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On Commutativity and Mediality of Polyagroup Cross Isomorphs

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Abstract. The notion of *cross isotopy* (*cross isomorphism*) of n-ary operations can be got from the well-known notion of *isotopy* (*isomorphism*) by replacing one of its components with a k-ary m-invertible operation [1, 2]. The idea of consideration of cross isotopy belongs to V.D. Belousov [3], who defined it for binary quasigroups.
In the paper necessary and sufficient conditions for commutativity and mediality of a polyagroup cross isomorph (when $n > 2k$) are determined. A neutrality criterion of an arbitrary element is stated.

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1 Introduction

V.D. Belousov [3] introduced left and right cross isotopy notions for binary quasigroups by replacing the left (right) component of the common isotopy with a left (right) invertible binary operation.

For the first time the corresponding notion for multiary operations was proposed in [1] and was based on the same idea. Namely, the notion of *i-cross isotopy of an $(n+1)$-ary operation* can be received from the well-known notion of isotopy by replacing its i-th nonprincipal component with an m-invertible operation depending on variables having indices in $\vec{\imath} := (i_0, \ldots, i_k)$, where $0 \leqslant i_0 < \cdots < i_k \leqslant n$ and $i_m = i$. The pair $(m; \vec{\imath})$ is called a *type* of the cross isotopy. If all its components coincide, except the i-th one, then the cross isotopy is called a *cross isomorphism*.

General properties of cross isotopy were studied in [1]. The set of all cross isotopies of fixed type of a set Q forms a group acting on the set of all operations of Q. It follows that the set of all cross autotopies of an operation is its subgroup; cross autotopy groups of cross isotopic operations are isomorphic; cross isotopy is an equivalence relation and so on. The same results were proved for cross isomorphism. Some other results were observed in the mentioned work too. For example, every two quasigroup operations defined on the same set are cross isomorphic if its type is maximal, i.e. if $n = k$, but there exists a pair of quasigroup operations (irreducible and completely reducible) which is not cross isotopic for every nonmaximal type.

In [2] the study of cross isotopy and cross isomorphism was continued: the structure of polyagroup nonmaximal type cross isotopism was found if the type is a segment of integers or the polyagroup is medial (i.e. a decomposition group of the polyagroup is commutative); an associate being i-cross isotopic to a quasigroup is a polyagroup if i is one of the integers 1, …, $n-1$; the notions of strong cross isomorphism and the well-known notion of isomorphism coincide if its type does not contain 1 or $n-1$ or the polyagroup is medial and the integers from the set $\{1,\dots,n-1\}$, which is not in the cross isotomorphism type are relatively prime.

E.A.Kuznetsov [4] used the notion of cross isotopy for describing some classes of loops. It is proposed a description of all cross isotopies between the given class of loops and well-studied class of loops, for example, the class of groups.

The same problem exists for multiary operations. Now the most developed operations are polyadic groups. So, the problem is *to describe the structure of cross isotopies assuring a polyagroup cross isotope belongs to the given class* $\mathfrak{A}$.

Here, we consider the problem in the case if the cross isotopy is a cross isomorphism and if the class $\mathfrak{A}$ is a class of commutative or medial operations. We also determine the neutrality criterion for an element of polyagroup cross isomorph.

2 General notion

All the operations below are defined on the same fixed set Q. We recall that $(n+1)$-ary operation f is called

- (i,j)-*associative* if for arbitrary x_0, …, $x_n \in Q$ the identity

$$f(x_0,\dots,x_{i-1},f(x_i,\dots,x_{i+n}),x_{i+n+1},\dots,x_{2n}) =$$

$$= f(x_0,\dots,x_{j-1},f(x_j,\dots,x_{j+n}),x_{j+n+1},\dots,x_{2n})$$

  is true;

- i-*invertible* if for any a_0,…, a_n of Q the equation

$$f(a_0,\dots,a_{i-1},x,a_{i+1},\dots,a_n) = a_i \tag{1}$$

  has a unique solution;

- *invertible* or a *quasigroup operation* if it is i-invertible for all i=0, 1, …, n.

A groupoid $(Q;f)$ is called (see [5]) an *associate of the kind* (s,n), where $s|n$, if the operation f is (i,j)-associative for all pairs (i,j) such that $i \equiv j \pmod{s}$; a *quasigroup* if f is invertible; a *polyagroup of the kind* (s,n) if it is an associate of the kind (s,n) and a quasigroup; an $(n+1)$-*group* if it is a polyagroup and $s=1$.

Theorem 1 [6]. *If a groupoid $(Q;f)$ is a polyagroup of the kind (s,n), then for arbitrary element $0 \in Q$ there exists a unique triple of operations $(+,\varphi,a)$ of the arities 2, 1, 0 respectively such that the following conditions are true:*

1) $(Q;+)$ *is a group,* φ *is its automorphism, 0 is its neutral element and the identities*

$$\varphi^n x + a = a + x, \qquad \varphi^s a = a \tag{2}$$

are valid;

2) a decomposition of the operation f *has the following form*

$$f(x_0, \dots, x_n) = x_0 + \varphi x_1 + \dots + \varphi^{n-1} x_{n-1} + \varphi^n x_n + a. \tag{3}$$

And vice versa, if the conditions 1) hold, then the groupoid $(Q; f)$ *defined by (3) is a polyagroup of the kind* (s, n)*.*

In that case, the group $(Q;+)$ is called a *decomposition group*, and the triple $(+, \varphi, a)$ is a *decomposition* of the polyagroup $(Q; f)$.

Let $\overset{k}{a}$ denote a sequence $a, \dots, a$ (k times).

An operation g of the arity $n+1$ is called *weak* i*-cross isomorphic of the type* $\vec{\imath} := (i_0, \dots, i_k)$, where $0 \leqslant i_0 < \dots < i_k \leqslant n$, or *weak cross isomorphic of the type* $(m, \vec{\imath})$ to $(n+1)$-ary operation f if $i_m = i$ and there exist a substitution α and an m-invertible operation h of the arity $k+1$ such that the equality

$$g(x_0, \dots, x_n) = \alpha^{-1} f(\alpha x_0, \dots, \alpha x_{i-1}, \alpha h(x_{i_0}, \dots, x_{i_k}), \alpha x_{i+1}, \dots, \alpha x_n) \tag{4}$$

holds for all $x_0, \dots, x_n \in Q$. The pair $(\alpha; h)$ is called a *weak cross isomorphism of the type* $(m, \vec{\imath})$ *of the arity* $k+1$. A cross isomorphism is called *principal* if $\alpha = \varepsilon$. If $k = n$, then i-th cross isomorphism is called i-th cross isomorphism of the *maximal type.*

A weak cross isomorphism $(\alpha; h)$ is called *strong* if h is a selector-like operation, i.e. if for arbitrary substitution τ of Q and for all $x \in Q$ the equality

$$h(\overset{m}{x}, \tau x, \overset{k-m}{x}) = \tau x \tag{5}$$

holds.

An operation g is called *commutative* if for all permutation σ of the set $\{0, 1, \dots, n\}$ the identity

$$g(x_{\sigma 0}, x_{\sigma 1}, \dots, x_{\sigma n}) = g(x_0, x_1, \dots, x_n) \tag{6}$$

is true.

Lemma 2 [7]. *If there exist transformations* α*,* β*,* γ*,* δ *of the group* $(Q;+)$ *such that the equality*

$$\alpha x + \beta y = \gamma y + \delta x$$

holds for all x*,* $y \in Q$ *and at least one element of each of the sets* $\{\alpha, \delta\}$ *and* $\{\beta, \gamma\}$ *is a substitution of* Q*, then the group* $(Q;+)$ *is commutative.*

The relation (4) implies the following identity

$$g_0(x_0,\dots,x_n)=f(x_0,\dots,x_{i-1},h_0(x_{i_0},\dots,x_{i_k}),x_{i+1},\dots,x_n),$$

where

$$g_0(x_0,\dots,x_n)=\alpha g(\alpha^{-1}x_0,\dots,\alpha^{-1}x_n),\ h_0(x_{i_0},\dots,x_{i_k})=\alpha h(\alpha^{-1}x_{i_0},\dots,\alpha^{-1}x_{i_k}),$$

i.e. groupoids $(Q;g)$ and $(Q;g_0)$ are isomorphic. To clarify the truth of a formula for cross isotopes it is enough to clarify it for principal cross isotopes. So from here on we will consider principal cross isotopes only.

Let $(n+1)$-ary groupoid $(Q;g)$ be a principal cross isomorph of the type $(m,\vec{\imath})$ of an $(n+1)$-ary polyagroup $(Q;f)$ with a decomposition $(+,\varphi,a)$, where $\vec{\imath}:=(i_0,\dots,i_k)$. Combining the identities (3) and (4) we obtain a decomposition of the operation g

$$\begin{aligned} g(x_0,\dots,x_n)= \ & x_0+\varphi x_1+\dots+\varphi^{i-1}x_{i-1}+\varphi^i h(x_{i_0},\dots,x_{i_k})+ \\ & +\varphi^{i+1}x_{i+1}+\dots+\varphi^n x_n+a. \end{aligned} \tag{7}$$

3 Commutativity

The next theorem gives a criterion when a cross isomorphism of a polyagroup is commutative.

Theorem 3. *Let $(Q;f)$ be a polyagroup with a decomposition $(+,\varphi,a)$ and let $(\varepsilon,h)f$ be a principal cross isomorph of a nonmaximal type $(m,\vec{\imath})$ of the operation f, where $\vec{\imath}:=(i_0,\dots,i_k)$ and $i:=i_m$. Then the operation $(\varepsilon,h)f$ is commutative if and only if the following relationships are true:*

1) *the group $(Q;+)$ is commutative;*

2) *$p\equiv \ell \pmod{|\varphi|}$ if $p,\ell\notin\vec{\imath}$, where $|\varphi|$ denotes the order of the automorphism φ;*

3) *a decomposition of the operation h is the following*

$$\begin{aligned} h(x_{i_0},\dots,x_{i_k})=\varphi^{-i}\big((\varphi^p-\varphi^{i_0})x_{i_0}+\dots+(\varphi^p-\varphi^{i_{m-1}})x_{i_{m-1}}+\varphi^p x_i+ \\ +(\varphi^p-\varphi^{i_{m+1}})x_{i_{m+1}}+\dots+(\varphi^p-\varphi^{i_k})x_{i_k}\big)+b \end{aligned} \tag{8}$$

for some $p\notin\vec{\imath}$ and $b\in Q$.

Proof. Let the operation g be commutative, then the identities (6) and (7) are true. Since the type $\vec{\imath}$ is not maximal then there exists a nonnegative integer $p\leqslant n$, which does not belong to the cross isomorphism type $\vec{\imath}$.

We replace all variables, except x_i and x_p, with the neutral element 0 of $(Q;+)$ in (6). From the commutativity of the operation g we have:

$$g(\overset{p}{0}, x_p, \overset{i-p}{0}, x_i, \overset{n-i}{0}) = g(\overset{p}{0}, x_i, \overset{i-p}{0}, x_p, \overset{n-i}{0}), \quad \text{if } p < i,$$

$$g(\overset{i}{0}, x_i, \overset{p-i}{0}, x_p, \overset{n-p}{0}) = g(\overset{i}{0}, x_p, \overset{p-i}{0}, x_i, \overset{n-p}{0}), \quad \text{if } p > i.$$

Taking into account (7), we obtain

$$\begin{aligned} &\varphi^p x_p + \varphi^i \lambda x_i = \varphi^p x_i + \varphi^i \lambda x_p, \quad \text{when } p < i, \\ &\varphi^i \lambda x_i + \varphi^p x_p = \varphi^i \lambda x_p + \varphi^p x_i, \quad \text{when } p > i, \end{aligned} \tag{9}$$

where $\lambda x := h(\overset{m}{0}, x, \overset{k-m}{0})$. Therefore, according to Lemma 2, the group $(Q;+)$ is commutative.

We denote $b := h(0, \dots, 0)$ and put $x_p = 0$ in (9):

$$\varphi^i \lambda x = \varphi^i b + \varphi^p x. \tag{10}$$

We notice that the commutativity of g implies the identity

$$g(x_0^{p-1}, x_p, x_{p+1}^{q-1}, x_q, x_q^n) = g(x_0^{p-1}, x_q, x_{p+1}^{q-1}, x_p, x_q^n) \tag{11}$$

for arbitrary numbers p, q.

To find a decomposition of the operation h we set $q = i_r$ for some $r \in \{0, \dots, m+1, m-1, \dots, k\}$ and $p \notin \vec{\imath}$ in (11) and replace the operation g with its decomposition (7):

$$\begin{aligned} &\varphi^i h(x_{i_0}, x_{i_1}, \dots, x_{i_k}) + \sum_{j=0, j\neq i}^{n} \varphi^j x_j + a = \\ &= \varphi^i h(x_{i_0}, x_{i_1}, \dots, x_{i_{r-1}}, x_p, x_{i_{r+1}}, \dots, x_{i_k}) + \\ &+ \sum_{j=0, j\neq i, i_r, p}^{n} \varphi^j x_j + \varphi^p x_{i_r} + \varphi^{i_r} x_p + a. \end{aligned} \tag{12}$$

After canceling the same summands and setting $x_p = 0$, we obtain

$$\varphi^i h(x_{i_0}, x_{i_1}, \dots, x_{i_k}) + \varphi^{i_r} x_{i_r} = \varphi^i h(x_{i_0}, x_{i_1}, \dots, x_{i_{r-1}}, 0, x_{i_{r+1}}, \dots, x_{i_k}) + \varphi^p x_{i_r}.$$

Thence

$$\varphi^i h(x_{i_0}, x_{i_1}, \dots, x_{i_k}) = \varphi^i h(x_{i_0}, x_{i_1}, \dots, x_{i_{r-1}}, 0, x_{i_{r+1}}, \dots, x_{i_k}) + (\varphi^p - \varphi^{i_r}) x_{i_r}.$$

We shall use this equality successively for $r = 0, \dots, m+1, m-1, \dots, k$:

$$\begin{aligned} \varphi^i h(x_{i_0}, x_{i_1}, \dots, x_{i_k}) &= \varphi^i h(0, x_{i_1}, \dots, x_{i_k}) + (\varphi^p - \varphi^{i_0}) x_{i_0} = \\ &= \varphi^i h(0, 0, x_{i_2}, \dots, x_{i_k}) + (\varphi^p - \varphi^{i_0}) x_{i_0} + (\varphi^p - \varphi^{i_1}) x_{i_1} = \dots = \\ &= \varphi^i h(\overset{m}{0}\ x_{i_m}, \overset{k-m}{0}) + \sum_{r=0, j\neq m}^{k} (\varphi^p - \varphi^{i_r}) x_{i_r} = \\ &\overset{(10)}{=} \varphi^p x_{i_m} + \varphi^i b + \sum_{j=0, r\neq m}^{k} (\varphi^p - \varphi^{i_r}) x_{i_r}. \end{aligned}$$

Thence we obtain the equality (8).

Let us set up in the equality (11) all variables with 0, except x_p and x_ℓ, if $q=\ell$. Taking into account decomposition (7) and commutativity of the decomposition group after respective cancellation we obtain $\varphi^p x_p+\varphi^\ell x_\ell=\varphi^p x_\ell+\varphi^\ell x_p$. Setting $x_p=0$ in the preceding equality, we obtain $\varphi^p=\varphi^\ell$. It follows that $\varphi^{p-\ell}=\varepsilon$, therefore $|\varphi|$ divides $p-\ell$, i.e. $p\equiv\ell \pmod{|\varphi|}$.

It is easy to prove the inverse statement. □

Putting $b=0$ in Theorem 3, we obtain a theorem for polyagroup strong cross isomorphs.

4 Mediality

We shall clarify the conditions when a principal cross isomorph $(Q;g)$ is medial, i.e. when the following identity

$$\begin{aligned} &g(g(x_{00},x_{01},\dots,x_{0n}),g(x_{10},x_{11},\dots,x_{1n}),\dots,g(x_{n0},x_{n1},\dots,x_{nn}))=\\ &=g(g(x_{00},x_{10},\dots,x_{n0}),g(x_{01},x_{11},\dots,x_{n1}),\dots,g(x_{0n},x_{1n},\dots,x_{nn}))\end{aligned} \tag{13}$$

is true. The next theorem can give an answer to this question.

Theorem 4. *Let a pair (ε,h) be a principal weak cross isomorphism of a nonmaximal type $(m,\vec{\imath})$, where $\vec{\imath}:=(i_0,i_1,\dots,i_k)$, between an $(n+1)$-ary groupoid $(Q;g)$ and $(n+1)$-ary polyagroup $(Q;f)$ with a decomposition $(+,\varphi,a)$ and let $n>2k$. A groupoid $(Q;g)$ is medial if and only if there exist endomorphisms $\lambda_0,\ \dots,\ \lambda_{m-1}$, $\lambda_{m+1},\ \dots,\ \lambda_k$, an automorphism λ_m and an element b of the group $(Q;+)$ such that:*

1) *$(Q;+)$ is commutative;*

2) *the relation*

$$h(y_0,y_1,\dots,y_n)=\lambda_0y_0+\lambda_1y_1+\dots+\lambda_ky_k+b \tag{14}$$

holds for all y_0, $y_1,\dots$, $y_k\in Q$;

3) *for arbitrary $r=0,1,\dots,k$ and $p\notin\vec{\imath}$ the following relations are true*

$$\lambda_r\varphi^p=\varphi^p\lambda_r, \tag{15}$$

$$(\lambda_r\varphi^i+\varphi^{i_r})\lambda_m=\lambda_m(\varphi^i\lambda_r+\varphi^{i_r}), \tag{16}$$

4) *for arbitrary $i_{r_1},i_{r_2}\in\vec{\imath}$ and $i_{r_1}\ne i_{r_2}\ne i$ the following equality is valid*

$$\lambda_{r_1}(\varphi^i\lambda_{r_2}+\varphi^{i_{r_2}})+\varphi^{i_{r_1}}\lambda_{r_2}=\lambda_{r_2}(\varphi^i\lambda_{r_1}+\varphi^{i_{r_1}})+\varphi^{i_{r_2}}\lambda_{r_1}. \tag{17}$$

Proof. Suppose that the groupoid $(Q;g)$ is medial, i.e. (13) holds, the equality (7) implies

$$g(0,\dots,0)=\varphi^i b+a, \tag{18}$$

where $b:=h(0,\dots,0)$. Nonmaximality of the type $\vec{\imath}$ means the existence of a number p not belonging to $\vec{\imath}$. We replace all variables in (13), except x_{pi} and x_{ip}, with the neutral element 0 of the group $(Q;+)$. Then there exist transformations μ_1, μ_2, μ_3, μ_4, which are compositions of translations of $(Q;+)$ and m-th translations of h such that

$$\mu_1 x_{pi}+\mu_2 x_{ip}=\mu_1 x_{ip}+\mu_2 x_{pi}.$$

The cross isotopy definition means m-invertibility of h, so that these transformations are substitutions of Q. So, according to Lemma 2 the operation $(+)$ is commutative.

We replace the operation h and the element a with the operation h_0 and the element a_0, determined with the following equalities

$$h_0(y_0,\dots,y_k):=h(y_0,\dots,y_k)-h(0,\dots,0),$$
$$a_0:=\varphi^i h(0,\dots,0)+a=\varphi^i b+a.$$

Therefore, taking into account commutativity of $(Q;+)$, the decomposition (7) of g can be written in the form:

$$g(x_0,\dots,x_n)=\varphi^i h_0(x_{i_0},\dots,x_{i_k})+\sum_{j=0,j\neq i}^{n}\varphi^j x_j+a_0. \tag{19}$$

We recall that $h_0(0,\dots,0)=0$.

Now we replace the first occurence of g in left and right sides of (13) with its decomposition (19):

$$\varphi^i h_0(g(x_{i_0 0},\dots,x_{i_0 n}),\dots,g(x_{i_k 0},\dots,x_{i_k n}))+$$
$$+\sum_{j=0,j\neq i}^{n}\varphi^j g(x_{j0},\dots,x_{jn})+a_0=$$
$$=\varphi^i h_0(g(x_{0i_0},\dots,x_{ni_0}),\dots,g(x_{0i_k},\dots,x_{ni_k}))+$$
$$+\sum_{j=0,j\neq i}^{n}\varphi^j g(x_{0j},\dots,x_{nj})+a_0. \tag{20}$$

Let us consider the second summands in the left and right sides of this equality. The summand of the left side is equal to

$$\sum_{j=0,j\neq i}^{n}\varphi^j g(x_{j0},\dots,x_{jn})=$$
$$\overset{(19)}{=}\sum_{j=0,j\neq i}^{n}\varphi^j\left(\varphi^i h_0(x_{ji_0},\dots,x_{ji_k})+\sum_{u=0,u\neq i}^{n}\varphi^u x_{ju}+a_0\right)=$$
$$=\sum_{j=0,j\neq i}^{n}\varphi^{i+j}h_0(x_{ji_0},\dots,x_{ji_k})+\sum_{j=0,j\neq i}^{n}\sum_{u=0,u\neq i}^{n}\varphi^{j+u}x_{ju}+\sum_{j=0,j\neq i}^{n}\varphi^j a_0.$$

By analogy we obtain a decomposition of the second summand of the right side:

$$\sum_{j=0,j\neq i}^{n} \varphi^j g(x_{0j},\dots,x_{nj}) =$$
$$= \sum_{j=0,j\neq i}^{n} \varphi^{i+j} h_0(x_{i_0 j},\dots,x_{i_k j}) + \sum_{j=0,j\neq i}^{n} \sum_{u=0,u\neq i}^{n} \varphi^{j+u} x_{uj} + \sum_{j=0,j\neq i}^{n} \varphi^j a_0.$$

We notice that the following equality is obvious

$$\sum_{j=0,j\neq i}^{n} \sum_{u=0,u\neq i}^{n} \varphi^{j+u} x_{ju} + \sum_{j=0,j\neq i}^{n} \varphi^j a_0 = \sum_{j=0,j\neq i}^{n} \sum_{u=0,u\neq i}^{n} \varphi^{j+u} x_{uj} + \sum_{j=0,j\neq i}^{n} \varphi^j a_0,$$

therefore (20) can be cancelled on these summands and element a_0.

$$\varphi^i h_0(g(x_{i_0 0},\dots,x_{i_0 n}),\dots,g(x_{i_k 0},\dots,x_{i_k n})) + \sum_{j=0,j\neq i}^{n} \varphi^{i+j} h_0(x_{j i_0},\dots,x_{j i_k}) =$$
$$= \varphi^i h_0(g(x_{0 i_0},\dots,x_{n i_0}),\dots,g(x_{0 i_k},\dots,x_{n i_k})) + \sum_{j=0,j\neq i}^{n} \varphi^{i+j} h_0(x_{i_0 j},\dots,x_{i_k j}).$$

After mentioned transformations we can apply the automorphism φ^{-i} to the equality

$$h_0(g(x_{i_0 0},\dots,x_{i_0 n}),\dots,g(x_{i_k 0},\dots,x_{i_k n})) + \sum_{j=0,j\neq i}^{n} \varphi^{j} h_0(x_{j i_0},\dots,x_{j i_k}) =$$
$$= h_0(g(x_{0 i_0},\dots,x_{n i_0}),\dots,g(x_{0 i_k},\dots,x_{n i_k})) + \sum_{j=0,j\neq i}^{n} \varphi^{j} h_0(x_{i_0 j},\dots,x_{i_k j}).$$

We replace all occurences of g with its decomposition (19):

$$h_0\left(\varphi^i h_0(x_{i_0 i_0},\dots,x_{i_0 i_k}) + \sum_{j=0,j\neq i}^{n} \varphi^j x_{i_0 j} + a_0;\dots;\varphi^i h_0(x_{i_k i_0},\dots,x_{i_k i_k}) + \right.$$
$$\left. + \sum_{j=0,j\neq i}^{n} \varphi^j x_{i_k j} + a_0\right) + \sum_{j=0,j\neq i}^{n} \varphi^j h_0(x_{j i_0},\dots,x_{j i_k}) =$$
$$= h_0\left(\varphi^i h_0(x_{i_0 i_0},\dots,x_{i_k i_0}) + \sum_{j=0,j\neq i}^{n} \varphi^j x_{j i_0} + a_0;\dots;\varphi^i h_0(x_{i_0 i_k},\dots,x_{i_k i_k}) + \right.$$
$$\left. + \sum_{j=0,j\neq i}^{n} \varphi^j x_{j i_k} + a_0\right) + \sum_{j=0,j\neq i}^{n} \varphi^j h_0(x_{i_0 j},\dots,x_{i_k j}). \tag{21}$$

Let $p \notin \vec{\imath}$. We replace all variables, except $x_{i_0 p}$, …, $x_{i_k p}$, with 0. Inasmuch as $h_0(0,\dots,0) = 0$, then (21) can be given in the form

$$\varphi^p h_0(x_{i_0 p},\dots,x_{i_k p}) = h_0(\varphi^p x_{i_0 p} + a_0,\dots,\varphi^p x_{i_k p} + a_0) - h_0(a_0,\dots,a_0). \tag{22}$$

We add to the both sides of (21) the element $(n-k)(-h_0(a_0,\dots,a_0))$ and apply (22) to the last summands of (21). Then the equality (21) in the case $x_{uv} = 0$ for all $u, v \in \vec{\imath}$ gives

$$h_0\left(\sum_{j=0, j\notin\vec{\imath}}^{n} \varphi^j x_{i_0 j} + a_0; \dots; \sum_{j=0, j\notin\vec{\imath}}^{n} \varphi^j x_{i_k j} + a_0\right) + \\ + \sum_{j=0, j\notin\vec{\imath}}^{n} h_0\left(\varphi^j x_{j i_0} + a_0; \dots; \varphi^j x_{j i_k} + a_0\right) = \\ = h_0\left(\sum_{j=0, j\notin\vec{\imath}}^{n} \varphi^j x_{j i_0} + a_0; \dots; \sum_{j=0, j\notin\vec{\imath}}^{n} \varphi^j x_{j i_k} + a_0\right) + \\ + \sum_{j=0, j\notin\vec{\imath}}^{n} h_0\left(\varphi^j x_{i_0 j} + a_0; \dots; \varphi^j x_{i_k j} + a_0\right).$$

Let us replace $\varphi^j y + a_0$ with y for all variables y appearing in the last identity:

$$h_0\left(\sum_{j=0, j\notin\vec{\imath}}^{n} x_{i_0 j}; \dots; \sum_{j=0, j\notin\vec{\imath}}^{n} x_{i_k j}\right) + \sum_{j=0, j\notin\vec{\imath}}^{n} h_0(x_{j i_0}, \dots, x_{j i_k}) = \\ = h_0\left(\sum_{j=0, j\notin\vec{\imath}}^{n} x_{j i_0}; \dots; \sum_{j=0, j\notin\vec{\imath}}^{n} x_{j i_k}\right) + \sum_{j=0, j\notin\vec{\imath}}^{n} h_0(x_{i_0 j}, \dots, x_{i_k j}). \tag{23}$$

Inasmuch as $n > 2k$, then there exist at least $k+1$ numbers which do not belong to $\vec{\imath}$. We denote them by p_0, p_1,..., p_k and replace all variables in (23), except $x_{p_0 i_0}$, $x_{p_1 i_1}$, ..., $x_{p_k i_k}$, with 0:

$$h_0(x_{p_0 i_0}, \overset{k}{0}) + h_0(0, x_{p_1 i_1}, \overset{k-1}{0}) + \dots + h_0(\overset{k}{0}, x_{p_k i_k}) = h_0(x_{p_0 i_0}, x_{p_1 i_1}, \dots, x_{p_k i_k}).$$

Denoting $y_j := x_{p_j i_j}$ and $\lambda_j x := h_0(\overset{j}{0}, x, \overset{k-j}{0})$ for all $j = 0, 1, \dots, k$, we obtain

$$h_0(y_0, y_1, \dots, y_k) = \lambda_0 y_0 + \lambda_1 y_1 + \dots + \lambda_k y_k. \tag{24}$$

It implies a decomposition (14) of h.

In the identity (23) we replace all variables, except $x_{i_r p_0}$ and $x_{i_r p_1}$, with 0 and replace h_0 with its decomposition:

$$\lambda_r(x_{i_r p_0} + x_{i_r p_1}) = \lambda_r x_{i_r p_0} + \lambda_r x_{i_r p_1},$$

i.e. λ_r is an endomorphism of the group $(Q; +)$.

Let us replace h_0 with its decomposition (24) in (21). Since every of λ_0, ..., λ_k is an endomorphism of commutative group $(Q; +)$, then left and right sides of the obtained equality can be cancelled on $\lambda_0 a_0 + \dots + \lambda_k a_0$:

$$\lambda_0 \left(\varphi^i(\lambda_0 x_{i_0 i_0} + \dots + \lambda_k x_{i_0 i_k}) + \sum_{j=0, j\neq i}^{n} \varphi^j x_{i_0 j} \right) + \dots$$
$$\dots + \lambda_k \left(\varphi^i(\lambda_0 x_{i_k i_0} + \dots + \lambda_k x_{i_k i_k}) + \sum_{j=0, j\neq i}^{n} \varphi^j x_{i_k j} \right) +$$
$$+ \sum_{j=0, j\neq i}^{n} \varphi^j(\lambda_0 x_{j i_0} + \dots + \lambda_k x_{j i_k}) =$$
$$= \lambda_0 \left(\varphi^i(\lambda_0 x_{i_0 i_0} + \dots + \lambda_k x_{i_k i_0}) + \sum_{j=0, j\neq i}^{n} \varphi^j x_{j i_0} \right) + \dots$$
$$\dots + \lambda_k \left(\varphi^i(\lambda_0 x_{i_0 i_k} + \dots + \lambda_k x_{i_k i_k}) + \sum_{j=0, j\neq i}^{n} \varphi^j x_{j i_k} \right) +$$
$$+ \sum_{j=0, j\neq i}^{n} \varphi^j(\lambda_0 x_{i_0 j} + \dots + \lambda_k x_{i_k j}). \tag{25}$$

Let us consider the equality (25). Let $i_r \in \vec{\imath}$, $p \notin \vec{\imath}$. Replacing all variables, except $x_{i_r p}$, with 0 we obtain the relationship (15).

We replace all variables in (25) with 0, except $x_{i_r i}$:

$$\lambda_r \varphi^i \lambda_m + \varphi^{i_r} \lambda_m = \lambda_m(\varphi^i \lambda_r + \varphi^{i_r}).$$

It implies (16).

Let $r_1, r_2 \in \{0, \dots, m-1, m+1, \dots, k\}$. If we replace all variables in (25) with 0, except $x_{i_{r_1} i_{r_2}}$, then in the left side of the equality we have $\lambda_{r_1}(\varphi^i \lambda_{r_2} + \varphi^{i_{r_2}}) + \varphi^{i_{r_1}} \lambda_{r_2}$, and in the right side we obtain $\lambda_{r_2}(\varphi^i \lambda_{r_1} + \varphi^{i_{r_1}}) + \varphi^{i_{r_2}} \lambda_{r_1}$. i.e. (17) is true.

Vice versa, let $(Q;+)$ be commutative group, $\lambda_0, \dots, \lambda_k$ be its endomorphisms; λ_m be its automorphism; b be an arbitrary element of Q; an operation h be determined by (14), and let the relationships (15)–(17) be valid. Thus, the operation g has a decomposition

$$g(x_0, \dots, x_n) = \sum_{j=0, j\notin\vec{\imath}}^{n} \varphi^j x_j + \sum_{r=0, r\neq m}^{k} (\varphi^{i_r} + \varphi^i \lambda_r) x_r + \varphi^i \lambda_m x_i + \varphi^i b + a. \tag{26}$$

All coefficients of g's decomposition are endomorphisms of the group $(Q;+)$. From the relationships (15), (16), (17) it follows that the coefficients pairwise commute. It is easy to prove that every such operation is medial. □

If a cross isomorphism is strong, then the operation h is idempotent (it follows that $b = h(0, \dots, 0) = 0$). Hence, Theorem 4 with $b = 0$ states a mediality criterion for a polyagroup strong cross isomorph.

5 Neutral elements

Every group isotop with a neutral element is derived. But for group cross isotop it is not true. The set of all identity elements of derived group is a subgroup of its decomposition group center. The structure of the set of all neutral elements of a cross isotop is still unknown. Here we consider a neutrality criterion in medial polyagroup cross isotopes only, i.e. having commutative decomposition groups.

Theorem 5. *Let $(Q; g)$ be i-cross isomorph of the type $\vec{\imath} := (i_0, i_1, \ldots, i_k)$, where $i = i_m$, of an $(n+1)$-ary medial polyagroup $(Q; f)$ with decomposition $(+, \varphi, a)$ and let (ε, h) be respective cross isomorphism. Then element e of the set Q is neutral in $(Q; g)$ if and only if*

1) $\varphi^p = \varepsilon$, *when* $p \notin \vec{\imath}$;
2) *for all* $r = 0, \ldots, m-1, m+1, \ldots, k$

$$h(\overset{r}{e}, x, \overset{k-r}{e}) = b + \varphi^{-i}(x - e) - \varphi^{i_r - i}(x - e); \tag{27}$$

3) *for all x from Q*

$$h(\overset{m}{e}, x, \overset{k-m}{e}) = b + \varphi^{-i}(x - e); \tag{28}$$

4) $g(e, \ldots, e) = e$.

Proof. Let e be a neutral element of $(Q; g)$, i.e. for arbitrary $j = 0, \ldots, n$

$$g(\overset{j}{e}, x, \overset{n-j}{e}) = x \tag{29}$$

holds for all $x \in Q$. In particular, when $x = e$ we obtain item 4) of Theorem 5. Taking into account the decomposition (7) and the relation $\varphi^n x + a = a + x$, we have

$$\varphi^i b + c = 0, \tag{30}$$

where $b := h(e, \ldots, e)$, $c := a + \sum\limits_{j=0, j\neq i}^{n-1} \varphi^j e$.

If p does not belong to $\vec{\imath}$, then (29) with decomposition (7) implies the equality

$$\varphi^i b + c + e - \varphi^p e + \varphi^p x = x$$

for all $x \in Q$. Taking into account (30), we obtain

$$\varphi^p x = x + \varphi^p e - e.$$

If $x = 0$ then $\varphi^p e = e$, which together with the previous equality give $\varphi^p = \varepsilon$, i.e. item 1) of Theorem 5 is valid.

We suppose that r is one of the numbers $0, 1, \ldots, k$, then (29) means that

$$\varphi^i h(\overset{r}{e}, x, \overset{k-r}{e}) + c - \varphi^{i_r} e + \varphi^{i_r} x + e = x.$$

Taking into account (30), we have

$$\varphi^i h(\overset{r}{e}, x, \overset{k-r}{e}) - \varphi^i b - \varphi^{i_r} e + \varphi^{i_r} x + e = x.$$

It implies (27).

If $j = i$, the equality (29) has the form

$$g(e,\dots,e) + \varphi^i h(\overset{m}{e}, x, \overset{k-m}{e}) - \varphi^i h(e,\dots,e) = x.$$

This equality is equivalent to $\varphi^i h(\overset{m}{e}, x, \overset{k-m}{e}) - \varphi^i b = x - e$. Hence (28) is valid.

Vice versa, let the conditions 1) – 4) of the theorem for some element e be true. We shall show that e is a neutral element. If $j \in \vec{\imath}$, then

$$g(\overset{j}{e}, x, \overset{n-j}{e}) \overset{(20)}{=} e + \varphi e + \ldots + \varphi^{i-1} e + \varphi^i h(\overset{k+1}{e}) + \varphi^{i+1} e + \ldots + \varphi^n e +$$

$$+ a - \varphi^j e + \varphi^j x \overset{1),3)}{=} g(e,\dots,e) - e + x = e - e + x = x.$$

If $j \neq i$ and $j \in \vec{\imath}$, i.e. $j = i_r$ for some number $r \in \{0,\dots,m-1,m+1,\dots,k\}$, then

$$g(\overset{j}{e}, x, \overset{n-j}{e}) \overset{(7)}{=} e + \varphi e + \ldots + \varphi^{i-1} e + \varphi^i h(\overset{r}{e}, x, \overset{k-r}{e}) + \varphi^{i+1} e + \ldots + \varphi^n e +$$

$$+ a + \varphi^{i_r} x - \varphi^{i_r} e = g(e,\dots,e) - \varphi^i b + \varphi^i h(\overset{r}{e}, x, \overset{k-r}{e}) + \varphi^{i_r} x - \varphi^{i_r} e$$

$$\overset{(27)}{=} e - \varphi^i b + \varphi^i b + x - e - \varphi^{i_r}(x - e) + \varphi^{i_r}(x - e) = x.$$

At last, let $j = i = i_m$, then

$$g(\overset{i}{e}, x, \overset{n-i}{e}) \overset{(7)}{=} e + \varphi e + \ldots + \varphi^{i-1} e + \varphi^i h(\overset{m}{e}, x, \overset{k-m}{e}) + \varphi^{i+1} e + \ldots + \varphi^n e + a =$$

$$= g(e,\dots,e) - \varphi^i b + \varphi^i h(\overset{m}{e}, x, \overset{k-m}{e}) \overset{(28)}{=} e - \varphi^i b + \varphi^i b + x - e = x. \qquad \square$$

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