BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(48), 2005, Pages 94–98
ISSN 1024–7696

On a criterion of normality for mappings

N. Gashitsoi

Abstract. In this paper we present a criterion of normality for mappings on complex manifolds of a special form.

Mathematics subject classification: 32A20, 32A40.
Keywords and phrases: Kobayashi pseudodistance, normal family, $\mathcal{K}$-normal mapping.

In 1957 Lehto and Virtanen introduced the notion of normal meromorphic function in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Later on the theory of normal functions was intensively developed and the generalization of this theory in the case of several complex variables was essentially begun in Dovbush's works [1, 2].

The aim of the present paper is to prove a criterion of normality for mappings on complex manifolds of a special form.

Let M be a complex manifold of complex dimension n, p be a point from M, $v \in T_pM$ a tangent vector, and $T(M)$ the complex tangent bundle of M. Let us denote the set of all holomorphic mappings from the disc U to the manifold M by $H(U, M)$.

The Kobayashi length of $v \in T_pM$ is defined as follows

$$K_M(p, v) = \inf \left\{ \frac{1}{r} : f \in H(U, M), f(0) = p, f'(0) = r \cdot v, r > 0 \right\}. \qquad (1)$$

The complex manifold N is called *Hermitian* if a bilinear Hermitian form $ds^2_N : T(N) \times T(N) \to \mathbb{C}$ is given in the fibers of its tangent bundle and this form is positively defined and twice smooth for the vector fields of the $C^2(N)$ class.

Let the set of all holomorphic mappings from the complex manifold M to the Hermitian manifold N be denoted by $H(M, N)$.

Definition 1. *Let M be a complex manifold and N be a Hermitian manifold with the Hermitian metric ds^2_N. The mapping $f \in H(M, N)$ is called $\mathcal{K}$-normal if there exists a constant $L > 0$ such that*

$$f^* ds^2_N \leq L \cdot (K_M)^2 \text{ on } T(M). \qquad (2)$$

The sequence $\{f_j\} \subset H(M, N)$ is called *compactly divergent* if for every compacts $K \subset M$ and $K' \subset N$ there exists such an index j_0 that $f_j(K) \cap K' = \emptyset$ for all $j > j_0$.

A family $\mathcal{F} \subset H(M,N)$ is called *normal* if every sequence $\{f_j\} \subset \mathcal{F}$ has either a uniformly convergent subsequence $\{f_{j_k}\}$ on compact subsets of M, or a compactly divergent subsequence.

In 1931 Marty [4] proved the following criterion of normality for families of meromorphic functions in the case of one complex variable.

A family $\mathcal{F}$ of functions f, meromorphic on the domain D of the complex plane is normal on D if and only if the ratio $|f'(z)|(1+|f(z)|^2)^{-1}$ is uniformly bounded on every compact subset of the domain D for every function $f \in \mathcal{F}$.

For the first time the generalization of Marty's criterion for the case of families of holomorphic functions of several complex variables was obtained in Dovbush's work [1]. Hahn [5] observed that Marty's criterion can be transferred on families of holomorphic mappings on hyperbolic manifolds in compact manifolds essentially with the same proof as in [1].

Let Y be a relatively compact complex subspace of a Hermitian manifold N. Let us denote the subset of $H(M,N)$ such that $f(M) \subset Y$ by $H(M,Y)$.

Further on we shall need the following particular case of Marty's criterion [3, Lemma 2, p. 11–12].

Lemma 1. *Let M be a complex manifold such that for every point $p \in M$ there exist a neighborhood $U_p \subset M$ and a constant $c = c(U_p) > 0$ such that $K_M(q,v) \geq c(U_p)|v|$ for all $(q,v) \in T(U_p) \cong U_p \times \mathbb{C}^n$, Y be a relatively compact complex subspace of a Hermitian manifold N with the Hermitian metric ds_N^2.*

A family $\mathcal{F} \subset H(M,Y)$ is normal on M if and only if for every compact K from M there exists a constant $L(K) > 0$ such that for every mapping $f \in \mathcal{F}$ it holds that

$$f^* ds_N^2(p,v) \leq L(K) \cdot (K_M(p,v))^2 \quad \text{for all } p \in K, v \in T_pM, \tag{3}$$

where $f^* ds_N^2(p,v)$ is the pull-back of ds_N^2 by the mapping f.

Proof. Suppose that (3) holds. Let's fix the points p_1 and p_2 in an arbitrary way on the manifold M and integrate the inequality (3) over all smooth curves $\gamma : [0,1] \to M$ of the C^1 class such that $\gamma(0) = p_1$ and $\gamma(1) = p_2$. Then we'll take the infimum of both sides of obtained inequalities and by using the definition of spherical distance and distance in the Kobayashi metric we will obtain that

$$s_N(f(p_1), f(p_2)) \leq L_1 \cdot k_M(p_1,p_2).$$

From this relation follows that if for every positive number $\varepsilon > 0$ take $\delta(\varepsilon) = \dfrac{\varepsilon}{L_1}$, then for every points $p_1, p_2 \in M$ such that $k_M(p_1,p_2) < \delta$ holds the inequality $s_N(f(p_1), f(p_2)) < \varepsilon$. As a result the family $\mathcal{F}$ is equicontinuous. Since $\overline{Y}$ is compact then by Ascoli - Arzela theorem the family $\mathcal{F}$ is normal.

Conversely, suppose that the family $\mathcal{F}$ is normal, but (3) does not hold. Then there is a compact subset $K \subset M$, a sequence of points $p_j \in K$, a sequence of tangent vectors $v_j \in T_{p_j}M$ and a sequence of mappings $f_j \in \mathcal{F}$, $j = 1, 2, \ldots$, such that

$$f_j^* ds_N^2(p_j, v_j) \geq j \cdot (K_M(p_j,v_j))^2. \tag{4}$$

Passing to a subsequence we may consider that for $j \to \infty$, $(p_j, v_j) \to (p_0, v_0) \in T(M)$ and $f_j \to f_0 \in \mathcal{F}$ uniformly on compact subsets. Since $\overline{Y}$ is compact then for $j \to \infty$ the left – hand side of (4) tends to a finite number $f_0^* ds_N^2(p_0, v_0)$. According to lemma's conditions for the point p_0 there are a neighborhood U_{p_0} and a constant $c(U_{p_0}) > 0$ such that $K_M(q, v) > c(U_{p_0})|v|$ for all $q \in U_{p_0}$ and all $v \in T_q M$. Thus, when $j \to \infty$, $K_M(p_j, v_j)$ tends to a number which is not less than $c(U_{p_0}) \cdot |v_0|$. Therefore the right hand side of (4) tends to infinity and that yields a contradiction. □

In the case of one complex variable, Yosida and Noshiro with every function f meromorphic in the unit disc U associated the family of functions $\mathcal{F} = \{f \circ T, T \in Aut\, U\}$, where $Aut\, U$ is the group of biholomorphic automorphisms of the unit disc U and they examined those functions f for which the corresponding family $\mathcal{F}$ is normal. These functions were called by Lehto and Virtanen normal functions.

Developing Yosida and Noshiro's idea we may associate every mapping $f \in H(D, N)$, where D is a domain in $\mathbb{C}^n$, with the family $\mathcal{F} = \{f \circ g : g \in Aut\, D\}$. However, it is well known that there exist simply connected domains in $\mathbb{C}^n$ whose group of automorphisms consists only of the identical mapping.

Cima and Krantz [6] associated every mapping $f \in H(D, \overline{\mathbb{C}})$ with the family $\mathcal{F} = \{f \circ h : h \in H(U, D)\}$ and proved the following criterion of $\mathcal{K}$-normality of mappings in the arbitrary domains in $\mathbb{C}^n$.

Proposition 1. *A mapping $f \in H(D, \overline{\mathbb{C}})$ is $\mathcal{K}$-normal in the domain $D \subset \mathbb{C}^n$ if and only if the family $\mathcal{F} = \{f \circ h, h \in H(U, D)\}$ is normal.*

Any mapping $g \in H(U, M)$ such that $g(0) = p$ and $K_M(p, v) = \dfrac{v}{g'(0)}$ is called *extremal* for $K_M(p, v)$.

A complex manifold M is called *taut* if the family of mappings $H(U, M)$ is normal. If the manifold M is taut then the infimum in (1) is always attained for some mapping g.

From now on we shall consider M such a complex manifold that for every point $p \in M$ and for every tangent vector $v \in T_p M$ there exists an extremal mapping for $K_M(p, v)$. Let's denote the set of all extremal mappings by $E(M)$ and give the following definition of normal mapping.

Definition 2. *Let Y be a relatively compact complex subspace of a Hermitian manifold N. A mapping $f \in H(M, Y)$ holomorphic on the manifold M is called normal if the family $\mathcal{F} = \{f \circ g, g \in E(M)\}$ is normal.*

If $D = U$, the above definition is equivalent with the classical one, since every mapping $\phi \in AutU$ is an extremal in U.

Remarks.

1) Let $D = \{z \in \mathbb{C}^n : \phi(z) < 0\}$ be a bounded domain where the function ϕ is plurisubharmonic in D and continuous in $\overline{D}$ or is obtained from such a domain by eliminating an analytic subset A of codimension 1. Then the infimum in (1) is always attained [7].

2) Lempert [8] proved that the extremal mapping exists and is unique for all strictly linear convex domains.

3) It is well known that domains $D \subset \mathbb{C}^n$, whose every boundary point is a peak point (see [9, definition 15.2.1]), are taut and then for this type of domains extremal mappings exist.

Theorem 1. *A mapping $f \in H(M,N)$ is $\mathcal{K}$-normal on the manifold M if and only if the family $\mathcal{F} = \{f \circ g, g \in E(M)\}$ is normal.*

Proof. Let $\mathcal{F}$ be a normal family on U. Let's take as compact $K \subset U$ the point 0. According to Marty's criterion there is a constant $L > 0$ such that for all $g \in E(M)$

$$(f \circ g)^* ds_N^2(0,1) \leq L \cdot (K_U(0,1))^2. \tag{5}$$

For every point $p \in M$ and for every vector $v \in T_pM$ there is an extremal mapping $g : U \to M$ such that

$$g(0) = p \text{ and } K_M(p,v) = \frac{v}{|g'(0)|}.$$

Therefore

$$f^* ds_N^2(p,v) < L'(K_M(p,v))^2.$$

Since

$$(f \circ g)^* ds_N^2(0,1) \equiv f^* ds_N^2(g(0), g'(0) \cdot 1) = f^* ds_N^2 \left(p, \frac{v}{K_M(p,v)}\right) =$$

$$= \frac{1}{K_M^2(p,v)} f^* ds_N^2(p,v),$$

then from the previous inequality, taking into account that $K_U(0,1) = 1$, we obtain

$$\frac{1}{K_M^2(p,v)} f^* ds_N^2(p,v) \leq L.$$

Consequently

$$f^* ds_N^2(p,v) \leq L \cdot (K_M^2(p,v)).$$

We conclude that the mapping f is $\mathcal{K}$-normal on the manifold M from the above inequality and on the basis of Definition 1.

Conversely, let f be a $\mathcal{K}$-normal mapping on the manifold M. Then by Definition 1 there is a constant $L > 0$ such that

$$f^* ds_N^2(p,v) \leq L \cdot (K_M(p,v))^2 \tag{6}$$

for all $p \in M$ and all $v \in T_pM$. Let's fix a point $p \in M$, a vector $v \in T_pM$ and a point $\lambda \in U$ and let $g : U \to M$ be such an extremal mapping that $g(\lambda) = p$ and $g'(\lambda) = v$. Since

$$f^* ds_N^2(p,v) = f^* ds_N^2(g(\lambda), g'(\lambda)) = (f \circ g)^* ds_N^2(\lambda, 1),$$

then from (6) we obtain

$$f^* ds_N^2(p, v) \leq L \cdot (K_M(p, v))^2.$$

Hence, by the contracting property of the Kobayashi pseudometric $K_M(g(\lambda), g'(\lambda)) \leq K_U(\lambda, 1)$ it follows that

$$(f \circ g)^* ds_N^2(\lambda, 1) \leq L \cdot (K_U(\lambda, 1))^2.$$

Thus, taking into consideration Lemma 1 we conclude that the family $\mathcal{F} = \{f \circ g, g \in E(M)\}$ is normal. □

References

[1] Dovbush P.V. *Normal functions of several complex variables.* Vestnic Moskov. Univ., Ser. I, Mat. Mekh., 1981, N 1, p. 44–48 (in Russian).

[2] Dovbush P.V. *The Lindelöf theorem in* $\mathbb{C}^n$. Vestnic Moskov. Univ., Ser. I, Mat. Mekh., 1981, N 6, p. 33–36 (in Russian).

[3] Dovbush P.V. *$X-$normal mappings.* Thesis of doctor habilitat, Chishinau, 2004 (in Romanian).

[4] Marty F. *Recerches sur la répartition des valeurs d'une function méromorphe.* Ann. Fac. Sci. Univ. Toulouse, 1931, **28**, N 3, p. 183–261.

[5] Hahn K.T. *Asymptotic behavior of normal mappings of several complex variables.* Canad. J. Math., 1984, **36**, N 4, p. 718–746.

[6] Cima J.A., Krantz S.G. *The Lindelöf principle and normal functions of several complex variables.* Duke Math. J., 1983, **50**, p. 303–328.

[7] Paletskii E.A., Shabat B.V. *Invariant metrics.* In Encyclopædia of Mathematical Science, Vol. 9., Several complex variables III, Ed. G.M. Khenkin, Springer, Berlin, 1989, p. 63–111.

[8] Lempert L. *La metrique Kobayashi et les representation des domanes sur la boule.* Bull. Soc. Math. France., 1981, **109**, p. 427–474.

[9] Rudin W. *Function Theory in the Unit Ball of* $\mathbb{C}^n$. Springer–Verlag, Berlin, 1980.

State University "A. Russo"
str. Pushkin 38, MD-3121 Baltsy
Moldova
E-mail: *ngashitsoi@mail.ru*

Received June 23, 2005