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Multiobjective Games and Determining Pareto-Nash Equilibria

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Abstract. We consider the multiobjective noncooperative games with vector payoff functions of players. Pareto-Nash equilibria conditions for such class of games are formulated and algorithms for determining Pareto-Nash equilibria are proposed.

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1 Introduction and Problem Formulation

In this paper we consider multiobjective games, which generalize noncooperative ones [1–3] and Pareto multicriterion problems [4, 5]. The payoff functions of players in such games are presented as vector functions, where players intend to optimize them in the sense of Pareto on their sets of strategies. At the same time in our game model it is assumed that players are interested to preserve Nash optimality principle when they interact between them on the set of situations. Such statement of the game leads to a new equilibria notion which we call Pareto-Nash equilibria.

The multiobjective game with p players is denoted by $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$, where X_i is the set of strategies of player i, $i = \overline{1,p}$, and $\overline{F}_i = (F_i^1, F_i^2, \dots, F_i^{r_i})$ is the vector payoff function of player i, defined on set of situations $X = X_1 \times X_2 \times \cdots \times X_p$:

$$\overline{F}_i \ : \ X_1 \times X_2 \times \cdots \times X_p \to R^{r_i}, \ i = \overline{1,p}.$$

Each component F_i^k of $\overline{F}_i$ corresponds to a partial criterion of player i and represents a real function defined on set of situations $X = X_1 \times X_2 \times \cdots \times X_p$:

$$F_i^k \ : \ X_1 \times X_2 \times \cdots \times X_p \to R^1, \ k = \overline{1,r_i}, \ i = \overline{1,p}.$$

We call the solution of the multiobjective game $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$ Pareto-Nash equilibrium and define it in the following way.

Definition 1. The situation $x^* = (x_1^*, x_2^*, \dots, x_p^*) \in X$ is called Pareto-Nash equilibrium for the multiobjective game $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$ if for every

$i \in \{1, 2, \ldots, p\}$ the strategy x_i^* represents Pareto solution for the following multicriterion problem:

$$\max_{x_i \in X_i} \to \overline{f}^i_{x^*}(x_i) = (f^{i1}_{x^*}(x_i), f^{i2}_{x^*}(x_i), \ldots, f^{ir_i}_{x^*}(x_i)),$$

where

$$f^{ik}_{x^*}(x_i) = F_i^k(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_p^*), \ k = \overline{1, r_i}, \ i = \overline{1, p}.$$

This definition generalizes well-known Nash equilibria notion for classical noncooperative games (single objective games) and Pareto optimum for multicriterion problems. If $r_i = 1$, $i = \overline{1, p}$, then $\overline{G}$ becomes classical noncooperative game, where x^* represents Nash equilibria solution; in the case $p = 1$ the game $\overline{G}$ becomes Pareto multicriterion problem, where x^* is Pareto solution.

An important special class of multiobjective games represents zero-sum games of two players. This class is obtained from general case of the multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$ when $p = 2$, $r_1 = r_2 = r$ and $\overline{F}_2(x_1, x_2) = -\overline{F}_1(x_1, x_2)$.

Zero-sum multiobjective game is denoted $\overline{G} = (X_1, X_2, \overline{F})$, where $\overline{F}(x_1, x_2) = \overline{F}_2(x_1, x_2) = -\overline{F}_1(x_1, x_2)$. Pareto-Nash equilibrium for this game corresponds to saddle point $x^* = (x_1^*, x_2^*) \in X_1 \times X_2$ for the following max-min multiobjective problem:

$$\max_{x_1 \in X_1} \min_{x_2 \in X_2} \to \overline{F}(x_1, x_2) = (F^1(x_1, x_2), F^2(x_1, x_2), \ldots, F^r(x_1, x_2)). \tag{1}$$

Strictly we define the saddle point $x^* = (x_1^*, x_2^*) \in X_1 \times X_2$ for zero-sum multiobjective problem (1) in the following way.

Definition 2. The situation $(x_1^*, x_2^*) \in X_1 \times X_2$ is called the saddle point for max-min multiobjective problem (1) (i.e. for zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$) if x_1^* is Pareto solution for multicriterion problem:

$$\max_{x_1 \in X_1} \to \overline{F}(x_1, x_2^*) = (F^1(x_1, x_2^*), F^2(x_1, x_2^*), \ldots, F^r(x_1, x_2^*)),$$

and x_2^* is Pareto solution for multicriterion problem:

$$\min_{x_2 \in X_2} \to \overline{F}(x_1^*, x_2) = (F^1(x_1^*, x_2), F^2(x_1^*, x_2), \ldots, F^r(x_1^*, x_2)).$$

If $r = 1$ this notion corresponds to classical saddle point notion for min-max problems, i.e. we obtain saddle point notion for classical zero-sum games of two players.

In this paper we show that theorems of J. Nash [2] and J. Neumann [1] related to classical noncooperative games can be extended for our multiobjective case of games. Moreover, we show that all results related to discrete multiobjective games, especially matrix games can be developed in analogous way as for classical ones. Algorithms for determining the optimal strategies of players in considered games will be developed.

2 The main results

First we formulate the main theorem which represents an extension of the Nash theorem for our multiobjective version of the game.

Theorem 1. *Let $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$ be a multiobjective game, where $X_1, X_2, \dots, X_p$ are convex compact sets and $\overline{F}_1, \overline{F}_2, \dots, \overline{F}_p$ represent continuous vector payoff functions. Moreover, let us assume that for every $i \in \{1, 2, \dots, p\}$ each component $F_i^k(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_p)$, $k \in \{1, 2, \dots, r_i\}$, of the vector function $\overline{F}_i(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_p)$ represents a concave function with respect to x_i on X_i for fixed $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_p$. Then for multiobjective game $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$ there exists Pareto-Nash equilibria situation $x^* = (x_1^*, x_2^*, \dots, x_p^*) \in X_1 \times X_2 \times \dots \times X_p$.*

Proof. Let $\alpha_{11}, \alpha_{12}, \dots, \alpha_{1r_1}, \alpha_{21}, \alpha_{22}, \dots, \alpha_{2r_2}, \dots, \alpha_{p1}, \alpha_{p2}, \dots, \alpha_{pr_p}$ be an arbitrary set of real numbers which satisfy the following condition

$$\begin{cases} \sum_{k=1}^{r_i} \alpha_{ik} = 1, & i = \overline{1,p}; \\ \alpha_{ik} > 0, & k = \overline{1,r_i},\ i = \overline{1,p}. \end{cases} \tag{2}$$

We consider an auxiliary noncooperative game (single objective game) $G = (X_1, X_2, \dots, X_p, f_1, f_2, \dots, f_p)$, where

$$f_i(x_1, x_2, \dots, x_p) = \sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1, x_2, \dots, x_p),\ i = \overline{1,p}.$$

It is evident that $f_i(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_p)$ for every $i \in \{1, 2, \dots, p\}$ represents a continuous and concave function with respect to x_i on X_i for fixed $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_p$ because $\alpha_{11}, \alpha_{12}, \dots, \alpha_{1r_1}, \alpha_{21}, \alpha_{22}, \dots, \alpha_{2r_2}, \dots, \alpha_{p1}, \alpha_{p2}, \dots, \alpha_{pr_p}$ satisfy condition (2) and $F_i^k(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_p)$ is a continuous and concave function with respect to x_i on X_i for fixed $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_p$, $k = \overline{1,r_i}$, $i = \overline{1,p}$.

According to Nash theorem [2] for the noncooperative game $G = (X_1, X_2, \dots, X_p, f_1, f_2, \dots, f_p)$ there exists Nash equilibria situation $x^* = (x_1^*, x_2^*, \dots, x_p^*)$, i.e.

$$f_i(x_1^*, x_2^*, \dots, x_{i-1}^*, x_i, x_{i+1}^*, \dots, x_p^*) \le$$
$$\le f_i(x_1^*, x_2^*, \dots, x_{i-1}^*, x_i^*, x_{i+1}^*, \dots, x_p^*)$$
$$\forall x_i \in X_i,\ i = \overline{1,p}.$$

Let us show that $x^* = (x_1^*, x_2^*, \dots, x_p^*)$ is Pareto-Nash equilibria solution for multiobjective game $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$. Indeed, for every $x_i \in X_i$ we have

$$\sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1^*, x_2^*, \dots, x_{i-1}^*, x_i, x_{i+1}^*, \dots, x_p^*) =$$

$$= f_i(x_1^*, x_2^*, \dots, x_{i-1}^*, x_i, x_{i+1}^*, \dots, x_p^*) \leq$$
$$\leq f_i(x_1^*, x_2^*, \dots, x_{i-1}^*, x_i^*, x_{i+1}^*, \dots, x_p^*) =$$
$$= \sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1^*, x_2^*, \dots, x_{i-1}^*, x_i^*, x_{i+1}^*, \dots, x_p^*)$$
$$\forall x_i \in X_i, \ i = \overline{1,p}.$$

So,

$$\sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1^*, x_2^*, \dots, x_{i-1}^*, x_i, x_{i+1}^*, \dots, x_p^*) \leq$$
$$\leq \sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1^*, x_2^*, \dots, x_{i-1}^*, x_i^*, x_{i+1}^*, \dots, x_p^*), \tag{3}$$
$$\forall x_i \in X_i, \ i = \overline{1,p},$$

for given $\alpha_{11}, \alpha_{12}, \dots, \alpha_{1r_1}, \alpha_{21}, \alpha_{22}, \dots, \alpha_{2r_2}, \dots, \alpha_{p1}, \alpha_{p2}, \dots, \alpha_{pr_p}$, which satisfy (2).

Taking in account that the functions $f_{x^*}^{i_k} = F_i^k(x_1^*, x_2^*, \dots, x_{i-1}^*, x_i, x_{i+1}^*, \dots, x_p^*)$, $k = \overline{1, r_i}$, are concave functions with respect to x_i on convex set X_i and $\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ik}$ satisfy the condition $\sum\limits_{k=1}^{r_i} \alpha_{ik} = 1$, $\alpha_{ik} > 0$, $k = \overline{1, r_i}$, then according to Theorem 1 from [6] (see also [7–9]) the condition (3) implies that x_i^* is Pareto solution for the following multicriterion problem:

$$\max_{x_i \in X_i} \to \overline{f}^i_{x^*}(x_i) = (f_{x^*}^{i1}(x_i), f_{x^*}^{i2}(x_i), \dots, f_{x^*}^{ir_i}(x_i)), \ i \in \{1, 2, \dots, p\}.$$

This means that $x^* = (x_1^*, x_2^*, \dots, x_p^*)$ is Pareto-Nash equilibria solution for multiobjective game $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$. □

So, if conditions of Theorem 1 are satisfied then Pareto-Nash equilibria solution for multiobjective game can be found by using the following algorithm.

Algorithm 1

1. Fix an arbitrary set of real numbers $\alpha_{11}, \alpha_{12}, \dots, \alpha_{1r_1}, \alpha_{21}, \alpha_{22}, \dots, \alpha_{2r_2}, \dots, \alpha_{p1}, \alpha_{p2}, \dots, \alpha_{pr_p}$, which satisfy condition (2);

2. Form the single objective game $G = (X_1, X_2, \dots, X_p, f_1, f_2, \dots, f_p)$, where

$$f_i(x_1, x_2, \dots, x_p) = \sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1, x_2, \dots, x_p), \ i = \overline{1,p};$$

3. Find Nash equilibria $x^* = (x_1^*, x_2^*, \dots, x_p^*)$ for noncooperative game $G = (X_1, X_2, \dots, X_p, f_1, f_2, \dots, f_p)$ and fix x^* as a Pareto-Nash equilibria solution for multiobjective game $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$.

Remark 1. Algorithm 1 finds only one of the solutions for multiobjective game $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$. In order to find all solutions in Pareto-Nash sense it is necessary to apply algorithm 1 for every $\alpha_{11}, \alpha_{12}, \dots, \alpha_{1r_1}, \alpha_{21}, \alpha_{22}, \dots, \alpha_{2r_2}, \dots, \alpha_{p1}, \alpha_{p2}, \dots, \alpha_{pr_p}$ which satisfy (2) and then to form the union of all obtained solutions.

Note that the proof of Theorem 1 is based on reduction the multiobjective game $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F_1}, \overline{F_2}, \dots, \overline{F_p})$ to the auxiliary one $G = (X_1, X_2, \dots, X_p, f_1, f_2, \dots, f_p)$ for which Nash theorem from [2] can be applied. In order to reduce multiobjective game $\overline{G}$ to auxiliary one G linear convolution criteria for vector payoff functions in the proof of Theorem 1 have been used. Perhaps similar reduction of the multiobjective game to classical one can be used also applying other convolution procedures for vector payoff functions of players, as example the standard procedure for multicriterion problem from [6–9].

For zero-sum multiobjective game of two players the following theorem holds.

Theorem 2. *Let $\overline{G} = (X_1, X_2, \overline{F})$ be a zero-sum multiobjective game of two players, where X_1, X_2 are convex compact sets and $\overline{F}(x_1, x_2)$ is a continuous vector function on $X_1 \times X_2$. Moreover, let us assume that each component $F^k(x_1, x_2)$, $k \in \{1, 2, \dots, r\}$, of $\overline{F}(x_1, x_2)$ for fixed $x_1 \in X_1$ represents a convex function with respect to x_2 on X_2 and for every fixed $x_2 \in X_2$ it is a concave function with respect to x_1 on X_1. Then for zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$ there exists saddle point $x^* = (x_1^*, x_2^*) \in X_1 \times X_2$, i.e. x_1^* is Pareto solution for multicriterion problem:*

$$\max_{x_1 \in X_1} \to \overline{F}(x_1, x_2^*) = (F^1(x_1, x_2^*), F^2(x_1, x_2^*), \dots, F^r(x_1, x_2^*))$$

and x_2^* is Pareto solution for multicriterion problem:

$$\min_{x_2 \in X_2} \to \overline{F}(x_1^*, x_2) = (F^1(x_1^*, x_2), F^2(x_1^*, x_2), \dots, F^r(x_1^*, x_2)).$$

Proof. The proof of Theorem 2 can be obtained as a corollary from Theorem 1 if we will regard our zero-sum game as a game of two players of the form $\overline{G} = (X_1, X_2, \overline{F}_1(x_1, x_2), \overline{F}_2(x_1, x_2)$, where $\overline{F}_2(x_1, x_2) = -\overline{F}_1(x_1, x_2) = \overline{F}(x_1, x_2)$.

The proof of Theorem 2 can be obtained also by reducing our zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$ to classical single objective case $G = (X_1, X_2, f)$ and applying J. Neumann theorem from [1], where

$$f(x_1, x_2) = \sum_{k=1}^{r} \alpha_k F^k(x_1, x_2)$$

and $\alpha_1, \alpha_2, \dots, \alpha_r$ are arbitrary real numbers such that

$$\sum_{k=1}^{r} \alpha_k = 1; \ \alpha_k > 0, \ k = \overline{1, r}.$$

It is easy to show that if $x^* = (x_1^*, x_2^*)$ is a saddle point for zero-sum game $G = (X_1, X_2, f)$ then $x^* = (x_1^*, x_2^*)$ represents a saddle point for zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$. □

So, if conditions of Theorem 2 are satisfied then a solution of zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$ can be found by using the following algorithm.

Algorithm 2

1. Fix an arbitrary set of real numbers $\alpha_1, \alpha_2, \dots, \alpha_r$ such that $\sum_{k=1}^{r} \alpha_k = 1$; $\alpha_k > 0,\ k = \overline{1,r}$;

2. Form the zero-sum game $G = (X_1, X_2, f)$, where $f(x_1, x_2) = \sum_{k=1}^{r} \alpha_k F^k(x_1, x_2)$.

3. Find a saddle point $x^* = (x_1^*, x_2^*)$ for single zero-sum game $G = (X_1, X_2, f)$. Then fix $x^* = (x_1^*, x_2^*)$ as a saddle point for zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$.

Remark 2. Algorithm 2 finds only a solution for given zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$. In order to find all saddle points it is necessary to apply algorithm 2 for every $\alpha_1, \alpha_2, \dots, \alpha_r$ satisfying conditions $\sum_{k=1}^{r} \alpha_k = 1$; $\alpha_k > 0$, $k = \overline{1,r}$, and then to form the union of obtained solutions.

Note that for reducing the zero-sum multiobjective games to classical ones also can be used other convolution criteria for vector payoff functions, i.e. the standard procedure from [7–9].

3 Discrete and matrix multiobjective games

Discrete multiobjective games are determined by the discrete structure of sets of strategies $X_1, X_2, \dots, X_p$. If $X_1, X_2, \dots, X_p$ are finite sets then we may consider $X_i = J_i,\ J_i = \{1, 2, \dots, q_i\},\ i = \overline{1,p}$. In this case the multiobjective game is determined by vectors

$$\overline{F}_i = (F_i^1, F_i^2, \dots, F_i^{r_i}),\ i = \overline{1,p},$$

where each component $F_i^k,\ k = \overline{1,r_i}$, represents p-dimensional matrix of size $q_1 \times q_2 \times \dots \times q_p$.

If $p = 2$ then we have bimatrix multiobjective game and if $F_2 = -F_1$ then we obtain matrix multiobjective one. In analogous way as for single objective matrix games here we can interpret the strategies $j_i \in J_i,\ i = \overline{1,p}$, of players as pure strategies.

It is evident that for such matrix multiobjective games Pareto-Nash equilibria may not exist because Nash equilibria may not exist for bimatrix and matrix games in pure strategies. But to each finite discrete multiobjective game we can associate a continuous multiobjective game $\overline{\overline{G}} = (Y_1, Y_2, \ldots, Y_p, \overline{f}_1, \overline{f}_2, \ldots, \overline{f}_p)$ by introducing mixed strategies $y_i = (y_{i1}, y_{i2}, \ldots, y_{ir_i}) \in Y_i$ of player i and vector payoff functions $\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_p$, which we define in the following way:

$$Y_i = \{y_i = (y_{i1}, y_{i2}, \ldots, y_{ir_i}) \in R^{r_i} \Big| \sum_{j=1}^{r_i} y_{ij} = 1, y_{ij} \geq 0, j = \overline{1, r_i}\};$$

$$\overline{f}_i = (f_i^1, f_i^2, \ldots, f_i^{r_i}),$$

where

$$f_i^k(y_{11}, y_{12}, \ldots, y_{1r_1}, y_{21}, y_{22}, \ldots, y_{2r_2}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pr_p}) =$$
$$= \sum_{j_1=1}^{r_1} \sum_{j_2=1}^{r_2} \cdots \sum_{j_p=1}^{r_p} F^k(j_1, j_2, \ldots, j_p) y_{ij_1} y_{ij_2} \ldots y_{ij_p}; \quad k = \overline{1, r_i}, \ i = \overline{1, p}.$$

It is easy to observe that for auxiliary multiobjective game $\overline{\overline{G}} = (Y_1, Y_2, \ldots, Y_p, \overline{f}_1, \overline{f}_2, \ldots, \overline{f}_p)$ conditions of theorem 1 are satisfied and therefore Pareto-Nash equilibria $y^* = (y^*_{11}, y^*_{12}, \ldots, y^*_{1r_1}, y^*_{21}, y^*_{22}, \ldots, y^*_{2r_2}, \ldots, y^*_{p1}, y^*_{p2}, \ldots, y^*_{pr_p})$ exist.

In the case of matrix games the auxiliary zero-sum multiobjective game of two players is defined as follows: $\overline{\overline{G}} = (Y_1, Y_2, \overline{f})$;

$$Y_1 = \{y_1 = (y_{11}, y_{12}, \ldots, y_{1r}) \in R^r \Big| \sum_{j=1}^{r} y_{1j} = 1, y_{1j} \geq 0, j = \overline{1, r}\};$$
$$Y_2 = \{y_2 = (y_{21}, y_{22}, \ldots, y_{2r}) \in R^r \Big| \sum_{j=1}^{r} y_{2j} = 1, y_{2j} \geq 0, j = \overline{1, r}\};$$

$$\overline{f} = (f^1, f^2, \ldots, f^r),$$
$$f^k(y_{11}, y_{12}, \ldots, y_{1r}, y_{21}, y_{22}, \ldots, y_{2r}) = \sum_{j_1=1}^{r} \sum_{j_2=1}^{r} F^k(j_1, j_2) y_{1j_1} y_{2j_2};$$
$$k = \overline{1, r}.$$

The game $\overline{\overline{G}} = (Y_1, Y_2, \overline{f})$ satisfies conditions of theorem 2 and therefore a saddle point $y^* = (y_1^*, y_2^*) \in Y_1 \times Y_2$ exists.

So, the results related to discrete and matrix game can be extended for multiobjective case of the game and can be interpreted in analogous way as for single objective games. In order to solve these associated multiobjective games algorithms 1 and 2 can be applied.

4 Conclusion

The considered multiobjective games extend classical ones and represent a combination of cooperative and noncooperative games. Indeed, the player i in multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$ can be regarded as a union of r_i subplayers with payoff functions $F_i^1, F_i^2, \dots, F_i^{r_i}$ respectively. So, the game $\overline{G}$ represents a game with p coalitions $1, 2, \dots, p$ which interact between them on the set of situations $X_1 \times X_2 \times \cdots \times X_p$.

The introduced Pareto-Nash equilibria notion uses the concept of cooperative games because according to this notion subplayers of the same coalitions should optimize in the sense of Pareto their vector functions F_i on set of strategies X_i. On the other hand Pareto-Nash equilibria notion takes into account also the concept of noncooperative games because coalitions interact between them on the set of situations $X_1 \times X_2 \times \cdots \times X_p$ and are interested to preserve Nash equilibria between coalitions.

The obtained results allow us to describe a class of multiobjective games for which Pareto-Nash equilibria exists. Moreover, a suitable algorithm for finding Pareto-Nash equilibria is proposed.

References

[1] Neumann J., Morgenstern O. *Theory of games and economic behaviour.* Princeton, Princeton Univ. Press., 1953.

[2] Nash j. *Non-cooperative games.* Ann. Math., 1951, **54**, p. 286–295.

[3] Moulin H. *Theorie des Jeux pour l'economie et la politique.* Paris, 1981.

[4] Pareto V. *Manuel d'economie politique.* Giard, Paris, 1904.

[5] Podinovsky V,, Noghin V. *Pareto optimal solution for multicriterion problem.* Moscow, Nauka, 1982.

[6] Emelichev V., Perepelitsa V. *The complexity of discrete multicriterion problems.* Discrete mathematics and applications, 1994, N 6(1), p. 5–33.

[7] Brucher P. *Discrete parameter optimization problem and essential efficient points.* Operat. Res., 1972, N 16(5), p. 189–197.

[8] Burkard R., Krarup J., Pruzan P. *Efficiency and optimality in minisum, minimax* $0-1$ *progrmming problems.* J. Oper. Res. Soc., 1982, N 33(2), p. 137–151.

[9] Burkard R., Krarup J., Pruzan P. *A relationship between optimality and efficiency in multicriteria* $0-1$ *programming problems.* Comp. Oper. Res., 1981, N 8(2), p. 241–247.

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