BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(49), 2005, Pages 123–140
ISSN 1024–7696

# On Bruck-Belousov Problem

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**Abstract.** In this paper on the language of subgroups of the multiplication group of a quasigroup (of the associated group of a quasigroup) necessary and sufficient conditions of normality of congruences of a left (right) loop are given. These conditions can be considered as a partial answer to the problem posed in books of R. H. Bruck and V.D. Belousov about conditions of normality of all congruences of quasigroups. Results on the regularity of congruences of quasigroups and the behavior of quasigroup congruences by isotopy are given.

**Mathematics subject classification:** 20N05.
**Keywords and phrases:** Quasigroup, congruence, normal congruence.

## 1 Introduction

The main purpose of this paper is an attempt to promote in solving the following Bruck-Belousov problem: *What loops $G$ have the property that every image of $G$ under a multiplicative homomorphism is also a loop [9, p. 92]? What quasigroups or loops in which all congruences are normal [5, Problem 20, p. 221]?*

We notice it is well known (see [3]), if homomorphic image of a multiplicative homomorphism $\varphi$ of a loop is also a loop then congruence $\theta$ which corresponds to $\varphi$, is a normal congruence.

This article is an extended variant of the paper [27]. See also [28]. We shall use standard quasigroup notations and definitions from [5,6,11,23]. Information on lattices and universal algebras can be found in [10,20,30], on groups in [14,19], on semigroups in [12].

For convenience of readers we recall some well known definitions.

A groupoid $(Q,\cdot)$ in which for any fixed elements $a, b$ from the set $Q$ the equations $a\cdot x = b$ and $y \cdot a = b$ have unique solutions is called a *quasigroup*.

A quasigroup $(Q,\cdot)$ that has an element $f$ such that $f \cdot x = x$ for all $x \in Q$ is called a *left loop*.

A quasigroup $(Q,\cdot)$ that has an element $e$ such that $x \cdot e = x$ for all $x \in Q$ is called a *right loop*.

A quasigroup with the identity of associativity ($x \cdot yz = xy \cdot z$) is a group [19].

It is known (see [5]) that in a quasigroup left ($L_a : x \to a \cdot x$), right ($R_a : x \to x \cdot a$) translations, as well as its inverse, are permutations. Let $\mathbb{L} = \{L_a \,|\, a \in Q\}$, $\mathbb{R} = \{R_b \,|\, b \in Q\}$, $\mathbb{T} = \{L_a, R_b \,|\, a, b \in Q\}$.

By $\Pi(Q)$ or by $\Pi$ for the short we shall designate a semigroup generated by the left and right translations of a quasigroup $Q$, i.e. elements of a semigroup $\Pi(Q)$ are words of the form $T_1^{\alpha_1}T_2^{\alpha_2}\dots T_n^{\alpha_n}$, where $T_i \in \mathbb{T}$, $\alpha_i \in \mathbb{N}$.

The group generated by all left and right translations of a quasigroup $Q$ will be denoted by $M(Q)$, or by $M$ for the short. Elements of the group $M$ are words of the form $T_1^{\alpha_1}T_2^{\alpha_2}\dots T_n^{\alpha_n}$, where $T_i \in \mathbb{T}$, $\alpha_i \in \mathbb{Z}$.

A binary relation $\varphi$ on a set $Q$ is a subset of the cartesian product $Q \times Q$ [10,22].

As it is known ([10,12,20,30]), a binary relation $q$ is an equivalence if and only if $\varepsilon \subseteq q$, $q^{-1} = q$, $q^2 = q$, where $\varepsilon = \{(x,x) \mid x \in Q\}$. We shall use both definitions: this definition and definition of equivalence as reflexive, symmetric and transitive relation on the language of pairs of elements.

A class of an equivalence $\theta$ that contains an element $a$ will be denoted by $\theta(a)$.

An equivalence $\theta$ of a quasigroup $(Q,\cdot)$ such that from $a\theta b$ follows $(c\cdot a)\theta(c\cdot b)$ for all $a,b,c \in Q$ is called a *left congruence* of a quasigroup $(Q,\cdot)$.

An equivalence $\theta$ of a quasigroup $(Q,\cdot)$ such that from $a\theta b$ follows $(a\cdot c)\theta(b\cdot c)$ for all $a,b,c \in Q$ is called a *right congruence* of a quasigroup $(Q,\cdot)$.

A left and right congruence $\theta$ of a quasigroup $(Q,\cdot)$ is called a *congruence* of a quasigroup $(Q,\cdot)$ [5,6].

A congruence $\theta$ of a quasigroup $(Q,\cdot)$ is called *normal,* if from $(a\cdot c)\theta(b\cdot c)$ follows $a\theta b$, from $(c\cdot a)\theta(c\cdot b)$ follows $a\theta b$ for all $a,b,c \in Q$.

We shall call a binary relation $\theta$ of a groupoid $(Q,\cdot)$ *stable from the left* (accordingly from the right) if from $x\theta y$ it follows $(a\cdot x)\theta(a\cdot y)$, (accordingly $(x\cdot a)\theta(y\cdot a)$) for all $a \in Q$.

It is easy to see that a stable from the left (from the right) equivalence of a quasigroup $(Q,\cdot)$ is called a *left (right) congruence.* A congruence is an equivalence relation which is stable from the left and from the right.

**Definition 1.** *If $\theta$ is a binary relation on a set $Q$, $\alpha$ is a permutation of the set $Q$ and from $x\theta y$ it follows $\alpha x\theta\alpha y$ for all $(x,y) \in \theta$, then we shall say that the permutation $\alpha$ is* ***semi-admissible relative to the relation*** *$\theta$.*

**Remark 1.** We notice in [13] a permutation with such property is called an admissible permutation.

**Definition 2.** *If $\theta$ is a binary relation on a set $Q$, $\alpha$ is a permutation of the set $Q$ and from $x\theta y$ it follows $\alpha x\theta\alpha y$ and $\alpha^{-1}x\theta\alpha^{-1}y$ for all $(x,y) \in \theta$, then we shall say that the permutation $\alpha$ is an* ***admissible permutation relative to the binary relation*** *$\theta$ [5].*

We recall any element of the group $M(Q)$ of a quasigroup $Q$ is admissible relative to any normal congruence of the quasigroup $Q$; any element the semigroup $\Pi(Q)$ of a quasigroup $Q$ is semi-admissible relative to any congruence of the quasigroup $Q$ [5,6].

As it is known (see [5, 6]) to each homomorphism $\varphi$ of a quasigroups $Q$ it is possible to associate a congruence $\theta$ by the rule: $a\theta b$ if and only if $\varphi a = \varphi b$. In this case $\varphi Q \cong Q/\theta$.

As is was proved in [3], see also [5,11], homomorphic image of a quasigroups $Q$ can be a quasigroup or a groupoid with division. If homomorphic image of a quasigroup is a quasigroup, then the congruence $\theta$ which corresponds to this homomorphism is normal, if $\varphi Q$ is a proper division groupoid, then congruence $\theta$ is not normal [6].

If $\varphi$ and $\psi$ are binary relations on $Q$, then their product is defined in the following way: $(a, b) \in \varphi \circ \psi$ if there is an element $c \in Q$ such that $(a, c) \in \varphi$ and $(c, b) \in \psi$ [12, 30]. The operation of product of binary relations is associative [12, 22, 24].

Below we shall designate product of binary relations and quasigroup operation by a point, by the letter $Q$ we shall designate a quasigroup $(Q, \cdot)$ and a set on which this quasigroup is defined.

**Lemma 1.** *For all binary relations $\varphi, \psi, \theta \subseteq Q^2$ from $\varphi \subseteq \psi$ follows $\varphi\theta \subseteq \psi\theta$, $\theta\varphi \subseteq \theta\psi$, i.e. it is possible to say that a binary relation of set-theoretic inclusion of binary relations is stable from the left and from the right relative to the multiplication of binary relations.*

**Proof.** If $(x, z) \in \varphi\theta$, then there exists an element $y \in Q$, such that $(x, y) \in \varphi$ and $(y, z) \in \theta$. Since $\varphi \subseteq \psi$, then we have $(x, y) \in \psi$, $(x, z) \in \psi\theta$. □

**Remark 2.** Translations of a quasigroup can be considered as binary relations: $(x, y) \in L_a$, if and only if $y = a \cdot x$, $(x, y) \in R_b$, if and only if $y = x \cdot b$.

**Remark 3.** To coordinate the multiplication of translations with their multiplication as binary relations, we use the following multiplication of translations: if $\alpha, \beta$ are translations, $x$ is an element of the set $Q$, then $(\alpha\beta)(x) = \beta(\alpha(x))$, i.e. $(\alpha\beta)x = \beta\alpha\, x$.

A partially ordered set $(L, \subseteq)$ is called a lower (an upper) semilattice if any its two-element subset has exact lower (upper) bound, i.e. in a set $L$ exists $\inf(a, b)$ $(\sup(a, b))$ for all $a, b \in L$ [10, 20].

If a partially ordered set is simultaneously the lower and upper semilattice, then it is called a *lattice*.

We can define a lattice as algebra $(L, \vee, \wedge)$ satisfying the following axioms ([10]):

$$\begin{array}{ll} (a \vee b) \vee c = a \vee (b \vee c); & a \vee b = b \vee a; \\ a \vee a = a; & (a \vee b) \wedge a = a; \\ (a \wedge b) \wedge c = a \wedge (b \wedge c); & a \wedge b = b \wedge a; \\ a \wedge a = a; & (a \wedge b) \vee a = a. \end{array}$$

We notice similarly as for quasigroups, which are defined in a signature with one and three binary operations, for the lattices which are defined in a signature with one binary operation $\leq$ and with two binary operations $\vee$ and $\wedge$, the concepts of a sublattice do not coincide. Namely, the sublattice of a lattice $(L, \vee, \wedge)$ always is a sublattice of a lattice $(L, \leq)$, but an inverse is not always correct [20].

## 2 Congruences of a quasigroup and its associated group

Connections between normal subloops of a loop $Q$ and normal subgroups of the group $M(Q)$ were studied by A. Albert [1,2]. Generalizations of Albert results on some classes of quasigroups can be found in works of V.A. Beglaryan [4] and K.K. Shchukin [29]. In these works also questions of the lattice embedding of lattices of some normal congruences of a quasigroup $Q$ into the lattice of normal subgroups of the group $M(Q)$ are studied.

**Proposition 1.** *An equivalence $\theta$ is a congruence of a quasigroup $Q$ if and only if $\theta\omega \subseteq \omega\theta$ for all $\omega \in \mathbb{T}$.*

**Proof.** Let $\theta$ be an equivalence, $\omega = L_a$. It is clear that $(x, z) \in \theta L_a$ is equivalent to that there exists an element $y \in Q$ such that $(x, y) \in \theta$ and $(y, z) \in L_a$. But if $(y, z) \in L_a$, $z = ay$, then $y = L_a^{-1}z$. Therefore, from the relation $(x, z) \in \theta L_a$ it follows that $(x, L_a^{-1}z) \in \theta$.

Let us prove that from $(x, L_a^{-1}z) \in \theta$ it follows $(x, z) \in \theta L_a$. We have $(x, L_a^{-1}z) \in \theta$ and $(L_a^{-1}z, z) \in L_a$, $(x, z) \in \theta L_a$. Thus $(x, z) \in \theta L_a$ is equivalent to $(x, L_a^{-1}z) \in \theta$.

Similarly, $(x, z) \in L_a\theta$ is equivalent to $(ax, z) \in \theta$. Now we can say that the inclusion $\theta\omega \subseteq \omega\theta$ by $\omega = L_a$ is equivalent to the following implication:

$$(x, L_a^{-1}z) \in \theta \Longrightarrow (ax, z) \in \theta$$

for all suitable $a, x, z \in Q$.

If we replace in the last implication $z$ with $L_a z$, we shall obtain the following implication:

$$(x, z) \in \theta \Longrightarrow (ax, az) \in \theta$$

for all $a \in Q$.

Thus, the inclusion $\theta L_a \subseteq L_a\theta$ is equivalent to the stability of the relation $\theta$ from the left relative to an element $a$. Since the element $a$ is an arbitrary element of the set $Q$, we have that the inclusion $\theta\omega \subseteq \omega\theta$ by $\omega \in \mathbb{L}$ is equivalent to the stability of the relation $\theta$ from the left.

Similarly, the inclusion $\theta\omega \subseteq \omega\theta$ for any $\omega \in \mathbb{R}$ is equivalent to the stability from the right of relation $\theta$. Uniting the last two statements, we obtain required equivalence. □

Let us remark Proposition 1 can be deduced from results of the article of Thurston [26].

The following proposition is almost obvious corollary of Theorem 5 from [21].

**Proposition 2.** *An equivalence $\theta$ is a congruence of a quasigroup $Q$ if and only if $\omega\theta(x) \subseteq \theta(\omega x)$ for all $x \in Q$, $\omega \in \mathbb{T}$.*

**Proof.** Let $\theta$ be an equivalence relation and for all $\omega \in \mathbb{T}$, $\omega\theta(x) \subseteq \theta(\omega x)$. We shall prove that from $a\theta b$ follows $ca\theta cb$, $ac\theta bc$ for all $c \in Q$.

By definition of the equivalence $\theta$, $a\theta b$ is equivalent to $a \in \theta(b)$. Then $ca \in c\theta(b) \subseteq \theta(cb)$, $ca\theta cb$. Similarly, from $a\theta b$ follows $ac\theta bc$.

Converse. Let $\theta$ be a congruence. We shall prove that $c\,\theta(a) \subseteq \theta(ca)$ for all $c, a \in Q$. Let $x \in c\,\theta(a)$. Then $x = cy$, where $y \in \theta(a)$, that is $y\theta a$. Then, since $\theta$ is a congruence, we obtain $cy\theta ca$. Therefore $x = cy \in \theta(ca)$. Thus, $L_c\theta \subseteq \theta(ca)$. It is similarly proved that $R_c\theta(a) \subseteq \theta(ac)$. □

**Corollary 1.** *An equivalence $\theta$ of a quasigroup $(Q, \cdot)$ is a congruence if and only if $\theta\omega \subseteq \omega\theta$ for all $\omega \in \Pi$.*

**Proof.** The multiplication of binary relations is associative, therefore, if $\theta\omega_1 \subseteq \omega_1\theta$, $\theta\omega_2 \subseteq \omega_2\theta$, where $\omega_1, \omega_2 \in \Pi$, then $\theta(\omega_1\omega_2) = (\theta\omega_1)\omega_2 \subseteq (\omega_1\theta)\omega_2 = \omega_1(\theta\omega_2) \subseteq \omega_1(\omega_2\theta) = (\omega_1\omega_2)\theta$. □

**Corollary 2.** *An equivalence $\theta$ is a congruence of a quasigroup $Q$ if and only if $\omega\theta(x) \subseteq \theta(\omega x)$ for all $x \in Q$, $\omega \in \Pi$.*

**Proof.** The proof is similar with the previous one. □

**Corollary 3.** *A congruence $\theta$ of a quasigroup $Q$ is normal if and only if at least one of the following conditions is fulfilled:*

*(i)* $\omega\theta \subseteq \theta\omega$ *for all* $\omega \in \mathbb{T}$;
*(ii)* $\omega\theta = \theta\omega$ *for all* $\omega \in \mathbb{T}$;
*(iii)* $\theta(\omega x) \subseteq \omega\theta(x)$ *for all* $\omega \in \mathbb{T}$, $x \in Q$;
*(vi)* $\theta(\omega x) = \omega\theta(x)$ *for all* $\omega \in \mathbb{T}$, $x \in Q$.

**Proof.** As it is proved in Proposition 1, the inclusion $\theta L_a \subseteq L_a\theta$ is equivalent to the implication $x\theta y \Longrightarrow ax\theta ay$.

Let us check up that the inclusion $L_a\theta \subseteq \theta L_a$ is equivalent to the implication

$$ax\theta ay \Rightarrow x\theta y.$$

Indeed, as it is proved in Proposition 1, $(x, z) \in \theta L_a$ is equivalent with $(x, L_a^{-1}z) \in \theta$. Similarly, $(x, z) \in L_a\theta$ is equivalent with $(ax, z) \in \theta$. The inclusion $\omega\theta \subseteq \theta\omega$ by $\omega = L_a$ has the form $L_a\theta \subseteq \theta L_a$ and it is equivalent to the following implication:

$$(ax, z) \in \theta \Longrightarrow (x, L_a^{-1}z) \in \theta$$

for all $a, x, z \in Q$. If we change in the last implication the element $z$ by the element $L_a z$, we shall obtain that the inclusion $\theta L_a \supseteq L_a\theta$ is equivalent to the implication $ax\theta ay \Rightarrow x\theta y$. Therefore, the equivalence $\theta$ is cancellative from the left.

Similarly, the inclusion $R_b\theta \subseteq \theta R_b$ is equivalent to the implication:

$$(xa, za) \in \theta \Longrightarrow (x, z) \in \theta.$$

If a congruence $\theta$ is cancellative from the left and from the right, then, by definition, $\theta$ is a normal congruence.

Cases *(ii)*, *(iii)*, *(iv)* are proved similarly. □

**Corollary 4.** *A congruence $\theta$ of a quasigroup $Q$ is normal if and only if $\omega\theta = \theta\omega$ for all $\omega \in \Pi$.*

*A congruence $\theta$ of a quasigroup $Q$ is normal if and only if $\omega\theta = \theta\omega$ for all $\omega \in M$.*

**Proof.** The proof is obvious. □

It is easy to see that an equivalence $q$ of a set $M$ is a congruence of the group $M$ if and only if $q$ is admissible relative to all elements of the set $\mathbb{T} \cup \mathbb{T}^{-1}$, where $\mathbb{T}^{-1} = \{L_x^{-1}, R_x^{-1} \,|\, \forall\, x \in Q\}$.

**Theorem 1.** *The lattice of congruences $(L(Q), \leq_1)$ of one-sided loop (in particular, of a loop) $Q$ is isomorphically embedded in the lattice $(L(M(Q)), \leq_2)$ of the left congruences of group $M$, which are semi-admissible from the right relative to all permutations of the semigroup $\Pi$.*

**Proof.** The proof of this theorem in some parts repeats the proof of the theorem on an isomorphic embedding of normal congruences of a quasigroup $Q$ in the lattice of congruences of the group $M(Q)$ [26].

By a quasigroup $Q$ during the proof of this theorem we shall understand a quasigroup with the right unit, i.e. right loop.

Let $q$ be a congruence of a quasigroup $Q$. We shall define the relation $q^{\top}$ in group $M$ as follows: $\theta q^{\top} \varphi \Longleftrightarrow \theta^{-1}\varphi \subseteq q$ for all $\theta, \varphi \in M$.

We prove that $q^{\top}$ is a left congruence of the group $M$ which is admissible from the right relative to all permutations $\alpha$, $\alpha \in \Pi$.

Reflexivity of $q^{\top}$. Since $\varepsilon \subseteq q$, $\alpha q^{\top} \alpha$ for all $\alpha \in M$.

Symmetry of $q^{\top}$. The equivalence $\theta q^{\top} \varphi \leftrightarrow \varphi q^{\top} \theta$ is equivalent to the equivalence $\theta^{-1}\varphi \subseteq q \leftrightarrow \varphi^{-1}\theta \subseteq q$. The last equivalence is true since, if $\theta^{-1}\varphi \subseteq q$, then $(\theta^{-1}\varphi)^{-1} \subseteq q^{-1}$, $\varphi^{-1}\theta \subseteq q^{-1} = q$. It is clear that in the same way it is possible to receive also an inverse implication: $(\varphi^{-1}\theta \subseteq q) \rightarrow (\theta^{-1}\varphi \subseteq q)$.

Transitivity of $q^{\top}$. An implication $\theta q^{\top} \varphi \wedge \varphi q^{\top} \psi \rightarrow \theta q^{\top} \psi$ is equivalent with the implications $\theta^{-1}\varphi \subseteq q \wedge \varphi^{-1}\psi \subseteq q \rightarrow \theta^{-1}\psi \subseteq q$. We shall show that the last implication is fulfilled. Indeed, if $\theta^{-1}\varphi \subseteq q \wedge \varphi^{-1}\psi \subseteq q$, $\theta^{-1}\varphi\varphi^{-1}\psi = \theta^{-1}\psi \subseteq q^2 = q$.

Let us show that $q^{\top}$ is semi-admissible from the left relative to any permutation $\alpha \in M$. Indeed, the condition "if $\theta q^{\top} \varphi$, then $\alpha\theta q^{\top} \alpha\varphi$" is equivalent with the following condition: if $\theta^{-1}\varphi \subseteq q$, then $\theta^{-1}\alpha^{-1}\alpha\varphi = \theta^{-1}\varphi \subseteq q$.

Let us show that the binary relation $q^{\top}$ (we have already proved that $q^{\top}$ is a left congruence of $M$) is semi-admissible from the right relative to any permutation $\alpha \in \Pi$. For this purpose we shall show that $\theta\alpha q^{\top} \varphi\alpha$ for all $\alpha \in \Pi$. We shall pass, using Proposition 1, to the needed inclusions.

Then we have $\theta q^{\top} \varphi \leftrightarrow \theta^{-1}\varphi \subseteq q$, $\theta\alpha q^{\top} \varphi\alpha \leftrightarrow \alpha^{-1}\theta^{-1}\varphi\alpha \subseteq q$. Since $q$ is a congruence, then by Corollary 1 we have $\alpha^{-1}q\alpha \subseteq q$ for all $\alpha \in \Pi$. Therefore, if $\theta^{-1}\varphi \subseteq q$, then $\alpha^{-1}\theta^{-1}\varphi\alpha \subseteq \alpha^{-1}q\alpha \subseteq q$.

Thus, we have proved that an arbitrary congruence of a quasigroup $Q$ corresponds the left congruence $q^\top$ of the group $M$ which is semi-admissible from the right relative to all permutations of the semigroup $\Pi$.

Let $p$ be a left congruence of the group $M$, that is semi-admissible from the right relative to all $\alpha \in \Pi$. We shall define a binary relation on a quasigroup $Q$ in the following way: $p^\perp = \cup\, \theta^{-1}\varphi$ for all $\theta, \varphi \in M$, such that $\theta\, p\, \varphi$.

We demonstrate that $p^\perp$ is a congruence of a quasigroup $Q$.

Reflexivity of $p^\perp$. Since $\theta\, p\, \theta$ for all $\theta \in M$, then $\theta^{-1}\theta = \varepsilon \subseteq p^\perp$.

Symmetry of $p^\perp$. $(p^\perp)^{-1} = p^\perp$, since $p^{-1} = p$ and $p^\perp = \cup\, \theta^{-1}\varphi$ for all $\theta, \varphi \in M$, such that $\theta\, p\, \varphi$.

Transitivity of $p^\perp$. Let $(a, b) \in (p^\perp)^2$, i.e. there exists element $c$ such that $ap^\perp c$ and $cp^\perp b$. Hence, there exist $\theta, \varphi, \psi, \xi \in M$, $\theta p \varphi$, $\psi p \xi$, such that $a\theta^{-1}\varphi c$ and $c\psi^{-1}\xi b$. Then $c = (\theta^{-1}\varphi)\, a, b = (\psi^{-1}\xi)\, c$, and $b = (\theta^{-1}\varphi\psi^{-1}\xi)\, a$, i. e. $(a, b) \in (\varphi^{-1}\theta)^{-1}\psi^{-1}\xi$.

We need to prove that $\varphi^{-1}\theta\, p\, \psi^{-1}\xi$. If $\theta\, p\, \varphi$, then taking into account that the binary relation $p$ is stable from the left relative to any permutation $\alpha \in M$, we obtain, $\varphi^{-1}\theta\, p\, \varphi^{-1}\varphi$, $\varphi^{-1}\theta\, p\, \varepsilon$.

Similarly, $\varepsilon\, p \psi^{-1}\xi$, and by transitivity of the relation $p$ we have: $\varphi^{-1}\theta\, p\, \psi^{-1}\xi$. Thus, we have proved that $p$ is an equivalence on $Q$.

Let us show that $p^\perp$ is a congruence of a quasigroup $Q$. For this purpose it is sufficient, taking into account Corollary 1, to prove that for all $\omega \in \Pi$, $\omega^{-1}\, p^\perp\, \omega \subseteq p^\perp$.

Let $(a, b) \in \omega^{-1}p^\perp\omega$. Then there exist $\varphi, \theta \in M$, $\theta p \varphi$ such that $(a, b) \in \omega^{-1}\theta^{-1}\varphi\omega = (\theta\omega)^{-1}\varphi\omega$.

Since $\theta\, p\, \varphi$ then for all $\omega \in \Pi$, $\theta\omega\, p\, \varphi\omega$, and then $(\theta\omega)^{-1}\varphi\omega \subseteq p^\perp$.

Thus $(a, b) \in p^\perp$, $\omega^{-1}\, p^\perp\, \omega \subseteq p^\perp$ for all $\omega \in \Pi$, i.e. $p^\perp$ is a congruence of a quasigroup $Q$.

We prove if $q$ is a congruence of a quasigroup $Q$, then $q^{\top\perp} = q$, i.e. we establish that the map $\top$ is a bijective map and that $(\top)^{-1} = \perp$.

It is easy to understand that $q^{\top\perp} \subseteq q$. Indeed, if $(a, b) \in (q^\perp)^\top$ there is a pair of permutations $\varphi, \theta \in M$ such that $\theta q^\top \varphi$, $(a, b) \in \theta^{-1}\varphi$.

By definition of the relation $q^\top$, $\varphi q^\top \theta$ if and only if $\varphi^{-1}\theta \subseteq q$, and then $(a, b) \in q$.

Let us prove a converse inclusion. Now we use property that the quasigroup $Q$ has the right unit.

Let $(a, b) \in q$. Then for all $x$ from $Q$ the relation $ax\, q\, bx$ is equivalent with $L_a x\, q\, L_b x$. Having replaced $x$ by $L_a^{-1}x$, we obtain $x\, q\, (L_a^{-1}L_b)x$, i.e. $L_a^{-1}L_b \subseteq q$, and then $L_a q^\top L_b$, $L_a^{-1}L_b \subseteq (q^\top)^\perp$. From the last relation we have $(a, (L_a^{-1}L_b)a) = (a, L_b e_a) = (a, b) \in (q^\top)^\perp$, since $e_a = e_b$. Therefore $q \subseteq (q^\top)^\perp$.

If we have quasigroup with the left unit, then instead of translations $L_a, L_b$ we take translations $R_a, R_b$. Thus $(q^\top)^\perp \supseteq q$, the map $\top$ is a bijective map, $(\top)^{-1} = \perp$.

Let us recall lattices $\mathcal{L}_1 = (L_1, \leq_1)$ and $\mathcal{L}_2 = (L_2, \leq_2)$ are called isomorphic if there is a bijective map $\sigma$ such that $a \leq_1 b$ in $\mathcal{L}_1$ if and only if $\sigma(a) \leq_1 \sigma(b)$ in $\mathcal{L}_2$ [10].

In order to prove that $\top$ is a lattice isomorphism, we need to prove: if $q_1 \subseteq q_2$, then $q_1^\top \subseteq q_2^\top$, if $p_1 \subseteq p_2$, then $p_1^\perp \subseteq p_2^\perp$, where $q_1, q_2$ are congruences of a quasigroup $Q$, $p_1, p_2$ are the left congruences of group $M$ that are semi-admissible relative to multiplication from the right on permutations $\alpha$ from the semigroup $\Pi$. These two implications, taking into account the definition of maps $\perp, \top$, are obvious. □

**Proposition 3.** *If the lattice of congruences is considered as an algebra of the form $(L, \vee, \wedge)$, i.e. in a signature with two binary operations, then $(q_1 \wedge q_2)^\top = q_1^\top \wedge q_2^\top$.*

**Proof.** Indeed, the operation $\wedge$ both in a lattice of congruences of a quasigroup and in a lattice of the left congruences of group $M$ coincides with the set-theoretic intersection of congruences. Therefore, if $(\alpha, \beta) \in (q_1 \wedge q_2)^\top$, then $\alpha^{-1}\beta \subseteq q_1 \wedge q_2$, $\alpha^{-1}\beta \subseteq q_1 \cap q_2$, $\alpha^{-1}\beta \subseteq q_1$, $\alpha^{-1}\beta \subseteq q_2$, $(\alpha, \beta) \in q_1^\top$, $(\alpha, \beta) \in q_2^\top$, $(\alpha, \beta) \in q_1^\top \cap q_2^\top = q_1^\top \wedge q_2^\top$.

Conversely, let $(\alpha, \beta) \in q_1^\top \wedge q_2^\top$. Then $\alpha^{-1}\beta \subseteq q_1$, $\alpha^{-1}\beta \subseteq q_2$, $\alpha^{-1}\beta \subseteq q_1 \cap q_2 = q_1 \wedge q_2$, $(\alpha, \beta) \in (q_1 \wedge q_2)^\top$. Thus, $(q_1 \wedge q_2)^\top = q_1^\top \wedge q_2^\top$. □

**Remark 4.** It is easy to see that $q_1^\top \vee q_2^\top \subseteq (q_1 \vee q_2)^\top$. Indeed, $q_1^\top \subseteq (q_1 \vee q_2)^\top$, $q_2^\top \subseteq (q_1 \vee q_2)^\top$, $q_1^\top \vee q_2^\top \subseteq (q_1 \vee q_2)^\top \vee (q_1 \vee q_2)^\top = (q_1 \vee q_2)^\top$.

Probably, in general, there exist examples such that $q_1^\top \vee q_2^\top \subsetneqq (q_1 \vee q_2)^\top$. Results from [1, 2, 4, 29] strengthen our guess.

Since in Theorem 1 it is proved that the map $\top$ is bijective, we can formulate the following theorem.

**Theorem 2.** *The lower semilattice of congruences of an one-sided loop $Q$ is isomorphically embedded in the lower semilattice of the left congruences of the group $M(Q)$ that are semi-admissible relative to all elements of the semigroup $\Pi(Q)$.*

**Corollary 5.** *The lower semilattice of congruences of an one-sided loop $Q$ is isomorphically embedded in the lower semilattices of congruences: of the semigroup $L\Pi$, of the semigroup $\Pi$ and of the left congruences of the group $LM$.*

**Proof.** By the intersection of the left congruences of group $M$ with the set $L\Pi \times L\Pi$, for example, we obtain some binary relations of semigroup $L\Pi$.

It is easy to understand that these binary relations are equivalences which are semi-admissible relative to multiplication from the left and from the right by elements of the semigroup $L\Pi$, i.e. these equivalences are congruences of semigroup $L\Pi$.

Now we should prove: if $p_1 \subset p_2$ and $p_1^\perp \subset p_2^\perp$, then $p_1, p_2$ are elements of lattice of the left congruences of the group $M$, $p_1 \cap (L\Pi)^2 \subset p_2 \cap (L\Pi)^2$.

If $p_1 \subset p_2$ and $p_1^\perp \subset p_2^\perp$, then there is a pair $(a, b)$, such that $(a, b) \in p_2^\perp$ and $(a, b) \notin p_1^\perp$. Then $(L_a, L_b) \in p_2 = ((p_2)^\perp)^\top$ and $(L_a, L_b) \notin p_1$.

If we suppose that $(L_a, L_b) \in p_1$, then $L_a^{-1}L_b \subseteq p_1^{\perp}$, $(a, L_a^{-1}L_b a) = (a, b) \in p_1^{\perp}$. We have received a contradiction. Thus $p_1 \cap (L\Pi)^2 \subset p_2 \cap (L\Pi)^2$.

The remaining inclusion maps are proved similarly. □

**Corollary 6.** *A lower semilattice of congruences of a loop is isomorphically embedded in the lower semilattices of congruences of semigroups $L\Pi$, $R\Pi$, $\Pi$, the left congruences of groups $LM$, $RM$.*

**Theorem 3.** *In an one-sided loop $Q$ all congruences are normal if and only if in the group $M$ all left congruences, which are semi-admissible from the right relative to all elements of the semigroup $\Pi$, are congruences.*

**Proof.** We suppose that in the group $M$ all left congruences, which are semi-admissible from the right relative to permutations from the semigroup $\Pi$, are congruences. We shall show that then they induce in $Q$ only normal congruences. Indeed, let $p$ be a congruence of the group $M$. We demonstrate that then $p^{\perp}$ is a normal congruence of a quasigroup $Q$.

For this purpose it is enough to prove, taking into account Theorem 1, Corollary 1, that $\omega p^{\perp}\omega^{-1} \subseteq p^{\perp}$ for all $\omega \in \Pi$.

Let $(a, b) \in \omega p^{\perp}\omega^{-1}$. Then there exist $\theta, \varphi \in M$, $\theta\, p\, \varphi$ such that $(a, b) \in \omega\theta^{-1}\varphi\omega^{-1} = (\theta\omega^{-1})^{-1}\varphi\omega^{-1}$. Since $p$ is a congruence of the group $M$, then from $\theta\, p\, \varphi$ follows $\theta\omega^{-1}\, p\, \varphi\omega^{-1}$ for all $\omega \in M$. Thus, $(a, b) \in p^{\perp}$, $\omega p^{\perp}\omega^{-1} \subseteq p^{\perp}$.

Converse. Let in an one-sided loop $Q$ all congruences be normal. We shall prove that then in the group $M$ all left congruences, which are semi-admissible from the right relatively to permutations from $\Pi$, are congruences.

We suppose converse, that in the group $M$ there exists a left congruence $p$ which is not semi-admissible relative to multiplication on the right by at least one element from the set $\mathbb{T}^{-1}$. We denote such element by $R_c^{-1}$. In other words there exist elements $\alpha, \beta$ such that $\alpha\, p\, \beta$, but $\alpha R_c^{-1}$ is not congruent with the element $\beta R_c^{-1}$.

Passing to the congruence $p^{\perp}$, we obtain $\alpha^{-1}\beta \subseteq p^{\perp}$, but $R_c\alpha^{-1}\beta R_c^{-1} \nsubseteq p^{\perp}$, i.e. there is an element $x \in Q$ such that $(x, (R_c\alpha^{-1}\beta R_c^{-1})x) \notin p^{\perp}$.

Since $p^{\perp}$ is a normal congruence of an one-sided loop $Q$, then: if $(a, b) \notin p^{\perp}$, then for all $x \in Q$ we obtain $(a\,x, b\,x) \notin p^{\perp}$.

Thus, if $(x, (R_c\alpha^{-1}\beta R_c^{-1})x) \notin p^{\perp}$, $(R_c x, (R_c\alpha^{-1}\beta)x) \notin p^{\perp}$ or $(xc, (\alpha^{-1}\beta)xc) \notin p^{\perp}$, i.e. $\alpha^{-1}\beta \nsubseteq p^{\perp}$. We have a contradiction.

Therefore left but not right congruence $\theta$ of the group $M$, which is semi-admissible from the right relative to permutations of the semigroup $\Pi$ defines a non-normal congruence of an one-sided loop. □

**Definition 3.** *A subgroup $H$ of a group $M$ will be called $A$-invariant relative to a set $A$ of elements of the group $M$, if $a^{-1}Ha \subseteq H$ for all $a \in A$.*

In the language of Definition 3 any normal subgroup $H$ of a group $G$ is $G$-invariant subgroup of the group $G$ [14].

We reformulate Theorems 2 and 3 as follows.

**Theorem 4.** *The lower semilattice of congruences of an one-sided loop is isomorphically embedded in the lower semilattice of* $\Pi$*-invariant subgroups of the group* $M$*.*

**Proof.** We shall show that the kernel of a left congruence $\theta$ of the group $M$ is some its subgroup $H$, but the left congruence $\theta$ is a partition of the group $M$ in left coset classes by this subgroup. Indeed, if $\alpha\,\theta\,\varepsilon$ and $\beta\,\theta\,\varepsilon$, then $\alpha\beta\,\theta\,\alpha$, whence, $\alpha\beta\,\theta\,\varepsilon$.

If $\alpha\,\theta\,\varepsilon$, then $\alpha^{-1}\alpha\,\theta\,\alpha^{-1}\varepsilon$, $\alpha^{-1}\,\theta\,\varepsilon$. Thus, the kernel of left congruence $\theta$ is a subgroup of a group $M$.

We notice that various left congruences of the group $M$ define various kernels. Indeed, if we suppose converse, that $\alpha\,\theta_1\,\beta$ and it is not true that $\alpha\,\theta_2\,\beta$, but $\beta^{-1}\alpha\,\theta_1\,\varepsilon$ and $\beta^{-1}\alpha\,\theta_2\,\varepsilon$, then $\beta(\beta^{-1}\alpha)\,\theta_2\,\beta\varepsilon$, $\alpha\,\theta_2\,\beta$. We have received a contradiction.

Since any subgroup $H$ of a group $M$ defines the left congruence ($\alpha \sim \beta \Longleftrightarrow \alpha H = \beta H$), we proved that there is a bijection between the left congruences of the group $M$ and its subgroups.

We shall show, that the left congruence $\theta$ of the group $M$ is semi-admissible from the right relatively all permutations of the semigroup $\Pi$ if and only if its kernel $H$ fulfill the relation $H\gamma \subseteq \gamma H$ for all elements $\gamma \in \Pi$, or, equivalently, $\gamma^{-1}H\gamma \subseteq H$.

Indeed, if the left congruence $\theta$ of the group $M$ is semi-admissible from the right relatively permutations of the semigroup $\Pi$, then for the kernel $H$ of the congruence $\theta$ we have: let $\alpha \in H$, i.e. $\alpha\,\theta\,\varepsilon$.

Then, taking into consideration the semi-admissibility from the right of the congruence $\theta$, we obtain $\alpha\gamma\,\theta\,\gamma$ for all $\gamma \in \Pi$. Since $\theta$ is a left congruence, then $\gamma^{-1}\alpha\gamma\,\theta\,\gamma^{-1}\gamma$, $\gamma^{-1}\alpha\gamma\,\theta\,\varepsilon$. Therefore, for all $\gamma \in \Pi$ we have $\gamma^{-1}H\gamma \subseteq H$.

Converse. Let kernel the $H$ of a congruence $\theta$ satisfy the relation $\gamma^{-1}H\gamma \subseteq H$ for all $\gamma \in \Pi$. If $\alpha\,\theta\,\beta$, then $\beta^{-1}\alpha\,\theta\,\varepsilon$, whence $\gamma^{-1}\beta^{-1}\alpha\gamma\,\theta\,\varepsilon$, $\alpha\gamma\,\theta\,\beta\gamma$ for all $\gamma \in \Pi$. □

**Theorem 5.** *Congruences of an one-sided loop are normal if and only if* $\Pi$*-invariant subgroups of the group* $M$ *are normal in* $M$*.*

We can give sufficient conditions of normality of all congruences of a quasigroup.

**Proposition 4.** *If a quasigroup* $Q$ *satisfies the condition* $\mathbb{T}^{-1} \subseteq \Pi$*, then in* $Q$ *all congruences are normal.*

**Proof.** If $\theta$ is a congruence of a quasigroup $Q$, then, obviously, from $a\theta b$ follows $\alpha a\,\theta\,\alpha b$ for all $\alpha \in \Pi$.

Since $\mathbb{T}^{-1} \subseteq \Pi$, then from $ab\,\theta\,ac$ follows $L_a^{-1}(ab)\theta L_a^{-1}(ac)$, $b\theta c$, from $ca\,\theta\,ba$ follows $R_a^{-1}(ca)\theta R_a^{-1}(ba)$, $c\theta b$. □

**Corollary 7.** *If in a quasigroup* $Q$ *the condition* $M = \Pi$ *is fulfilled, then in the quasigroup* $Q$ *all congruences are normal.*

**Proof.** It is easy to see that conditions $\mathbb{T}^{-1} \subseteq \Pi$ and $M = \Pi$ are equivalent. □

Conditions of Proposition 4 and Corollary 7 can be used for concrete classes of quasigroups. See some examples below.

But, in general, these conditions are only sufficient, since there exists an example of a quasigroup, in which all congruences are normal, but $\mathbb{T} \subsetneqq \Pi$, or, that is equivalent, $M \neq \Pi$.

**Example 1.** Let $A = \{\frac{a}{2^n} \,|\, a \in \mathbb{Z}, n \in \mathbb{N}\}$, where $\mathbb{Z}$ is the set of integers, and $\mathbb{N}$ is the set of natural numbers.

The set $A$ forms a torsion-free abelian group of rank 1 relative to the operation of addition of elements of the set $A$ [19].

Using the group $(A, +)$ we define on the set $A$ a new quasigroup operation $\circ$. Let $\varphi$ be a map of the set $A$ into itself such that $\varphi x = \frac{1}{2}x$ for all $x \in A$.

It is easy to check that $\varphi$ is an automorphism of the group $(A, +)$. Then $(A, \cdot)$ with the form $x \cdot y = \varphi x + y$ for all $x, y \in A$ is a left loop with the left identity 0. Indeed, $0 \cdot x = \varphi 0 + x = x$.

We prove that in the quasigroup $(A, \cdot)$ $M(A, \cdot) \neq \Pi(A, \cdot)$, and all congruences are normal.

For this purpose in the beginning we calculate the form of translations of a quasigroup $(A, \cdot)$. We have $R^{\cdot}_a x = x \cdot a = \varphi x + a = (\varphi R^{+}_a)x$, $L^{\cdot}_a x = a \cdot x = \varphi a + x = L^{+}_{\varphi a} x$. Using results from [14, 29] further it is possible to deduce the following relations

$$\begin{aligned}
&LM(A, \cdot) = LM(A, +) \cong (A, +),\\
&RM(A, \cdot) \cong RM(A, +) \rightthreetimes \langle \varphi \rangle \cong (A, +) \rightthreetimes (\mathbb{Z}, +),\\
&L\Pi(A, \cdot) = L\Pi(A, +),\\
&R\Pi(A, \cdot) = \{(\varphi^n R^{+}_a) \,|\, a \in A, n \in \mathbb{N}\}
\end{aligned}$$

It is easy to see that $M(A, \cdot) = RM(A, \cdot) = \{(\varphi^n R^{+}_a) \,|\, a \in A, n \in \mathbb{Z}\}$, $\Pi(A, \cdot) = \{(\varphi^n R^{+}_a) \,|\, a \in A, n \in \mathbb{N} \cup \{0\}\}$. Thus, $\Pi(A, \cdot) \subsetneqq M(A, \cdot)$. Moreover, if we denote by $\Pi^{-1}(A)$ the set $\{(\varphi^n R^{+}_a) \,|\, a \in A, n \in -\mathbb{N}\}$, then $M(A) = \Pi(A) \cup \Pi^{-1}(A)$.

Since $(A, \cdot)$ is a left loop, we can use Theorem 4. As it follows from Theorem 4, the subgroups of the group $M(A, \cdot)$ that are invariant relative to all permutations of the semigroup $\Pi(A, \cdot)$ correspond to congruences of the quasigroup $(A, \cdot)$.

We demonstrate that any $\Pi$-invariant subgroup of the group $M(A, \cdot)$ is a normal subgroup of the group $M(A, \cdot)$.

We notice that following our agreements we have $(R^{+}_a \varphi)(x) = \varphi(x + a) = \varphi x + \varphi a = (\varphi R^{+}_{\varphi a})(x)$. Below in this example we shall write $R_x$ instead of $R^{+}_x$. We have $(\varphi^k R_a)(\varphi^l R_b) = \varphi^{k+l} R_{\varphi^l a + b}$, $(\varphi^n R_a)^{-1} = \varphi^{-n} R_{-\varphi^{-n} a}$.

It is clear that any element of a subgroup $H$ of the group $M$ has the form $\varphi^k R_b$. If $H$ is a $\Pi$-invariant subgroup of the group $M$, then we have: if $\varphi^k R_b \in H$, then $\varphi^{-n} R_{-\varphi^{-n} a} \varphi^k R_b \varphi^n R_a = \varphi^k R_c \in H$ for all $\varphi^k R_b \in H$, $\varphi^n R_a \in \Pi$, where $c = -\varphi^k a + \varphi^n b + a$.

In other words, If $H$ is a $\Pi$-invariant subgroup of the group $M$, then: if $\varphi^k R_b \in H$, then $\varphi^k R_{\varphi^n b} R_{-\varphi^k a + a} \in H$ for all $\varphi^k R_b \in H$, $n \in \mathbb{N} \cup \{0\}$, $a \in A$.

If we change in the last implication $a$ by $-a$, then we obtain the following implication:

if $\varphi^k R_b \in H$, then $\varphi^k R_{\varphi^n b} R_{\varphi^k a - a} \in H$ for all $\varphi^k R_b \in H$, $n \in \mathbb{N} \cup \{0\}$, $a \in A$. $(*)$

From the implication $(*)$ by $a = 0$ it follows:

if $\varphi^k R_b \in H$, then $\varphi^k R_{\varphi^n b} \in H$ for all $\varphi^k R_b \in H$, $n \in \mathbb{N} \cup \{0\}$. $(**)$

We can write the condition that the $\Pi$-invariant subgroup $H$ of group $M$ is a normal subgroup of group $M$, in the form: if $\varphi^k R_b \in H$, then $\varphi^n R_a \varphi^k R_b \varphi^{-n} R_{-\varphi^{-n} a} = \varphi^k R_d \in H$, where $d = -\varphi^{-n}(-\varphi^k a - b + a)$, for all $\varphi^k R_b \in H$, $\varphi^n R_a \in \Pi$.

Applying to the last implication condition $(**)$, we obtain the following equivalent condition of normality of $\Pi$-invariant group $H$: if $\varphi^k R_b \in H$, then $\varphi^k R_h \in H$, where $h = \varphi^k a + b - a$ for all $\varphi^k R_b \in H$, $a \in A$.

The last implication we can re-write in the form:
if $\varphi^k R_b \in H$, then $\varphi^k R_b R_{\varphi^k a - a} \in H$ for all $\varphi^k R_b \in H$, $a \in A$.

It is easy to see that the last implication follows from the implication $(*)$ by $n = 0$.

**Example 2.** Using the group $(A, +)$ from Example 1 we define on the set $A$ a binary operation $*$ in the following way $x * y = 2 \cdot x + y$ for all $x, y \in A$. The operation $*$ is a quasigroup operation, since the map $2 : x \mapsto 2 \cdot x$ for all $x \in A$ is an automorphism of the group $(A, +)$, moreover, a left loop operation, see Example 1.

We denote by $H$ the following subgroup of the group $M(A, *)$: $H = \langle R_1^+ \mid 1 \in A \rangle = \{\dots R_{-2}^+, R_{-1}^+, R_0^+, R_1^+, R_2^+, \dots\}$. It is easy to see that $H \cong (\mathbb{Z}, +)$.

We check that the group $H$ is a $\Pi$-invariant non-normal subgroup of the group $M$.

We use results of Example 1 by $\varphi = 2$. Thus, if $H$ is a $\Pi$-invariant subgroup of the group $M$, then we have: $2^{-n} R_{-2^{-n} a} R_1 2^n R_a = R_{2^n} \in H$ for all $2^n R_a \in \Pi$.

We prove that the group $H$ is a non-normal subgroup of the group $M$. We have $2^n R_a R_1 2^{-n} R_{-2^{-n} a} = R_{2^{-n}} \notin H$ for all $2^n R_a \in M$ such that $n > 1$.

As it follows from Theorems 2 and 4, the subgroup $H$ of the group $M(A, *)$ induces a non-normal congruence of the quasigroup $(A, *)$.

**Remark 5.** The fact that the group $H$ induces a congruence of the quasigroup $(A, *)$ can be deduced from results of the article of T. Kepka and P. Nemec [16, Theorem 42], since the quasigroup $(A, *)$ is a T-quasigroup, moreover, it is a medial quasigroup.

## 3 On normality of congruences of some inverse quasigroups

**Definition 4.** *A quasigroup $Q$ is called $rst$-inverse quasigroup, if there exist a permutation $J$ of the set $Q$, some fixed integers $r, s, t$ such that in the quasigroup $Q$ for all $x, y \in Q$ the relation $J^r(x \circ y) \circ J^s x = J^t y$ is fulfilled [16].*

A $(0,1,0)$-inverse quasigroup is called a CI-quasigroup. A $(-1,0,-1)$-inverse quasigroup is called an WIP-quasigroup [5]. An $(m,m+1,m)$-inverse quasigroup is called an $m$-inverse quasigroup [15].

**Proposition 5.** *In $rst$-quasigroup $(Q,\cdot)$ all congruences are normal if permutations $J^r$ and $J^{-t}$ are semi-admissible relative to any congruence of $(Q,\cdot)$.*

**Proof.** In the language of translations we can re-write Definition 4 in the form $L_x J^r R_{J^s x} = J^t$. Then $L_x^{-1} = J^r R_{J^s x} J^{-t}$, $R_{J^s x}^{-1} = J^{-t} L_x J^r$.

Using Proposition 4 we obtain the required. □

**Corollary 8.** *In CI-quasigroup all congruences are normal.*

**Corollary 9.** *In WIP-quasigroup $(Q,\cdot)$ all congruences are normal if the permutation $J$ is admissible relative to any congruence of $(Q,\cdot)$.*

**Corollary 10.** *In m-inverse quasigroup $(Q,\cdot)$ all congruences are normal if the permutation $J^m$ is admissible relative to any congruence of $(Q,\cdot)$.*

In [17] Definition 4 is generalized in the following way.

**Definition 5.** *A quasigroup $Q,\circ)$ is called an $(\alpha,\beta,\gamma)$-inverse quasigroup if there exist permutations $\alpha,\beta,\gamma$ of the set $Q$ such that $\alpha(x\circ y)\circ \beta x = \gamma y$ for all $x,y\in Q$.*

**Proposition 6.** *In $(\alpha,\beta,\gamma)$-quasigroup $(Q,\cdot)$ all congruences are normal if permutations $\alpha$ and $\gamma^{-1}$ are semi-admissible relative to any congruence of $(Q,\cdot)$.*

**Proof.** The proof repeats the proof of Proposition 5. □

**Definition 6.** *A quasigroup $(Q,\circ)$ has the **$\lambda$-inverse-property** if there exist permutations $\lambda_1,\lambda_2,\lambda_3$ of the set $Q$ such that $\lambda_1 x\circ\lambda_2(x\circ y) = \lambda_3 y$ for all $x,y\in Q$ [8].*

**Definition 7.** *A quasigroup $(Q,\circ)$ has the **$\rho$-inverse-property** if there exist permutations $\rho_1,\rho_2,\rho_3$ of the set $Q$ such that $\rho_1(x\circ y)\circ\rho_2 y = \rho_3 x$ for all $x,y\in Q$ [8].*

**Definition 8.** *A quasigroup $(Q,\circ)$ that has $\lambda$-inverse-property and $\rho$-inverse-property is called $I$-inverse quasigroup [8].*

**Proposition 7.** *In an $I$-inverse quasigroup $(Q,\cdot)$ all congruences are normal if permutations $\lambda_2$, $\lambda_3^{-1}$, $\rho_1$ and $\rho_3^{-1}$ are semi-admissible relative to any congruence of $(Q,\cdot)$.*

**Proof.** From Definition 6 we have $L_x\lambda_2 L_{\lambda_1 x} = \lambda_3$. Then $L_x^{-1} = \lambda_2 L_{\lambda_1 x}\lambda_3^{-1}$. From Definition 7 we have $R_y\rho_1 R_{\rho_2 y} = \rho_3$. Therefore $R_y^{-1} = \rho_1 R_{\rho_2 y}\rho_3^{-1}$. Further we can apply Proposition 4. □

If in $I$-inverse quasigroup $(Q,\circ)$ $\lambda_2=\lambda_3=\rho_1=\rho_3=\varepsilon$, then $(Q,\circ)$ is called an $IP$-quasigroup.

**Corollary 11.** *In IP-quasigroup all congruences are normal [5].*

**Proof.** The proof follows from the definition of IP-quasigroup and Proposition 7. □

## 4 On regularity of congruences of quasigroups

**Definition 9.** *A congruence is called regular if it is uniquely defined by any its coset, the coset of a congruence is called regular if it is a coset of only one congruence.*

In [21] A.I. Mal'tsev has given necessary and sufficient conditions that a normal complex $K$ of an algebraic systems $A$ is a coset of only one congruence, i.e. $K$ is a coset of only one congruence of a system $A$.

For this purpose for any set $S \subseteq A$ the congruence $\mathrm{mod\,S}$ is constructed. Elements $a, b$ are equivalent $a \sim b\,(\mathrm{mod\,S})$ if either $a = b$, or $a, b \in S$, or $a = \alpha u$, $b = \alpha v$, where $u, v \in S$, $\alpha$ is a translation of the algebraic system $A$.

A.I. Mal'tsev names elements $a$ and $b$ *comparable* if there exists a sequence $x_1, \dots, x_n$ of elements from $A$ such that: $a \sim x_1, x_1 \sim x_2, \dots, x_n \sim b\,(\mathrm{mod\,S})$.

The binary relation $(\mathrm{mod\,S})$ is a congruence on an algebraic system $A$, and the congruence $(\mathrm{mod\,S})$ is minimal among all congruences for which elements of the set $S$ are comparable with each other [21].

**Theorem 6.** *The normal complex $K$ is a coset of only one congruence of an algebraic system $A$ if and only if elements $a, b \in A$, for which by any translation $\alpha$ the statements $\alpha a \in K$ and $\alpha b \in K$ are equivalent, are comparable* $(\mathrm{mod\,K})$ *[21].*

We notice if in Theorem 6 $A$ is a binary quasigroup, then $\alpha$ is an element of $\Pi(A)$.

If in Mal'tsev theorem we pass from a quasigroup $A$ to its homomorphic image $\bar{A} = A/\mathrm{mod\,K}$, then we shall have the following conditions of regularity of a normal complex $K$ of a quasigroup $A$.

**Proposition 8.** *The normal complex $K$ is a coset of only one congruence of a quasigroup $A$ if and only if for each pair of elements $\bar{a}, \bar{b} \in \bar{A}$ for which by any translation $\bar{\alpha} \in \bar{A}$ the statements $\bar{\alpha}\bar{a} = \bar{k}$ and $\bar{\alpha}\bar{b} = \bar{k}$ are equivalent the equality $\bar{a} = \bar{b}$ is fulfilled.*

**Remark 6.** Let's remark if $\bar{A}$ is a binary quasigroup, then conditions of Proposition 8 are fulfilled. Indeed, if we take translation $\bar{\alpha}$ such that $\bar{\alpha} = \bar{L_c}$ and $\bar{c} \cdot \bar{a} = \bar{k}$, then we have $\bar{c} \cdot \bar{b} = \bar{k}$ by conditions of the proposition. Then $\bar{a} = \bar{c} \backslash \bar{k} = \bar{b}$.

**Example 3.** It is possible to construct division groupoid in which the conditions of Proposition 8 are satisfied. We denote by $(\mathbb{Q}, +)$ the group of rational numbers relative to the operation of addition, and by $(\mathbb{Z}, +)$ the group of integers relative to the operation of addition. On the factor group $\bar{A} = (\mathbb{Q}/\mathbb{Z}, +)$ we define operation $x \circ y = 2x + y$ for all $x, y \in \bar{A}$.

It is easy to check up that $(\bar{A}, \circ)$ is a division groupoid. We shall show that this groupoid satisfies conditions of Proposition 8. Since $(\bar{A}, \circ)$ is a division groupoid, then for any $\bar{k} \in \bar{A}$ there exists $\bar{c} \in \bar{A}$ such that $\bar{c} \circ \bar{a} = \bar{k}$, and then by conditions of this proposition also $\bar{c} \circ \bar{b} = \bar{k}$. Therefore $2\bar{c} + \bar{a} = \bar{k}$, $2\bar{c} + \bar{b} = \bar{k}$, $\bar{a} = \bar{b} = \bar{k} - 2\bar{c}$.

**Proposition 9.** *There exist a quasigroup $Q$ and its subset $K$ such that $K$ is a coset of more than one congruence.*

**Proof.** It is known (see [9. p. 10]) that any division groupoid is a homomorphic image of some quasigroups.

From Mal'tsev theorem it follows that to give an example of a quasigroup in which not all congruences are regular it is necessary to find a pair of elements $a, b \in Q$ such that $a \nsim b\,(\mathrm{mod\,K})$, where $K$ is a coset of some congruence, but for which by any translation $\alpha$ statements $\alpha a \in K$ and $\alpha b \in K$ are equivalent.

We pass to homomorphic image $P = Q/\mathrm{modK}$ of quasigroups $Q$. Then conditions that the coset $K$ is not regular are the following: $\bar{a} \neq \bar{b}$, but for any $c \in Q$ the equality $\bar{c} \cdot \bar{a} = \bar{k}$ is equivalent with the equality $\bar{c} \cdot \bar{b} = \bar{k}$, the equality $\bar{a} \cdot \bar{c} = \bar{k}$ is equivalent to the equality $\bar{b} \cdot \bar{c} = \bar{k}$ where $\bar{k}$ is an image of the set $K$ in the groupoid $P$.

We construct the following division groupoid. Let $\mathbb{C}$ be a set of complex numbers, $x \circ y = (xy)^2$ for all $x, y \in \mathbb{C}$. It is easy to check that $(\mathbb{C}, \circ)$ is a commutative division groupoid.

Let $\bar{k} = 4$. Then the equation $a \circ y = 4 \Longleftrightarrow (ay)^2 = 4$, $ay = \pm 2$, $y = \pm\frac{2}{a}$ has two solutions. And, if one of radicals is a solution of the equations $a \circ y = 4$, so is the other, for any $a \in \mathbb{C}$. If we take in a quasigroup $Q$ pre-images of elements of 2 and $-2$, then we find the necessary pair. $\square$

## 5 On behavior of congruences by an isotopy

A quasigroup $(Q, \circ)$ is an isotope of a quasigroup $(Q, \cdot)$ if there exist permutations $\alpha, \beta, \gamma$ of the set $Q$ such that $x \circ y = \gamma^{-1}(\alpha x \cdot \beta y)$ for all $x, y \in Q$. We can also write this fact in the form $(Q, \circ) = (Q, \cdot)T$, where $T = (\alpha, \beta, \gamma)$ [5,6,23]. An isotopy $T = (\alpha, \beta, \gamma)$ is admissible relative to a binary relation $\theta$, if the permutations $\alpha, \beta, \gamma$ are admissible relative to $\theta$.

If $(Q, \cdot)$ is a quasigroup, then an isotopy of the form $(R_a^{-1}, L_b^{-1}, \varepsilon)$, where $R_a, L_b$ are some fixed translations of the quasigroup $(Q, \cdot)$ is called LP-isotopy. Any LP-isotopic image of a quasigroup is a loop [5,6].

In [5], p. 59 the following lemma is proved.

**Lemma 2.** *Let $\theta$ be a normal congruence of a quasigroup $(Q, \cdot)$. If a quasigroup $(Q, \circ)$ is isotopic to $(Q, \cdot)$ and the isotopy $(\alpha, \beta, \gamma)$ is admissible relative to $\theta$, then $\theta$ is a normal congruence also in $(Q, \circ)$.*

**Corollary 12.** *If $(Q, \cdot)$ is a quasigroup, $(Q, +)$ is a loop of the form $x + y = R_a^{-1}x \cdot L_b^{-1}y$ for all $x, y \in Q$, then $nCon(Q, \cdot) \subseteq nCon(Q, +)$, where $nCon(Q, \cdot)$ is the set of normal congruences of the quasigroup $(Q, \cdot)$, and $nCon(Q, +)$ is the set of normal congruences of the loop $(Q, +)$.*

**Proof.** If $\theta$ is a normal congruence of a quasigroup $(Q, \cdot)$, then, since $\theta$ is admissible relative to the isotopy $T = (R_a^{-1}, L_b^{-1}, \varepsilon)$, $\theta$ is also a normal congruence of a loop $(Q, +)$. $\square$

**Remark 7.** It is easy to see, if $x + y = R_a^{-1}x \cdot L_b^{-1}y$, then $x \cdot y = R_a x + L_b y$. If in conditions of Corollary 12 we shall in addition suppose, that the isotopy $T^{-1} = (R_a, L_b, \varepsilon)$ is admissible relative to any normal congruence of the loop $(Q, +)$, then we obtain the following equality $nCon(Q, \cdot) = nCon(Q, +)$.

**Proposition 10.** *The lattice $(L, \vee, \wedge)$ of normal congruences of a quasigroup $(Q, \cdot)$ is isomorphic to a sublattice of the lattice $(L_1, \vee, \wedge)$ of normal congruences of isotope loop $(Q, \circ)$ [9].*

**Proof.** By an LP-isotopy $T$ ($T = (R_a^{-1}, L_b^{-1}, \varepsilon)$) a normal congruence $\theta$ of quasigroup $(Q, \cdot)$ is also a normal congruence of a loop $(Q, \star)$, $(Q, \star) = (Q, \cdot)T$ (Corollary 12).

Since the operation $\wedge$ in sets of congruences of a quasigroup $(Q, \cdot)$ and loops $(Q, \star)$ coincides with the set-theoretic intersection, and the operation $\vee$ coincides, in view of the permutability of normal congruences, with their product as binary relations ([30]), we can state that the lattice of normal congruences of a quasigroup $(Q, \cdot)$ is a sublattice of the lattice of normal congruences of the loop $(Q, \star)$. This corollary is proved, since any isotopy between a loop and a quasigroup has the form $(R_a^{-1}, L_b^{-1}, \varepsilon)(\varphi, \varphi, \varphi)$. □

Obviously, any permutation of the semigroup $\Pi(Q, \cdot)$ is semi-admissible relative to any congruence of a quasigroup $(Q, \cdot)$. An isotopy is semi-admissible, if all permutations included in it are semi-admissible.

**Proposition 11.** *Let $\theta$ be a congruence of a quasigroup $(Q, \cdot)$. If a quasigroup $(Q, \circ)$ is isotopic to $(Q, \cdot)$, and the isotopy $T$ is semi-admissible relative to $\theta$, then $\theta$ is a congruence also in $(Q, \circ)$.*

**Proof.** We suppose that the isotopy $T$ has the form $T = (\alpha, \beta, \gamma)$. If $a\theta b$, then $\beta a \theta \beta b$, $\alpha c \cdot \beta a \theta \alpha c \cdot \beta b$, $\gamma^{-1}(\alpha c \cdot \beta a)\theta\gamma^{-1}(\alpha c \cdot \beta b)$.

Finally, we obtain $(c \circ a)\theta(c \circ b)$. Similarly, if $a\theta b$, then $a \circ c\theta b \circ c$. □

**Proposition 12.** *If in a quasigroup $(Q, \cdot)$ there exist elements $a, b$ such that $R_a^{-1}$, $L_b^{-1} \in \Pi$, then the lower semilattice $(L_1, \wedge)$ of congruences of a quasigroup $(Q, \cdot)$ is a subsemilattice of the semilattice $(L_2, \wedge)$ of congruences of the loop $(Q, \circ)$ which is an isotope of a quasigroup $(Q, \cdot)$ of the form $(R_a^{-1},\ L_b^{-1},\ \varepsilon)$.*

**Proof.** If $R_a^{-1}, L_b^{-1} \in \Pi$, then the isotopy $(R_a^{-1}, L_b^{-1}, \varepsilon)$ is admissible relative to any congruence of quasigroup $(Q, \cdot)$. The corollary is true, since the operations $\wedge$ in $(L_1, \wedge)$ and $(L_2, \wedge)$ coincide with the set-theoretic intersection of congruences. □

In any IP-loop $(Q, \circ)$ with the identity 1 the map $J : a \mapsto a^{-1}$ for all $a \in Q$, where $a \circ a^{-1} = 1$, is a permutation of the set $Q$, $J^2 = \varepsilon$ ([11]).

**Example 4.** If $(Q, \circ)$ is an IP-loop, $(Q, \cdot)$ is its isotope of the form $(\alpha J^\tau, \beta J^\kappa, \varepsilon)$, where $\alpha, \beta \in M(Q, \circ)$, $\tau, \kappa \in \{0, 1\}$, i.e. $x \cdot y = \alpha J^\tau x \circ \beta J^\kappa y$ for all $x, y \in Q$, then $Con(Q, \circ) = nCon(Q, \cdot)$.

**Proof.** The permutation $J$ is an antiautomorphism in $(Q,\circ)$ and any normal congruence in $(Q,\circ)$ is admissible relative to this permutation. Indeed, if $x\theta y$, then $1\theta x^{-1}\circ y$, $y^{-1}\theta(x^{-1}\circ y)\circ y^{-1}$, $y^{-1}\theta x^{-1}$, $x^{-1}\theta y^{-1}$ and in the similar way $x\theta y$ follows from $x^{-1}\theta y^{-1}$.

By Corollary 11 in an IP-loop all congruences are normal, i.e. $Con(Q,\circ) = nCon(Q,\circ)$. Then permutations $\alpha,\beta$ and $J$ are admissible relative to any congruence of the loop $(Q,\circ)$, by Lemma 2 $Con(Q,\circ) \subseteq nCon(Q,\cdot)$.

Since $(Q,\cdot) = (Q,\circ)(\alpha J^{\tau},\beta J^{\kappa},\varepsilon)$, then $(Q,\circ) = (Q,\cdot)((\alpha J^{\tau})^{-1},(\beta J^{\kappa})^{-1},\varepsilon)$. It is known ([5,6]) that every principal isotopy (the third component of such isotopy is an identity mapping) of a quasigroup $(Q,\cdot)$ to a loop $(Q,\circ)$ has the form $(R_a^{-1},L_b^{-1},\varepsilon)$, where $R_a x = x\cdot a$, $L_b x = b\cdot x$.

Thus, taking into consideration Corollary 12, we have: $nCon(Q,\cdot) \subseteq Con(Q,\circ)$. Therefore $nCon(Q,\cdot) = Con(Q,\circ)$. □

**Example 5.** If $(Q,\circ)$ is a CI-loop, $(Q,\cdot)$ is its isotope of the form $x\cdot y = \alpha J^{\tau}x\circ\beta J^{\kappa}y$ for all $x,y\in Q$, where $\alpha,\beta\in M(Q,\circ)$, $\tau,\kappa\in\{0,1\}$, then $Con(Q,\circ) = nCon(Q,\cdot)$.

**Proof.** The permutation $J$ is an automorphism in $(Q,\circ)$ ([5]) and any normal congruence in $(Q,\circ)$ is admissible relative to this permutation. Indeed, if $x\theta y$, then $1\theta y\circ Jx$, $Jy\theta(y\circ Jx)\circ Jy$, $Jy\theta Jx$, $Jx\theta Jy$.

In any CI-quasigroup $(Q,\circ)$ the following equality is true $x\circ(y\circ Jx) = y$ for all $x,y\in Q$ [16]. If $Jx\theta Jy$, then $y\circ Jx\,\theta\, y\circ Jy$, $y\circ Jx\,\theta\, 1$, $x\circ(y\circ Jx)\,\theta\, x$, $y\theta x$.

By Corollary 8 in the loop $(Q,\circ)$ all congruences are normal. Therefore, permutations $\alpha,\beta$ are admissible relative to any congruence of the loop $(Q,\circ)$. □

**Acknowledgement.** The author thanks Prof. G.B. Belyavskaya and Prof. E.A. Zamorzaeva for their helpful comments.

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*Received August 19, 2005*