BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(48), 2005, Pages 99–103
ISSN 1024–7696

Strong Stability of Linear Symplectic Actions and the Orbit Method

Z. Rzeszótko

Abstract. Using the orbit method we give necessary and sufficient conditions for a linear symplectic action of the group R^m to be strongly stable. This criterion generalizes the respective one stated for linear Hamiltonian systems by Cushman and Kelly.

Mathematics subject classification: 37J25.

Keywords and phrases: Linear Poisson action, Hamiltonian polyoperator, joint spectrum, centralizer, strong stability, orbit method.

It is well known that necessary and sufficient condition for Lyapunov stability of linear autonomous differential system is purely imaginary spectrum and absence of Jordan's blocks. In virtue of semi-continuity and symmetry of the spectrum, a Hamiltonian matrix with purely imaginary simple spectrum satisfies even a more strong stability condition, i.e. all sufficiently close linear Hamiltonian systems are stable. As M.G. Krein [1] has shown, the system does not lose its stability by perturbations, even if there exist multiple eigenvalues, provided they are definite.

Linear Hamiltonian differential systems have been considered also in [2–5], where various strong stability criteria have been obtained. The main result of [3] consists in the following: A linear system with the Hamiltonian matrix A is strongly stable if and only if its centralizer $\mathrm{C}(A)$ consists of stable matrices. In [4] a geometrical proof has been proposed.

A Hamiltonian system often admits an additional first integral. Such systems usually define an action of the group R^2 and are called bihamiltonian systems. In [6] bihamiltonian systems on four-dimensional manifolds are considered. Among other, a Poincaré-type classification of fixed points is given and some questions concerning structural stability of such systems have been examined.

In this connection the problem of extension of the "parametric resonance" methods, due to M.G. Krein, on systems with multi-dimensional time, including those with symmetry, arises.

In [7, 8] a generalization of strong stability notion on linear symplectic actions of groups R^m and Z^m is proposed and some sufficient conditions on the joint spectrum of the generators, ensuring strong stability, are given.

The purpose of the present paper is to find a general criterion of strong stability of linear symplectic actions of the group R^m, actions generated by linear completely integrable systems of the type

$$\frac{\partial x}{\partial t_j} = A_j x. \quad (x \in R^{2n},\ t_j \in R,\ j = 1, \dots, m). \tag{1}$$

The complete integrability means $[A_i, A_j] = A_iA_j - A_jA_i = 0$, $i, j = 1, 2, \dots, m$. We endow R^{2n} with the standard symplectic structure and denote by $Sp(2n, R)$ the Lie group of symplectic matrices and by $sp(2n, R)$ the corresponding Lie algebra of Hamiltonian matrices.

In virtue of commutativity, the fundamental matrix of the system (1) is $\mathrm{e}^{(\mathcal{A},t)} := \exp(A_1t_1 + \dots + A_mt_m)$. The system (1) is called *stable* if $\exists r > 0$ such that $\|\exp(\mathcal{A}, t)\| < r$ for all $t \in R^m$. It is called *strongly stable* if there exists $\varepsilon > 0$ such that for any polyoperator $\mathcal{B} = \{B_1, \dots, B_m\} \in (sp(2n, R))^m$, $B_iB_j = B_jB_i$, $\|B_i - A_i\| < \varepsilon$ $(i, j = 1, \dots, m)$, the inequality $\|\exp(\mathcal{B}, t)\| < r$ holds for some $r > 0$ and all $t \in R^m$.

To prove strong stability we use the orbit method [9]. For this we consider the submanifold $\mathcal{M} \subset sp(2n, R)^m$ defined by the equations $[A_i, A_j] = 0$ $(i, j = 1, 2, \dots, m)$. For a stable element $\mathcal{A} \in \mathcal{M}$ we construct a neighbourhood of $\mathcal{A}$ in $\mathcal{M}$ which is "almost quadrilateral". The "horizontal side" of this quadrilateral is situated on the orbit of $\mathcal{A}$ under the diagonal adjoint action of the Lie group $Sp(2n, R)$ on $sp(2n, R)^m$, another "side" is orthogonal to the "horizontal" one: orthogonality is defined by the scalar product $\mathrm{tr}AB$ on $sp(2n, R)$, extended to the direct product $sp(2n, R)^m$.

If $\mathcal{A}$ is stable, every generator A_j is stable as well, so each A_j is semi-simple. In virtue of commutativity the tuple $\{A_1, A_2, \dots, A_m\}$ is simultaneously diagonalizable. So does the tuple $\{\mathrm{ad}_{A_1}, \mathrm{ad}_{A_2}, \dots, \mathrm{ad}_{A_m}\}$, considered as a linear operator from $Sp(2n, R)$ to $Sp(2n, R)^m$ and denoted by $\mathrm{ad}_{\mathcal{A}}$. If $\mathcal{A}$ is stable so does every element $\mathcal{B} \in \mathrm{orb}(\mathcal{A})$ (see below). Thus, for $\mathcal{A}$ to be strongly stable, it is necessary and sufficient that any tuple $\{C_1, C_2, \dots, C_m\} \in \mathcal{M}$, sufficiently close to $\mathcal{A}$ and orthogonal to the orbit at the point $\mathcal{A}$, should consist of stable matrices. This is the main result of the paper. In proving it, we follow [3] and [4].

Consider the system (1) with the column tuple $\mathcal{A} = \{A_1, A_2, \dots, A_m\}^T$. The group $Sp(2n, R)$ acts "diagonally" on $sp(2n, R)^m$:

$$\pi(g; \{A_1, A_2, \dots, A_m\}) = \{gA_1g^{-1}, gA_2g^{-1}, \dots, gA_mg^{-1}\}.$$

The stabilizer of the point $\mathcal{A} = \{A_1, A_2, \dots, A_m\}^T$ is the Lie subgroup $\mathrm{Stab}(\mathcal{A}) = \{g \in Sp(2n, R) : [A_i, g] = 0, j = 1, 2, \dots, m\}$. Let $\mathrm{stab}(\mathcal{A})$ denote the corresponding Lie subalgebra of $Sp(2n, R)$.

Proposition 1. *The partial derivative D_1 of the action*

$$\pi : Sp(2n, R) \times sp(2n, R)^m \longrightarrow sp(2n, R)^m$$

at the point $(e, \mathcal{A})$ equals

$$D_1\pi(e, \mathcal{A})U = \{\mathrm{ad}_{A_1}U, \mathrm{ad}_{A_2}U, \dots, \mathrm{ad}_{A_m}U\} \quad (U \in sp(2n, R)),$$

where $\mathrm{ad}_AB = [A, B]$.

Proof. It is sufficient to consider the derivative for the equality of each "coordinate function" $g \longmapsto gAg^{-1}$. It is known (see, e.g.[2]), that this derivative equals $U \longmapsto \mathrm{ad}_A U$. □

In other words, the "velocity vector" of the action π at the point $\mathcal{A}$ and at the "moment" $U \in sp(2n,R)$ has the "coordinates" $(\mathrm{ad}_{A_1}U, \mathrm{ad}_{A_2}U, \ldots, \mathrm{ad}_{A_m}U)$.

Corollary 1. *If all $A_1, A_2, \ldots, A_m$ are semi-simple, then the orbit*

$$orb(\mathcal{A}) = \{\pi(g; \{A_1, A_2, \ldots, A_m\} : g \in Sp(2n,R)\}$$

is closed and the tangent space to the orbit at the point $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}^T$ is isomorphic (as a vector space) to the quotient space

$$\mathrm{imad}_{\mathcal{A}}/[\mathrm{stab}(\mathcal{A})]^m := ((\{\mathrm{ad}_{A_1}U, \mathrm{ad}_{A_2}U, \ldots, \mathrm{ad}_{A_m}U\}) + [\mathrm{stab}(\mathcal{A})]^m \quad (U \in sp(2n,R)))$$

.

Remark 1. *In the ordinary case $m = 1$, when A is semi-simple, one has the following equality:* $\mathrm{im}\ \mathrm{ad}_A \oplus \mathrm{stab}(A) = sp(2n,r)$. *In contrast with thus, for $m > 1$ with the commutativity conditions $[A_i, A_j] = 0$ $(i,j = 1,2,\ldots,m)$ the situation is more complicated.*

Let $\mathcal{M}$ denote the closed subset of $sp(2n,R)^m$ defined by the equations $[A_i, A_j] = 0$ $(i,j = 1,2,\ldots,m)$. If $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}^T \neq \{0, 0, \ldots, 0\} \in sp(2n,R)^m$, the subset $\mathcal{M}$ is a submanifold. It is easy to verify that the subset $\mathcal{M}$ is invariant under the action π.

Proposition 2. *The tangent space to $\mathcal{M}$ at the point $\mathcal{A}$ equals*

$$T_{\mathcal{A}}\mathcal{M} = \{\{C_1, \ldots, C_m\} \in sp(2n,R)^m : \mathrm{ad}_{A_i}C_j = \mathrm{ad}_{A_j}C_i; i,j = 1,2,\ldots,m\}.$$

Proof. One has

$$[A_i + \varepsilon C_i, A_j + \varepsilon C_j] = [A_i, A_j] + \varepsilon([A_i, C_j] - [A_j, C_i]) + \varepsilon^2[C_i, C_j].$$

Since $[A_i, A_j] = 0$, the linear equations determining the tangent space are $[A_i, C_j] - [A_j, C_i] = 0, \quad (i,j = 1,2,\ldots,m)$. □

Remark 2. *It is worth noting that tangency to the orbit of the vector $\mathcal{C} = (C_1, \ldots, C_m)^T$ implies its tangency to the manifold $\mathcal{M}$. In other words, the necessary condition for the solvability of the system of equations*

$$\mathrm{ad}_{A_j}U = C_j \quad (j = 1,2,\ldots,m) \tag{2}$$

with pairwise commuting operators of the left hand side ("exactness") is the condition

$$\mathrm{ad}_{A_i}C_j = \mathrm{ad}_{A_j}C_i \quad (i,j = 1,2,\ldots,m) \tag{3}$$

(i.e. "closeness").

Recall (see e.g.[2]) that $\langle A, B\rangle := Tr(AB)$ defines an inner product on $sp(2n, R)$. We will extend it up to an inner product on $sp(2n, R)^m$ as follows: if $\mathcal{A} = \{A_1, A_2, \dots, A_m\}^T$, $\mathcal{B} = \{B_1, B_2, \dots, B_m\}^T$, then

$$\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=1}^{m} \langle A_i, B_i\rangle = \sum_{i=1}^{m} \mathrm{tr}\, A_i B_i.$$

Using properties of the trace, one has for any $\mathcal{C} = \{C_1, C_2, \dots, C_m\} \in sp(2n, R)^m$ and any $U \in \ sp(2n, R)/\mathrm{stab}(\mathcal{A})$:

$$\langle \mathrm{ad}_{\mathcal{A}} U, \mathcal{C}\rangle = \sum_{i=1}^{m} \langle \mathrm{ad}_{A_i} U, C_i\rangle = \sum_{i=1}^{m} \langle [A_i, U], C_i\rangle =$$

$$= -\sum_{i=1}^{m} \langle U, [A_i, C_i]\rangle = -\left\langle U, \sum_{i=1}^{m} [A_i, C_i]\right\rangle.$$

From this we obtain the following orthogonality condition:

Proposition 3. *The tuple $\mathcal{C} = \{C_1, C_2, \dots, C_m\}^T \in sp(2n, R)^m$ is orthogonal to the orbit at the point $\mathcal{A}$ if and only if $\sum_{i=1}^{m} [A_i, C_i] = 0$.*

After denoting by $\mathrm{ad}_{\mathcal{A}^*}$ the line-operator $\{\mathrm{ad}_{A_1}, \mathrm{ad}_{A_2}, \dots, \mathrm{ad}_{A_m}\}$, one can identify the tangent space at $\mathcal{A} \in sp(2n, R)^m$ with $\mathrm{im}\, \mathrm{ad}_{\mathcal{A}}/\mathrm{stab}(\mathcal{A})^m \oplus \ker \mathrm{ad}_{\mathcal{A}^*}$, where $\mathrm{im}\, \mathrm{ad}_{\mathcal{A}}/\mathrm{stab}(\mathcal{A})^m$ coincides with the space tangent to the orbit at the point $\mathcal{A}$, while $\ker \mathrm{ad}_{\mathcal{A}^*}$ represents the orthogonal complement.

Lemma 1 (On tubular neighborhood)**.** *If $\mathcal{A} \in \mathcal{M}$ is semi-simple, then each element $\mathcal{B} \in \mathcal{M}$, close enough to $\mathcal{A}$, can be represented as a shift along the orbit of some element $\mathcal{D}$, from the intersection of the manifold $\mathcal{M}$ with the affine subspace of $sp(2n, R)^m$, orthogonal to the orbit at the point $\mathcal{A}$.*

Using this lemma we obtain the following generalization of the strong stability criterion of an ordinary linear Hamiltonian systems, criterion given by R.Cushman and A.Kelly [3].

Theorem 1. *A stable system (1), with the polyoperator $\mathcal{A}$ as the right hand side, is strongly stable if and only if each $\mathcal{D} \in \mathcal{M} \cap \ker \mathrm{ad}_{\mathcal{A}^*}$, close enough to $\mathcal{A}$, is stable.*

As consequences one obtains the sufficient and necessary conditions, given in [7] and [8]: if the union of centralizers $\bigcup C(A_j)$ consists of stable matrices, then the system (1) is strongly stable; respectively, if the system (1) is strongly stable, then the intersection of the centralizers of the right hand sides consists of stable matrices.

References

[1] Krein M.G. *Obobshchenie nekotorykh issledovanii A.M. Lyapunova o lineinykh differentsial'nykh uravneniyakh s periodicheskimi koeffitsientami.* DAN SSSR, 1950, **73**, N 3, p. 445–448.

[2] Arnold V.I., Givental A.B. *Simplekticheskaya geometriya.* Itogi nauki i tekhniki VINITI: Sovremennye problemy matematiki. Fundamental'nye napravleniya. Moskva, 1985, **4**, p. 5–139.

[3] Cushman R., Kelly A. *Strongly stable real infinitesimal symplectic mappings.* J. Diff. Eq., 1979, N 31(2), p. 200–223.

[4] Levi M. *On stability of symplectic maps.* J. Diff. Eq., 1983, N 50(3), p. 441–443.

[5] Wójtkowski M.P. *A remark on strong stability of linear Hamiltonian systems.* J. Diff. Eq., 1989, N 81(2), p. 313–316.

[6] Lerman L.M., Umanskiy Ya.L. *Four–dimensional integrable Hamiltonian systems with simple singular points (Topological aspects).* Transl. Mathem. Monographs, Vol. 176, AMS, Providence, 1998.

[7] Glavan V. *Strong stability criteria for linear Hamilton-Pfaff systems.* Proc. Second Intern. workshop "Mathematica system in teaching and research", Siedlce. Moscow, 2000, p. 70–72.

[8] Glavan V., Rzeszótko Z. *On strong stability of linear Poisson actions.* Buletinul Academiei de Stiinţe f Republicii Moldova, Matematica, 2003, N 2(42), p. 5–12.

[9] Kirilov A.A. *Lektsii po metodu orbit.* Novosibirsk, Naychnaya kniga, 2002.

University of Podlasie
Siedlce, Poland
E-mail: *zrzesz@ap.siedlce.pl*

Received September 15, 2005