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A loop transversal in a sharply 2-transitive permutation loop

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Abstract. The well-known theorem of M.Hall about the description of a finite sharply 2-transitive permutation group is generalized for the case of permutation loops. It is shown that the identity permutation with the set of all fixed-point-free permutations in a finite sharply 2-transitive permutation loop forms a loop transversal by its proper subloop – a stabilizator of one symbol.

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1 Introduction

In the theory of finite multiply transitive permutation groups the following M. Hall's theorem is well-known.

Theorem 1. *Let G be a sharply 2-transitive permutation group on a finite set of symbols E, i.e.*

1. *G is a 2-transitive permutation group on E;*
2. *only the identity permutation id fixes two symbols from the set E.*

Then

1. *the identity permutation id together with the set of all fixed-point-free permutations from the group G forms a transitive invariant subgroup A in the group G;*
2. *the group G is isomorphic to the group of linear transformations*

$$G_K = \{\alpha \,|\, \alpha(x) = x \cdot a + b, \quad a, b \in E, \quad a \neq 0\}$$

of some near-field $K = \langle E, +, \cdot, 0, 1\rangle$.

In the articles [11,12,14] the notion of a permutation loop on some set of symbols E is defined. Both for permutation groups, and for permutation loops the notions of transitivity, multiple transitivity and sharply multiple transitivity can be defined

[11,12,14]. The studying of a sharply 2-transitive permutation loop of permutations is the most interesting, because (see [6]) there exists a 1-1 correspondence between every finite projective plane and some sharply 2-transitive permutation loop.

Using the notion of a transversal in a loop to its subloop (see [11, 13]), the author of the present article proves a generalization of Hall's Theorem for the case of a sharply 2-transitive permutation loop.

Theorem 2. *Let L be a sharply 2-transitive permutation loop on a finite set of symbols E, i.e.*

1. *L is a 2-transitive set of permutations on the finite set of symbols E;*
2. *permutations from the set L form a loop by some operation "$\cdot$";*
3. *only the identity permutation id fixes two symbols from the set E.*

Then

1. *the identity permutation id together with the set of all fixed-point-free permutations from the loop L forms a transitive loop transversal A in the loop L to its proper subloop R_a, where R_a is a loop of all permutations from the loop L which fix some symbol $a \in E$;*
2. *this loop transversal A is a unique loop transversal in the loop L to its proper subloop R_a, i.e. any other loop transversal T in the loop L to its proper subloop R_a coincide with the transversal T.*

Let us give some necessary notations and prove some basic statements.

2 Necessary definitions and notations

Definition 1. *A system $\langle E, \cdot\rangle$ is called [2, 5] a* ***right (left) quasigroup*** *if for arbitrary $a, b \in E$ the equation $x \cdot a = b$ ($a \cdot y = b$) has a unique solution in the set E. If a system $\langle E, \cdot\rangle$ is both a right and left quasigroup, then it is called a* ***quasigroup****. If in a right (left) quasigroup $\langle E, \cdot\rangle$ there exists an element $e \in E$ such that*

$$x \cdot e = e \cdot x = e,$$

for any $x \in E$, then the system $\langle E, \cdot\rangle$ is called a ***right (left) loop*** *(the element e is called a* ***unit*** *or* ***identity element****). If a system $\langle E, \cdot\rangle$ is both a right and left loop, then it is called a* ***loop****.*

Definition 2. *Let G be a group and H be a subgroup in G. A complete system $T = \{t_i\}_{i\in E}$ of representatives of the left (right) cosets of H in G ($e = t_1 \in H$) is called [1] a* ***left (right) transversal in*** *G* ***to*** *H.*

Let $T = \{t_x\}_{x \in E}$ be a left transversal in G to H. We can define correctly (see [1,6]) the following operation (**transversal operation**) on the set E (E is an index set; left cosets of H in G are numbered by indexes from E):

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = t_z h, \quad h \in H. \tag{1}$$

In [5] it was proved that the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left loop with the unit 1.

Definition 3. *Let T be a left transversal in G to H. If the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a loop, then T is called a **left loop** (or simply "**loop**") **transversal** in G to H.*

3 A transversal in a loop to its subloop

The author of the present article generalized in [10,11] the well-known (in group theory) notion of a transversal in a group to its proper subgroup. Also the analogous generalization is studied in [3].

At the beginning let us define a partition of a loop by left (right) cosets to its proper subloop.

Definition 4. *Let $\langle L, \cdot\rangle$ be a loop and $\langle R, \cdot\rangle$ be its proper subloop. Then [13] a **left coset** of R is a set of the form*

$$xR = \{xr \mid r \in R\},$$

*and a **right coset** has the form*

$$Rx = \{rx \mid r \in R\}.$$

The cosets of a subloop do not necessarily form a partition of the loop. This leads to the following definition.

Definition 5. *A loop L has a **left (right) coset decomposition by its proper subloop** R [13], if the left (right) cosets form a partition of the loop L, i.e. for some set of indexes E*

1. $\bigcup\limits_{i \in E} (a_i R) = L$;
2. *for every $i, j \in E$, $i \neq j$*

$$(a_i R) \cap (a_j R) = \varnothing.$$

Lemma 1. *The following conditions are equivalent:*

1. *a loop L has a left coset decomposition by its proper subloop R;*

2. *the following condition take place (it can be named a* ***weak left Condition A****, see below): for every $a \in L$*

$$(aR)R = aR. \tag{2}$$

Proof. See in [13], Theorem I.2.12.

In order to define correctly the notion of a left (right) transversal in a loop to its proper subloop, it is necessary that the following condition be fulfilled.

Definition 6. *(**Left Condition A**) The multiplication to the left of an arbitrary element a of the loop L by an arbitrary left coset in the loop L to its proper subloop R is a left coset in the loop L to its proper subloop R too, i.e. for every $a, b \in L$ there exists an element $c \in L$ such that*

$$a(bR) = cR. \tag{3}$$

The **right Condition A** is defined analogously.

Lemma 2. *The following conditions are equivalent:*

1. *a left Condition A is fulfilled in the loop L to its proper subloop R;*

2. *for every $a, b \in L$*

$$a(bR) = (ab)R. \tag{4}$$

Proof. See in [11].

Remark 1. The condition (4) is called in [3] a **strong left coset decomposition of the loop L by its proper subloop R**. Also we can say that the subloop R is a **left invariant** subloop in the loop L.

Definition 7. *(See also [3]) Let $\langle L, \cdot, e\rangle$ be a loop and $\langle R, \cdot, e\rangle$ be its proper subloop. Let a left Condition A be fulfilled in the loop L to its proper subloop R. Then the loop L has a left coset decomposition by its proper subloop R. A* ***left transversal*** *$T = \{t_x\}_{x\in E}$ in the loop L to its proper subloop R is a set of representatives, one from each left coset; moreover, $t_1 = e \in R$.*

A right transversal $T = \{t_x\}_{x\in E}$ in the loop L to its proper subloop R is defined analogously.

Remark 2. If in the last definition we eliminate the condition $t_1 = e \in R$, then we obtain a definition of a **non-reduced left transversal** $T = \{t_x\}_{x\in E}$ in the loop L to its proper subloop R.

Let $T = \{t_x\}_{x \in E}$ be a left transversal in a loop L to its proper subloop R. We can define correctly the following operation (**transversal operation**) on the set E:

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x \cdot t_y = t_z \cdot r, \quad r \in R, \tag{5}$$

where $t_x, t_y, t_z \in T,\ r \in R$. In [11] it is proved that the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left loop with the unit 1.

Definition 8. *Let T be a left transversal in a loop L to its proper subloop R. If the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a loop, then T is called a **left loop** (or simply "**loop**") **transversal** in the loop L to its proper subloop R.*

4 Finite projective planes, DK-ternars and loop transversals in the group S_n to $St_{a,b}(S_n)$

Let us remember the basic facts from the theory of finite projective planes and their coordinatization (see [7]).

Definition 9. *The **projective plane** of order n is the incidence structure $\langle P, L, I\rangle$ which satisfies the following axioms:*

1. *Given any two distinct points from P there exists just one line from L incident with both of them;*

2. *Given any two distinct lines from L there exists just one point from P incident with both of them;*

3. *There exist four points such that a line incident with any two of them is not incident with either of the remaining two.*

4. *There exists a line in L which consists of exactly $n+1$ points.*

Definition 10. *A system $\langle E, (x,t,y), 0, 1\rangle$ is called [7] a **DK-ternar** (i.e. a set E with ternary operation (x,t,y) and distinguished elements $0, 1 \in E$) if the following conditions hold:*

1. $(x, 0, y) = x$,

2. $(x, 1, y) = y$,

3. $(x, t, x) = x$,

4. $(0, t, 1) = t$,

5. *if a, b, c, d are arbitrary elements from E and $a \neq b$, then the system*

$$\begin{cases} (x, a, y) = c \\ (x, b, y) = d \end{cases}$$

has an unique solution in $E \times E$.

Definition 11. *A set M of permutations on a set X is called [4]* ***sharply 2-transitive*** *if for any two pairs (a, b) and (c, d) of different elements from X there exists an unique permutation $\alpha \in M$ satisfying the following conditions:*

$$\alpha(a) = c, \qquad \alpha(b) = d.$$

Lemma 3. *Let π be an arbitrary finite projective plane. We can introduce on the plane π the coordinates $(a, b), (m), (\infty)$ for points and $[a, b], [m], [\infty]$ for lines (where the set E is a finite set with the distinguished elements $0, 1$ and $a, b, m \in E$) such that if we define a ternary operation (x, t, y) on the set E by the formula*

$$(x, t, y) = z \quad \stackrel{def}{\Longleftrightarrow} \quad (x, y) \in [t, z],$$

then the system $\langle E, (x, t, y), 0, 1\rangle$ be a DK-ternar.

Proof. See Lemma 1 in [7].

Now let a system $\langle E, (x, t, y), 0, 1\rangle$ be a DK-ternar. Let us define the following binary operation (x, ∞, y) on the set E:

$$(x, \infty, 0) \stackrel{def}{=} x,$$

$$\begin{cases} (x, \infty, y) = u \\ (x, y) \neq (u, 0) \end{cases} \quad \stackrel{def}{\Longleftrightarrow} \quad \begin{matrix} (x, t, y) \neq (u, t, 0) \\ \forall t \in E. \end{matrix}$$

Lemma 4. *Operation (x, ∞, y) satisfies the following conditions:*

1. $\begin{cases} (x, \infty, y) = (u, \infty, v) \\ (x, y) \neq (u, v) \end{cases} \quad \Longleftrightarrow \quad \begin{matrix} (x, t, y) \neq (u, t, v) \\ \forall t \in E. \end{matrix}$

2. $(x, \infty, x) = 0.$

3. *if a, b, c are arbitrary elements from E, then the system*

$$\begin{cases} (x, a, y) = b \\ (x, \infty, y) = c \end{cases}$$

has a unique solution in $E \times E$.

Proof. See Lemma 4 in [7].

Let $\langle E, (x,t,y), 0, 1\rangle$ be a finite DK-ternar. Let us introduce points $(a,b), (m), (\infty)$ and lines $[a,b], [m], [\infty]$ (where $a,b,m \in E$) and define the following incidence relation I between points and lines:

$$\begin{array}{l} (a,b)\ I\ [c,d] \iff (a,c,b)=d,\\ (a,b)\ I\ [d] \iff (a,\infty,b)=d,\\ (a)\ I\ [c,d] \iff a=c,\\ (a)\ I\ [\infty], \qquad (\infty)\ I\ [d], \qquad (\infty)\ I\ [\infty],\\ (a,b)\ I\ [\infty] \iff (a)\ I\ [d] \iff\\ (\infty)\ I\ [c,d] \iff false. \end{array} \tag{6}$$

Lemma 5. *The incidence system $\langle X, L, I\rangle$, where*

$$X = \{(a,b), (m), (\infty) \mid a,b,m \in E\},$$
$$L = \{[a,b], [m], [\infty] \mid a,b,m \in E\},$$
$$I \text{ is the incidence relation, defined above in (6),}$$

is a projective plane.

Proof. See Lemma 5 in [7].

Lemma 6. *(**Cell permutations**) Let the system $\langle E, (x,t,y), 0, 1\rangle$ be a finite DK-ternar. Let a,b be arbitrary elements from E and $a \neq b$. Then every unary operation $\alpha_{a,b}(t) = (a,t,b)$ is a permutation on the set E.*

Proof. See Lemma 6 in [7].

Lemma 7. *Cell permutations $\{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ of the finite DK-ternar $\langle E, (x,t,y), 0, 1\rangle$ satisfy the following conditions:*

1. *All cell permutations are distinct;*
2. *The set M of all cell permutations is sharply 2-transitive on the set E;*
3. *A permutation $\alpha_{a,b}$ is a fixed-point-free cell permutation on the set E iff the following condition holds*

$$(a,\infty,b) = (0,\infty,1).$$

4. *There exists the fixed-point-free permutation ν_0 on the set E such that we can represent the set A of all fixed-point-free cell permutations together with the identity cell permutation $\alpha_{0,1}$ in the following form:*

$$A = \{\alpha_{a,b} \mid b = \nu_0(a), \quad a \in E\} = \{\alpha_{a,\nu_0(a)}\}_{a\in E}.$$

Proof. See Lemma 7 in [7].

Lemma 8. *Let $M = \{\alpha_{a,b}\}_{a,b \in E,\ a \neq b}$ be a set of permutations on the set E (E is a finite set with distinguished elements 0 and 1), and the following conditions hold:*

1. $\alpha_{0,1} = id$.
2. $\alpha_{a,b}(0) = a,\ \alpha_{a,b}(1) = b$.
3. *The set M is a sharply 2-transitive set of permutations on E.*

   Let us suppose by definition:

$$(x,t,y) \stackrel{def}{=} \alpha_{x,y}(t) \quad if \quad x \neq y,$$
$$(x,t,x) \stackrel{def}{=} x.$$

Then the system $\langle E, (x,t,y), 0, 1\rangle$ is a finite DK-ternar.

Proof. See Lemma 8 in [7].

Next theorem shows a connection between finite sharply 2-transitive sets of permutations and loop transversals in the symmetric group S_n.

Theorem 3. *Let E be a finite set and $\operatorname{card} M = n$. Then the following conditions are equivalent:*

1. *A set T of permutations of degree n is a sharply 2-transitive set of permutations on the set E and $id \in T$.*
2. *A set T of permutations of degree n is a loop transversal in S_n to $St_{a,b}(S_n)$ (where a,b are arbitrary fixed elements from E and $a \neq b$).*
3. *A system $\langle E \times E - \{\triangle\}, \stackrel{(T)}{\cdot}, \langle a,b\rangle\rangle$ is a sharply 2-transitive permutation loop of degree n (a definition of permutation loop see in [11, 12, 14]).*

Proof. See Theorem 1 in [6].

Lemma 9. *Let $T_{a,b} = \{\alpha_{x,y}\}_{x,y \in E,\ x \neq y}$ be a loop transversal in S_n to $St_{a,b}(S_n)$ (where a,b are arbitrary fixed elements from E and $a \neq b$). Let a system $\langle E \times E - \{\triangle\}, \stackrel{(T_{a,b})}{\cdot}, \langle a,b\rangle\rangle$ be a loop transversal operation corresponding to the transversal $T_{a,b}$. Then*

$$\langle x,y\rangle \stackrel{(T_{a,b})}{\cdot} \langle u,v\rangle = \langle \alpha_{x,y}(u), \alpha_{x,y}(v)\rangle. \tag{7}$$

Proof. See Lemma 10 in [7].

5 A loop transversal in a sharply 2-transitive permutation loop

As it is shown above, there exist a 1-1 correspondences between

- a finite projective plane π of order n;
- a finite DK-ternar $\langle E, (x,t,y), 0, 1\rangle$ which gives a coordinatization of the projective plane π;
- a sharply 2-transitive permutation loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ of cell permutations of the DK-ternar $\langle E, (x,t,y), 0, 1\rangle$;
- a loop transversal $T_{a,b} = \{\alpha_{x,y}\}_{x,y\in E,\ x\neq y}$ in the symmetric group S_n to $St_{a,b}(S_n)$ (where a, b are arbitrary fixed elements from E and $a \neq b$);
- a loop transversal operation $\langle E\times E - \{\triangle\}, \overset{(T_{a,b})}{\cdot}, \langle a,b\rangle\rangle$ corresponding to the transversal $T_{a,b}$ (in [7] this loop is called a **loop of pairs** of the DK-ternar $\langle E, (x,t,y), 0, 1\rangle$).

Below for simplicity we shall consider that $\langle a,b\rangle = \langle 0,1\rangle$.

Lemma 10. *The set*

$$H_0^* = \{\langle 0,a\rangle \mid a \in E - \{0\}\}$$

forms a subloop in the loop of pairs $L^* = \langle E\times E - \{\triangle\}, \overset{(T_{0,1})}{\cdot}, \langle a,b\rangle\rangle$.

Proof. See Lemma 11 in [7].

Lemma 11. *A left Condition A is fulfilled for the loop of pairs L^* to its proper subloop H_0^*.*

Proof. Let us have

$$\begin{aligned} a_0 &= \langle a,b\rangle \in L, \qquad b_0 = \langle c,d\rangle \in L,\\ x &= \langle 0,u\rangle \in H_0^*, \quad\ y = \langle 0,v\rangle \in H_0^*, \end{aligned}$$

where $a,b,c,d \in E,\quad a\neq b,\quad c\neq d,\quad u,v \in E-\{0\}$. According to (7), we obtain

$$\begin{aligned} a_0 \overset{(T_{0,1})}{\cdot} (b_0 \overset{(T_{0,1})}{\cdot} x) &= \langle a,b\rangle \overset{(T_{0,1})}{\cdot} (\langle c,d\rangle \overset{(T_{0,1})}{\cdot} \langle 0,u\rangle) = \langle a,b\rangle \overset{(T_{0,1})}{\cdot} \langle \alpha_{c,d}(0), \alpha_{c,d}(u)\rangle =\\ &= \langle a,b\rangle \overset{(T_{0,1})}{\cdot} \langle c, \alpha_{c,d}(u)\rangle = \langle \alpha_{a,b}(c), \alpha_{a,b}\alpha_{c,d}(u)\rangle, \end{aligned}$$

since $\alpha_{x,y}(0) = x$ (see Lemma 8). By the analogous way we obtain

$$\begin{aligned} (a_0 \overset{(T_{0,1})}{\cdot} b_0) \overset{(T_{0,1})}{\cdot} y &= (\langle a,b\rangle \overset{(T_{0,1})}{\cdot} \langle c,d\rangle) \overset{(T_{0,1})}{\cdot} \langle 0,v\rangle = \langle \alpha_{a,b}(c), \alpha_{a,b}(d)\rangle \overset{(T_{0,1})}{\cdot} \langle 0,v\rangle =\\ &= \langle \alpha_{a,b}(c), \alpha_{\alpha_{a,b}(c),\alpha_{a,b}(d)}(v)\rangle. \end{aligned}$$

Because the function $\alpha_{a,b}(t)$ is a permutation on the set E, then for every $u \in E-\{0\}$ there exists $u \in E - \{0\}$ such that

$$\alpha_{a,b}\alpha_{c,d}(u) = \alpha_{\alpha_{a,b}(c),\alpha_{a,b}(d)}(v);$$

really

$$v = \alpha^{-1}_{\alpha_{a,b}(c),\alpha_{a,b}(d)}\alpha_{a,b}\alpha_{c,d}(u).$$

Let us note that

$$\alpha_{a,b}\alpha_{c,d}(0) = \alpha_{a,b}(c) = \alpha_{\alpha_{a,b}(c),\alpha_{a,b}(d)}(0).$$

Finally we obtain that for every $x \in H_0^*$ there exists $y \in H_0^*$ such that

$$a_0 \overset{(T_{0,1})}{\cdot} (b_0 \overset{(T_{0,1})}{\cdot} x) = (a_0 \overset{(T_{0,1})}{\cdot} b_0) \overset{(T_{0,1})}{\cdot} y$$

for every $a_0, b_0 \in L$. A left Condition A is fulfilled for the loop of pairs L^* to its proper subloop H_0^*. □

According to the last Lemma we obtain that the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ of cell permutations has a strong left coset decomposition by its proper subloop $H_0 = \{\alpha_{0,a} \,|\, a \in E - \{0\}\}$. So it is possible to define and investigate a left or right transversals in the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ to its proper subloop H_0.

Let us study the set $A = \{\alpha_{a,\nu(a)}\}_{a\in E} \subset L$ of all fixed-point-free permutations and the identity permutation (see Lemma 8).

Lemma 12. *The set $A = \{\alpha_{a,\nu(a)}\}_{a\in E}$ is a loop transversal in the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ to its proper subloop H_0.*

Proof. Let us study left cosets $(\alpha_{a,b} \overset{(T_{0,1})}{\cdot} H_0)$ in the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ to its subloop H_0. We have

$$\begin{aligned} \alpha_{c,d} &\in \alpha_{a,b} \overset{(T_{0,1})}{\cdot} H_0, \\ \alpha_{c,d} &= \alpha_{a,b} \overset{(T_{0,1})}{\cdot} \alpha_{0,u} \end{aligned}$$

for some $u \in E - \{0\}$. Then we obtain

$$\begin{cases} c = \alpha_{a,b}(0) = a, \\ d = \alpha_{a,b}(u) \neq a, \end{cases}$$

i.e.

$$\alpha_{a,b} \overset{(T_{0,1})}{\cdot} H_0 = \{\alpha_{a,v} \,|\, v \in E - \{a\}\}.$$

So for every $a \in E$ a left coset $H_a = (\alpha_{a,b} \overset{(T_{0,1})}{\cdot} H_0)$ in the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ to its subloop H_0 is a set of all permutations φ from L such that $\varphi(0) = a$.

Let us study the set $A = \{\alpha_{a,\nu(a)}\}_{a\in E}$ from the Lemma's condition. If $a = 0$ then

$$\alpha_{0,\nu(0)} = \alpha_{0,1} = id \in A \cap H_0,$$

i.e. the unit id of the loop L belongs to the set A. Further,

$$\alpha_{a,\nu(a)}(0) = a \quad \Rightarrow \quad \alpha_{a,\nu(a)} \in H_a,$$

i.e. for every $a \in E$ it is true that

$$A \cap H_a = \{\alpha_{a,\nu(a)}\}.$$

Then the set $A = \{\alpha_{a,\nu(a)}\}_{a\in E}$ is a left transversal in the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ to its proper subloop H_0.

Finally, let us consider a transversal operation $\langle E, \overset{(A)}{\cdot}, 1\rangle$ corresponding to the transversal A:

$$x \overset{(A)}{\cdot} y = z \quad \Leftrightarrow \quad \alpha_{x,\nu(x)} \overset{(T_{0,1})}{\cdot} \alpha_{y,\nu(y)} = \alpha_{z,\nu(z)} \overset{(T_{0,1})}{\cdot} \alpha_{0,u}, \tag{8}$$

where $\alpha_{0,u} \in H_0$. According to [11], the system $\langle E, \overset{(A)}{\cdot}, 1\rangle$ is a left loop with the unit 1. It is sufficient to prove that the system $\langle E, \overset{(A)}{\cdot}, 1\rangle$ is a right loop with the same unit 1 too. So let us study for every $a, b \in E$ the equation $x \overset{(A)}{\cdot} a = b$. According (8), we have

$$\begin{aligned} x \overset{(A)}{\cdot} a &= b, \\ \alpha_{x,\nu(x)} \overset{(T_{0,1})}{\cdot} \alpha_{a,\nu(a)} &= \alpha_{b,\nu(b)} \overset{(T_{0,1})}{\cdot} \alpha_{0,u}, \end{aligned}$$

where $u \in E - \{0\}$. It is equivalent to the following system

$$\begin{cases} \alpha_{x,\nu(x)}(a) = \alpha_{b,\nu(b)}(0) = b, \\ \alpha_{x,\nu(x)}(\nu(a)) = \alpha_{b,\nu(b)}(u). \end{cases}$$

It is easy to see that it is sufficient to show, that for every $a, b \in E$ there exists a unique permutation $\gamma \in A$ such that $\gamma(a) = b$. If $a = b$, then $\gamma = id = \alpha_{0,1}$. Let $a \neq b$; then according to Lemma 4 we obtain:

$$\begin{cases} \alpha_{x,\nu(x)}(a) = b, \\ \alpha_{x,\nu(x)} \text{ is a fixed-point-free permutation on the set } E, \end{cases}$$

$$\begin{cases} (x, a, \nu(x)) = b, \\ (x, t, \nu(x)) \neq t \quad \forall t \in E, \end{cases}$$

$$\begin{cases} (x, a, \nu(x)) = b, \\ (x, \infty, \nu(x)) = (0, \infty, 1). \end{cases}$$

According to Lemma 4 the last system has a unique solution in $E \times E$, i.e. there exists a unique such $\gamma = \alpha_{x,\nu(x)}$. $\square$

Lemma 13. *There exists a unique left loop transversal in the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ to its subloop H_0.*

Proof. According to the last Lemma there exists a such left loop transversal: the transversal $A = \{\alpha_{a,\nu(a)}\}_{a\in E}$ of all fixed-point-free permutations and the identity permutation. Let us prove that the transversal $A = \{\alpha_{a,\nu(a)}\}_{a\in E}$ is a unique left loop transversal in the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ to its subloop H_0.

Let $T = \{t_x\}_{x\in E}$ be a left loop transversal in the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ to its subloop H_0. Because the set T is a left transversal in the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ to its subloop H_0, then

$$T = \{\alpha_{x,\delta(x)}\}_{x\in E},$$

where δ is some function on the set E; $\delta(x) \neq x$ for every $x \in E$. Moreover,

$$t_1 = \alpha_{0,\delta(0)} = id = \alpha_{0,1} \in H_0,$$

i.e. $\delta(0) = 1$.

Let us study a transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$, corresponding to the transversal T in the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ to its subloop H_0. According to the definition of transversal operation, we have:

$$x \overset{(T)}{\cdot} y = z \quad \Leftrightarrow \quad \alpha_{x,\delta(x)} \overset{(T_{0,1})}{\cdot} \alpha_{y,\delta(y)} = \alpha_{z,\delta(z)} \overset{(T_{0,1})}{\cdot} \alpha_{0,u},$$

where $\alpha_{0,u} \in H_0$. So we obtain the following system

$$\begin{cases} \alpha_{x,\delta(x)}(y) = \alpha_{z,\delta(z)}(0) = z, \\ \alpha_{x,\delta(x)}(\delta(y)) = \alpha_{z,\delta(z)}(u). \end{cases}$$

Since the transversal $T = \{t_x\}_{x\in E}$ is a left loop transversal in the loop $L = \{\alpha_{a,b}\}_{a,b\in E,\ a\neq b}$ to its subloop H_0, then for every $a, b \in E$ the equation $x \overset{(T)}{\cdot} a = b$ has a unique solution in the set E; i.e. for every $a, b \in E$ the equation

$$\alpha_{x,\delta(x)}(a) = b$$

has a unique solution in the set E. It means that if $x_1, x_2 \in E$ and $x_1 \neq x_2$, then must be

$$\alpha_{x_1,\delta(x_1)}(a) \neq \alpha_{x_2,\delta(x_2)}(a).$$

It is true for every $a \in E$, so we obtain for every $a \in E$ and $x_1, x_2 \in E$, $x_1 \neq x_2$:

$$\alpha_{x_1,\delta(x_1)}(a) \neq \alpha_{x_2,\delta(x_2)}(a),$$

i.e. for every $a \in E$ and $x_1, x_2 \in E$, $x_1 \neq x_2$:

$$(x_1, a, \delta(x_1)) \neq (x_2, a, \delta(x_2)).$$

According to Lemma 4, we obtain

$$(x_1, \infty, \delta(x_1)) = (x_2, \infty, \delta(x_2)). \tag{9}$$

It means that for any different elements $\alpha_{x_1,\delta(x_1)}$ and $\alpha_{x_2,\delta(x_2)}$ of the left loop transversal $T = \{\alpha_{x,\delta(x)}\}_{x\in E}$ the formula (9) holds. Moreover,

$$t_1 = \alpha_{0,\delta(0)} = \alpha_{0,1} \in T,$$

since for every $x \in E - \{0\}$ we have

$$(x, \infty, \delta(x)) = (0, \infty, \delta(0)) = (0, \infty, 1).$$

According to Lemma 8 (statements 2 and 3), we obtain that $\delta(x) = \nu(x)$ for every $x \in E$, i.e.

$$T = \{\alpha_{x,\delta(x)}\}_{x\in E} = \{\alpha_{x,\nu(x)}\}_{x\in E} = A.$$

$\square$

Corollary 1. *There exist exactly $n - 2$ different non-reduced left loop transversals in the loop L to its subloop H_0.*

Proof. The proof is analogous to the proof of the last Lemma till the moment, when we obtain the following identity for the non-reduced left loop transversal $T = \{\alpha_{x,\delta(x)}\}_{x\in E}$ in the loop L to its subloop H_0:

$$(x_1, \infty, \delta(x_1)) = (x_2, \infty, \delta(x_2)) \tag{10}$$

for every $x_1, x_2 \in E,\quad x_1 \neq x_2$. Since $T = \{\alpha_{x,\delta(x)}\}_{x\in E}$ is a *non-reduced* left loop transversal in the loop L to its subloop H_0, then $T \cap H_0 = \{\alpha_{0,u_0}\}$ for some $u_0 \in E - \{0, 1\}$. So we obtain from (10) for every $x \in E$

$$(x, \infty, \delta(x)) = (0, \infty, \delta(0)) = (0, \infty, u_0).$$

Since $u_0 \neq 0, 1$, then there exist exactly $n - 2$ such elements u_0 in the set E. So there exist exactly $n - 2$ different non-reduced left loop transversals in the loop L to its subloop H_0. $\square$

Remark 3. We can note a correlation between the left loop transversal A in the loop L to its subloop H_0 and points of the line $[(0, \infty, 1)]$ in the projective plane π:

$$\alpha_{x,\nu(x)} \in A \quad \Leftrightarrow \quad (x, \nu(x)) \in [(0, \infty, 1)].$$

There exists an analogous correlation between non-reduced left loop transversals in the loop L to its subloop H_0 and points of the lines $[d]$ $(d \neq 0)$ in the projective plane π:

$$\alpha_{x,\delta(x)} \in T_c \quad \Leftrightarrow \quad (x, \delta(x)) \in [(0, \infty, c)], \quad c \neq 0, 1.$$

Corollary 2. *The following condition is fulfilled for the loop $\langle E, \overset{(A)}{\cdot}, 0\rangle$ and permutation ν: for every $x \in E$*

$$\nu(x) = x \overset{(A)}{\cdot} 1.$$

Proof. According to formula (8) we have for the transversal operation $\langle E, \overset{(A)}{\cdot}, 0\rangle$:

$$x \overset{(A)}{\cdot} y = z \quad \Leftrightarrow \quad \alpha_{x,\nu(x)}(y) = z.$$

If $y = 1$ then we obtain

$$x \overset{(A)}{\cdot} 1 = z \quad \Leftrightarrow \quad \alpha_{x,\nu(x)}(1) = z \quad \Leftrightarrow \quad \nu(x) = z,$$

i.e.

$$\nu(x) = z = x \overset{(A)}{\cdot} 1.$$

□

References

[1] Baer R. *Nets and groups. 1.* Trans. Amer. Math. Soc., 1939, **46**, p. 110–141.

[2] Belousov V.D. *Foundations of quasigroup and loop theory.* Moscow, Nauka, 1967 (in Russian).

[3] Foguel T., Kappe L.C. *On loops covered by subloops.* Expositiones Mathematicae, 2005, **23**, p. 255–270.

[4] Hall M. *Group theory.* Moscow, IL, 1962 (in Russian).

[5] Kuznetsov E.A *Transversals in groups. 1. Elementary properties.* Quasigroups and related systems, 1994, **1**, N 1, p. 22–42.

[6] Kuznetsov E.A. *Sharply k-transitive sets of permutations and loop transversals in* S_n. Quasigroups and related systems, 1994, **1**, N 1, p. 43–50.

[7] Kuznetsov E.A. *About some algebraic systems related with projective planes.* Quasigroups and related systems, 1995, **2**, p. 6–33.

[8] Kuznetsov E.A. *Sharply 2-transitive permutation groups. 1.* Quasigroups and related systems, 1995, **2**, p. 83–100.

[9] Kuznetsov E.A. *Loop transversals in* S_n *by* $St_{a,b}(S_n)$ *and coordinatizations of projective planes.* Bulletin of AS of RM, Mathematics, 2001, N 2(36), p. 125–135.

[10] Kuznetsov E.A. *Transversals in loops: Abstracts of International Conference "Loops'03".* Prague, August 10-17, 2003, p. 18–20.

[11] Kuznetsov E.A. *Transversals in loops. 1. Elementary properties and structural theorems* (to appear).

[12] Kuznetsov E.A *Permutation loops* (to appear).

[13] Pflugfelder H. *Quasigroups and Loops: Introduction.* Sigma Series in Pure Math., 7, Helderman Verlag, New York, 1972.

[14] Bonetti F., Lunardon G., Strambach K. *Cappi di permutazionni.* Rend. Math., 1979, **12**, N 3-4, p. 383–395.

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