : \beta \in B\}$. The mapping $\delta : \Pi\{X_\alpha : \alpha \in A\} \to \Pi\{\Delta_\alpha(X_\alpha) : \alpha \in A\}$, where $\delta(x_\alpha : \alpha \in A) = (\delta_\alpha(x_\alpha) : \alpha \in A)$ for every $(x_\alpha : \alpha \in A) \in \Pi\{X_\alpha : \alpha \in A\}$, is a homeomorphism of the space $\Pi\{X_\alpha : \alpha \in A\}$ onto the subspace $\Pi\{\Delta_\alpha(X_\alpha) : \alpha \in A\}$ of the space $\Pi\{E_\beta : \beta \in B\}$.

Let $f_\Phi(x) = (f_\alpha(x) : \alpha \in A)$ for every $x \in X$ and $e_\Phi X$ be the closure of the subspace $f_\Phi(X)$ in $\Pi\{X_\alpha : \alpha \in A\}$. Let $\varphi = \delta | e_\Phi X : e_\Phi X \to \Pi\{E_\beta : \beta \in B\}$. Then $\varphi(f_\Phi(x)) = (g_\beta(x) : \beta \in B) = f(x)$ for every $x \in X$. Thus $(Y, f) \leq (e_\Phi X, f_\Phi)$. Let $\Psi = h\,|\,Y$. Then $\Psi(f(x)) = f_\Phi(x)$ for every $x \in X$. Thus $(e_\Phi X, f_\Phi) \leq (Y, f)$. The proof is complete.

2.6. Remark. *By construction, $\varphi : e_\Phi X \to Y$ is an embedding and $\Psi : Y \to e_\Phi X$ is a retract, i.e. $\varphi(\Psi(y)) = y$ for every $y \in \varphi(e_\Phi X)$.*

2.7. Theorem. *Let $\Phi = \{f_\alpha : X \to Y_\alpha : \alpha \in A\} \subseteq \cup\{C(X,Y) : Y \in P\}$ and $G = \{g_\beta : X \to Z_\beta : \beta \in B\} \subseteq \cup\{C(X,Y) : Y \in P\}$ be two sets of mappings, $A \subseteq B$ and $Y_\alpha = Z_\alpha$, $\quad f_\alpha = g_\alpha$ for each $\alpha \in A$. Then $(e_\Phi X, f_\Phi) \leq (e_G X, f_G)$.*

Proof. Consider the projection $p : \Pi\{Z_\beta : \beta \in B\} \to \Pi\{Y_\alpha : \alpha \in A\} = \Pi\{Z_\alpha : \alpha \in A\}$, where $p(z_\beta : \beta \in B) = (z_\beta : \beta \in A)$. Then $p(f_G(x)) = f_\Phi(x)$ for every $x \in X$. Thus $p(e_G X) \subseteq e_\Phi X$ and $(e_\Phi X, f_\Phi) \leq (e_G X, f_G)$. The proof is complete.

2.8. Corollary. *Let A be a set, $\Phi_\alpha = \{g_\beta : X \to Z_\beta : \beta \in B_\alpha\} \subseteq \cup\{C(X,Y) : Y \in P\}$ be a set of continuous mappings for every $\alpha \in A$, $\quad B = \cup\{B_\alpha : \alpha \in A\}$ and $\Phi = \{g_\beta : X \to Z_\beta : \beta \in B\}$. Then $(e_\Phi X, f_\Phi) = \vee\{(e_{\Phi_\alpha} X, f_{\Phi_\alpha}) : \alpha \in A\}$.*

2.9. Corollary. *Let P be a small quasi-compactness. Then there exists a set $K \subseteq P$ of spaces such that:*

1. For every $(Y, f) \in PGE(X)$ there exist a set $\Phi \subseteq \cup\{C(X,Y) : Y \in K\}$, an embedding $\varphi : e_\Phi X \to Y$ and a retraction $\Psi : Y \to e_\Phi X$ such that $\varphi(\Psi(y)) = y$ for every $y \in \varphi(e_\Phi X)$ and $(Y, f) \sim (e_\Phi X, f_\Phi)$.

2. The class $PGE(X)$ is a lattice provided $PE(X) \neq \emptyset$.

3. For every space X there exists a maximal element $(\beta_p X, \beta_p)$, where $(\beta_p X, \beta_p) = (e_\Phi X, f_\Phi)$ for $\Phi = \cup\{C(X,Y) : Y \in K\}$.

2.10. Definition. *A quasi-compactness P is called a virtual small quasi-compactness if for every space X the class $PGE(X)$ is a lattice.*

Every small quasi-compactness is a virtual small quasi-compactness

2.11. Corollary. *Let P be a virtual small quasi-compactness. Then:*

1. For every space X there exists some maximal element $(\beta_P X, \beta_P)$ in $PGE(X)$.

2. For every continuous mapping $f : X \to Y$ there exists a continuous mapping $\beta f : \beta_P X \to \beta_P Y$ such that $\beta f(\beta_P(x)) = \beta_P(f(x))$ for every $x \in X$.

3. For every continuous mapping $f : X \to Y$ into a space there exists a continuous mapping $\beta f : \beta_P X \to Y$ such that $\beta f(\beta_P(x)) \to f(x)$ for every $x \in X$.

2.12. Remark. *If $Y \in P$ and $i_\varphi : Y \to Y$ is the identical mapping, then $(Y, i_\varphi) = (\beta_P Y, \beta_P)$ is one of the maximal elements from $PGE(X)$ and $PE(X) \neq \emptyset$.*

3 On E-compact spaces

Let E be a space and $|E| \geq 2$. Consider the small compactness $P = P(E) = P(\{E\})$. We put $EGE(X) = P(E)GE(X)$ and $EE(X) = P(E)E(X)$. If $(Y, f) \in EGE(X)$, then (Y, f) is called a $g - E$-compactification of the space X. If $(Y, f) \in EE(X)$, then (Y, f) or Y is called a E-compactification of the space.

The notion of E-compactification was introduced by S. Mrovka [7,4]. From Theorems 2.5, 2.7, 2.10 and Corollary 2.9 follow the next assertions.

3.1. Corollary. *For every space X the class $EGE(X)$ is a lattice with the maximal element $(\beta_E X, \beta_E) = (e_{C(X,E)} X, f_{C(X,E)})$.*

3.2. Corollary. *For every $(Y, f) \in EGE(X)$ there exists a set $\Phi \subseteq C(X, E)$ such that:*

1. $(e_\Phi X, f_\Phi) \approx (Y, f)$.

2. There exist a continuous mapping $\varphi : Y \to e_\Phi X$ and an embedding $\Psi : e_\Phi X \to Y$ such that $\Psi(f_\Phi(x)) = \varphi(f(x))$ and $\varphi(\Psi(y)) = y$ for all $x \in X$ and $y \in e_\Phi X$.

3.3. Corollary. *For every continuous mapping $\varphi : X \to Y$ there exists a continuous mapping $\beta\varphi : \beta_E X \to \beta_E Y$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad\varphi\quad} & Y \\ {\scriptstyle \beta_E}\downarrow & & \downarrow{\scriptstyle \beta_E} \\ \beta_E X & \xrightarrow{\quad\beta\varphi\quad} & \beta_E Y \end{array}$$

is commutative.

3.4. Corollary. *If Ind X=0, then $EE(X) \neq \emptyset$ and $(\beta_E X, \beta_E) \in EE(X)$.*

3.5. Corollary. *If E is a T_1-space, then there exists a regular space X such that $EE(X) = \emptyset$.*

3.6. Corollary. *If E is a T_0-space and E is not a T_1-space, then $EE(X) \neq \emptyset$ and $(\beta_E X, \beta_E) \subseteq EE(X)$ for every T_0-space X.*

4 Examples

4.1. Example. Let $F = \{0,1\}$ with the topology $\{\emptyset, \{0\}, \{0,1\}\}$. Then F is a T_0-space and F is not a T_1-space. In this case $FE(X) \neq \emptyset$ for every T_0-space X. The class $FE(X)$ is not a set and the class $FGE(X)$ is a lattice.The assertions of the preceding section are true for $FGE(X)$.

The space F^m is called the Alexandroff cube (see [3]).

Denote by (maX, m_X) the maximal element of the lattice $FGE(X)$ of a space X. We may suppose that $maX = e_{C(X,F)}X$. We identify $x \in X$ and $m_X(x) \in maX$. In this case X is a dense subspace of the T_0-compact space maX. If $\varphi : X \to Y$ is a continuous mapping, then there exists a continuous mapping $m\varphi : maX \to maY$ such that $\varphi = m\varphi|X$.

4.2. Example. Let D=$\{0,1\}$ with the discrete topology $\{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. In this case:

– $EGE(X)$ is a set for every space X;

– $EGE(X)$ is a lattice for every space X;

– $EE(X) \neq \emptyset$ if and only if ind X=0.

4.3. Example. Let $I = [0,1]$ be a subspace of reals. In this case:

– $EGE(X)$ is a set for every space X;

– $EGE(X)$ is a lattice for every space X;

– $EE(X) \neq \emptyset$ if and only if X is a completely regular space;

– for every completely regular space the compactification $\beta_E \in X$ is the Stone-Čech compactification βE of the space X.

4.4. Example. Let τ be an infinite cardinal and E be a space of cardinality τ with the topology $\{\emptyset, E\} \cup \{E \backslash F : F \text{ is a finite subset}\}$. The space E is a compact T_1-space and E is not a Hausdorff space. In this case:

– $EGE(X)$ is not a set for some T_1-space;

– $EGE(X)$ is a lattice for every space X;

– If X is a T_1-space and $|X| \leq \tau$, then $EE(X) \neq \emptyset$;

– If $c \leq \tau$, then $EE(X) \neq \emptyset$ for every completely regular space X.

4.5. Example. A class P of topological T_0-spaces is called a double compactness if the following conditions are fulfilled:

1. There exists a space $X \in D$ such that $|X| \geq 2$.

2. If Γ is the topology of the space $X \in P$, then there is determined the Hausdorff topology $d\Gamma$ on X such that $(X, d\Gamma) \in P$, $\Gamma \subseteq d\Gamma$ and $dd\Gamma = d\Gamma$. We say that $d\Gamma$ is the strong topology and Γ is the weak topology on X.

3. If $\{(X_\alpha, \Gamma_\alpha) \in P : \alpha \in A\}$ is a non-empty set of spaces, $X = \Pi\{X_\alpha : \alpha \in A\}$, Γ is the product of topologies Γ_α on X and Γ' is the product of topologies $d\Gamma_\alpha$ on X, then $\Gamma' \subseteq d\Gamma$.

4. If $(X, \quad \Gamma) \in P$, $Y \subseteq X$ and Y is a closed subset of the space $(X, d\Gamma)$, then $(Y, \Gamma) \in P$ and $d\Gamma|Y \subseteq d(\Gamma|Y)$, where $\Gamma|Y = \{U \cap Y : U \in \Gamma\}$ for the topology Γ on X.

Every double compactness is a quasi-compactness.

Let P be a double compactness. Then $PGE(X)$ is a set for every space X. Moreover, for every non-empty subset $L \subseteq PGE(X)$ there exists the supremum $\vee L \in PGE(X)$. In particular, there exists the maximal element $(\beta_p X, \beta_P)$.

A mapping $f : X \to Y$ of a space $(X, \Gamma) \in P$ into a space $(Y, \Gamma') \in P$ is double continuous if $f^{-1}\Gamma' \subseteq \Gamma$ and $f^{-1}d\Gamma' \subseteq d\Gamma$. For every continuous mapping $f : X \to Y$ of a space X into a space $Y \in P$ there exists a unique double continuous mapping $\beta f : \beta_p X \to P$ such that $f(x) = \beta f(\beta_P(x))$ for every point $x \in X$. In particular, for every continuous mapping $f : X \to Y$ there exists a unique double continuous mapping $\beta f : \beta_p X \to \beta_p Y$ such that $\beta_p(f(x)) = \beta f(\beta_P(x))$ for every $x \in X$.

4.6. Example. Let K be a class of triples $(X,\, T_X, T'_X)$, where X is a non-empty set, T_X and T'_X are topologies on X, $T_X \subseteq T'_X$ and T'_X is a Hausdorff topology. Then there exists a minimal double compactness P such that $(X,\, T_X) \in P$ and $T'_X = dT_X$ for every triple $(X,\, T_X, T'_X) \in K$. We say that the double compactness is generated by the class K. If P' is the quasi-compactness generated by the class $\{(X,\, T_X), (X, T'_X) : (X,\, T_X, T'_X) \in K\}$, then $P' \subseteq P$.

4.7. Example. Let $X_0 = \{0, 1\}, T_{X_0} = \{\emptyset, \{0\}, \{0, 1\}\}$, $T'_{X_0} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$, then there exists the minimal double compactness S such that $(X_0, T_{X_0}) \in S$ and $dT_{X_0} = T'_{X_0}$. The class S is the class of all spectral spaces (see [1]).

The class S satisfies the following properties:

1. For every T_0-space X the class $SGE(X)$ is a set, is a lattice, $SE(X) \neq \emptyset$ and the maximal element $(\beta_S X, \beta_S)$ is a compactification of X. We may consider that X is a subspace of $\beta_S X$ and X is dense in $\beta_S X$ in the strong topology on $\beta_S X$.

2. The class S is a virtual small quasi-compactness.

3. The class S is not a small quasi-compactness.

5 Non-existence of universal compactification

5.1. Definition. *A compactification $(bX,\, \varphi)$ of a space X is called a universal compactification of a space X if $(Y, f) \leq (bX,\, \varphi)$ for every compactification (Y, f) of X.*

5.2. Definition. *Let $i \in \{0, 1, 2\}$ and X be a T_i-space. A compactification (bX, φ) of the space X is called a universal T_i-compactification of X if bX is a T_i-space and $(Y, f) \leq (bX, \varphi)$ for every $T_i - g$-compactification (Y, f) of the space X.*

If X is a completely regular space, then the Stone-Čech compactification βX of X is a universal T_2-compactification of the space X.

5.3.Theorem. *Let X be a T_1-space. The following assertions are equivalent:*

1. For a space X there exists a universal compactification.

2. For a space X there exists a universal T_0-compactification.

3. For a space X there exists a universal T_1-compactification.

Proof. Part 1. Let Z be a space with the topology T. Denote by nT the topology on Z generated by the open base $\{U \backslash H : U \in T, H \text{ is finite subset of } Z\}$. The topology nT is called the T_1-modification of the topology T. Denote by nZ the set Z with the topology nT. The space Z is compact if and only if the space nZ is compact.

Part 2. Let (Y, f) be a compactification of the T_1-space X. Then $f : X \to Y$ is an embedding. It is obvious that the mapping $f : X \to nY$ is an embedding too. Thus (nY, f) is a T_1-compactification of the space X. By construction, $(Y, f) \leq (nY, f)$.

Part 3. For every g-compactification (Y, f) of the space X there exists a T_1-compactification (Z, g) of X such that $(Y, f) \leq (Z, g)$.

Let (Y_1, f_1) be some compactification of X. Consider the mapping $g : X \to Y \times Y_1$, where $g(x) = (f(x), f_1(x))$ for every $x \in X$. Then g is an embedding. Denote by Y_2 the closure of the set $g(X)$ in the space $Y \times Y_1$. Then (Y_2, g) is a compactification of X. We put $Z = nY_2$. Then (Z, g) is a T_1-compactification of X and $(Y, f) \leq (Y_2, g) \leq (Z, g)$.

Part 4. Let (Z, φ) be a universal compactification of the space X. Then (mZ, φ) is a universal compactification, a universal T_0-compactification and a universal T_1-compactification. The implications $1 \to 2$ and $1 \to 3$ are proved.

Part 5. Let (Z, φ) be a universal T_0-compactification. Then $(Z, g) \leq (Z, \varphi) \leq (nZ, \varphi)$ for every $T_1 - g$-compactification of X. Thus (nZ, φ) is a universal T_0-compactification, universal T_1-compactification. From Part 3 it follows that (nZ, φ) is a universal compactification. The implications $2 \to 1$ and $2 \to 3$ are proved.

Part 6. Let (Z, φ) be a universal T_1-compactification. From Part 3 it follows that (Z, φ) is a universal compactifiction and a universal T_0-compactification, too. The implications $3 \to 1$ and $3 \to 2$ and the theorem are proved.

5.4. Corollary. There exists a T_1-space X such that:

1. For X a universal T_1-compactification does not exist.
2. For X a universal T_0-compactification does not exist.
3. For X a universal compactification does not exist.

Proof. The existence of T_1-space X without universal T_1-compactification was proved by M. Hušec [5, 6]. Theorem 5.2. completes the proof.

6 The minimality of the compactification maX

Fix a T_0-space X. Let F be the space from Example 4.1.

6.1. Theorem. *Let P be a quasi-compactness and F be a subspace of some space from P. Then:*

1. There exists a compactification $(Y, f) \in PE(X)$ such that $(maX, m_X) \leq (Y, f)$.

2. If P is a small quasi-compactness then $(maX, m_X) \leq (\beta_p X, \beta_p)$.

Proof. Let $E \in P$ and F be a subspace of the space E. There exists an open subset U of E such that $0 \in U$ and $1 \notin U$. Consider the mapping $r : E \to F$, where $r^{-1}(0) = U$ and $r^{-1}(1) = E \backslash U$. The mapping r is a continuous retraction. By construction, $C(X, F) \subseteq C(X, E)$. We put $\Phi = C(X, F)$. Consider the mapping

$f_\Phi : X \to F^\Phi \subseteq E^\Phi$. By construction, maX is the closure of the set $f_\Phi(X)$ in the space F^Φ. Let Y be the closure of the set $f_\Phi(X)$ in the space E^Φ. Then $(Y, f_\Phi) \in P(E)E(P) \subseteq PE(X)$. Consider the continuous mapping $h : E^\Phi \to F^\Phi$, where $h(x_f : f \in \Phi) = (r(x_f) : f \in \Phi)$ for every point $x = (x_f : f \in \Phi) \in E^\Phi$. The mapping h is a retraction and $maX \subseteq h(Y)$. Thus maX is a subspace of the space Y and $(maX, m_X) = (maX, f_\Phi) \leq (Y, f_\Phi)$. The assertion 1 is proved. If P is a small quasi-compactness, then we may consider that $P = P(E)$. In this case $(Y, f_\Phi) \leq (\beta_p X, \beta_p)$. The proof is complete.

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