On a criterion of normality for mappings

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Abstract. In this paper we present a criterion of normality for mappings on complex manifolds of a special form.


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In 1957 Lehto and Virtanen introduced the notion of normal meromorphic function in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Later on the theory of normal functions was intensively developed and the generalization of this theory in the case of several complex variables was essentially begun in Dovbush’s works [1, 2].

The aim of the present paper is to prove a criterion of normality for mappings on complex manifolds of a special form.

Let $M$ be a complex manifold of complex dimension $n$, $p$ be a point from $M$, $v \in T_pM$ a tangent vector, and $T(M)$ the complex tangent bundle of $M$. Let us denote the set of all holomorphic mappings from the disc $U$ to the manifold $M$ by $H(U, M)$.

The Kobayashi length of $v \in T_pM$ is defined as follows

$$K_M(p, v) = \inf \left\{ \frac{1}{r} : f \in H(U, M), f(0) = p, f'((0) = r \cdot v, r > 0 \right\}. \quad (1)$$

The complex manifold $N$ is called Hermitian if a bilinear Hermitian form $ds_N^2 : T(N) \times T(N) \to \mathbb{C}$ is given in the fibers of its tangent bundle and this form is positively defined and twice smooth for the vector fields of the $C^2(N)$ class.

Let the set of all holomorphic mappings from the complex manifold $M$ to the Hermitian manifold $N$ be denoted by $H(M, N)$.

Definition 1. Let $M$ be a complex manifold and $N$ be a Hermitian manifold with the Hermitian metric $ds_N^2$. The mapping $f \in H(M, N)$ is called $K$-normal if there exists a constant $L > 0$ such that

$$f^*ds_N^2 \leq L \cdot (K_M)^2 \text{ on } T(M). \quad (2)$$

The sequence $\{f_j\} \subset H(M, N)$ is called compactly divergent if for every compacts $K \subset M$ and $K' \subset N$ there exists such an index $j_0$ that $f_j(K) \cap K' = \emptyset$ for all $j > j_0$. 

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A family $\mathcal{F} \subset H(M, N)$ is called normal if every sequence $\{f_j\} \subset \mathcal{F}$ has either a uniformly convergent subsequence $\{f_{j_k}\}$ on compact subsets of $M$, or a compactly divergent subsequence.

In 1931 Marty [4] proved the following criterion of normality for families of meromorphic functions in the case of one complex variable.

A family $\mathcal{F}$ of functions $f$, meromorphic on the domain $D$ of the complex plane is normal on $D$ if and only if the ratio $|f'(z)|(1 + |f(z)|^2)^{-1}$ is uniformly bounded on every compact subset of the domain $D$ for every function $f \in \mathcal{F}$.

For the first time the generalization of Marty’s criterion for the case of families of holomorphic functions of several complex variables was obtained in Dovbush’s work [1]. Hahn [5] observed that Marty’s criterion can be transferred on families of holomorphic mappings on hyperbolic manifolds in compact manifolds essentially with the same proof as in [1].

Let $Y$ be a relatively compact complex subspace of a Hermitian manifold $N$. Let us denote the subset of $H(M, N)$ such that $f(M) \subset Y$ by $H(M, Y)$.

Further on we shall need the following particular case of Marty’s criterion [3, Lemma 2, p. 11–12].

**Lemma 1.** Let $M$ be a complex manifold such that for every point $p \in M$ there exist a neighborhood $U_p \subset M$ and a constant $c = c(U_p) > 0$ such that $K_M(q, v) \geq c(U_p)|v|$ for all $(q, v) \in T(U_p) \cong U_p \times \mathbb{C}^n$, $Y$ be a relatively compact complex subspace of a Hermitian manifold $N$ with the Hermitian metric $ds_N^2$.

A family $\mathcal{F} \subset H(M, Y)$ is normal on $M$ if and only if for every compact $K$ from $M$ there exists a constant $L(K) > 0$ such that for every mapping $f \in \mathcal{F}$ it holds that

$$f^*ds_N^2(p, v) \leq L(K) \cdot (K_M(p, v))^2 \quad \text{for all } p \in K, v \in T_pM,$$

where $f^*ds_N^2(p, v)$ is the pull-back of $ds_N^2$ by the mapping $f$.

**Proof.** Suppose that (3) holds. Let’s fix the points $p_1$ and $p_2$ in an arbitrary way on the manifold $M$ and integrate the inequality (3) over all smooth curves $\gamma : [0, 1] \to M$ of the $C^1$ class such that $\gamma(0) = p_1$ and $\gamma(1) = p_2$. Then we’ll take the infimum of both sides of obtained inequalities and by using the definition of spherical distance and distance in the Kobayashi metric we will obtain that

$$s_N(f(p_1), f(p_2)) \leq L_1 \cdot k_M(p_1, p_2).$$

From this relation follows that if for every positive number $\varepsilon > 0$ take $\delta(\varepsilon) = \frac{\varepsilon}{L_1}$, then for every points $p_1, p_2 \in M$ such that $k_M(p_1, p_2) < \delta$ holds the inequality $s_N(f(p_1), f(p_2)) < \varepsilon$. As a result the family $\mathcal{F}$ is equicontinuous. Since $Y$ is compact then by Ascoli - Arzela theorem the family $\mathcal{F}$ is normal.

Conversely, suppose that the family $\mathcal{F}$ is normal, but (3) does not hold. Then there is a compact subset $K \subset M$, a sequence of points $p_j \in K$, a sequence of tangent vectors $v_j \in T_{p_j}M$ and a sequence of mappings $f_j \in \mathcal{F}$, $j = 1, 2, \ldots$, such that

$$f_j^*ds_N^2(p_j, v_j) \geq j \cdot (K_M(p_j, v_j))^2.$$

(4)
Passing to a subsequence we may consider that for \( j \to \infty \), \((p_j, v_j) \to (p_0, v_0) \in T(M)\) and \( f_j \to f_0 \in \mathcal{F} \) uniformly on compact subsets. Since \( f \) is compact then for \( j \to \infty \) the left-hand side of (4) tends to a finite number \( f_0 ds_N^2(p_0, v_0) \). According to lemma’s conditions for the point \( p_0 \) there are a neighborhood \( U_{p_0} \) and a constant \( c(U_{p_0}) > 0 \) such that \( K_M(q, v) > c(U_{p_0})|v| \) for all \( q \in U_{p_0} \) and all \( v \in T_q M \). Thus, when \( j \to \infty \), \( K_M(p_j, v_j) \) tends to a number which is not less than \( c(U_{p_0})|v_0| \). Therefore the right hand side of (4) tends to infinity and that yields a contradiction. \( \square \)

In the case of one complex variable, Yosida and Noshiro with every function \( f \) meromorphic in the unit disc \( U \) associated the family of functions \( \mathcal{F} = \{ f \circ T, T \in Aut U \} \), where \( Aut U \) is the group of biholomorphic automorphisms of the unit disc \( U \) and they examined those functions \( f \) for which the corresponding family \( \mathcal{F} \) is normal. These functions were called by Lehto and Virtanen normal functions.

Developing Yosida and Noshiro’s idea we may associate every mapping \( f \in H(D, N) \), where \( D \) is a domain in \( \mathbb{C}^n \), with the family \( \mathcal{F} = \{ f \circ g : g \in Aut D \} \). However, it is well known that there exist simply connected domains in \( \mathbb{C}^n \) whose group of automorphisms consists only of the identical mapping.

Cima and Krantz [6] associated every mapping \( f \in H(D, \overline{U}) \) with the family \( \mathcal{F} = \{ f \circ h : h \in H(U, D) \} \) and proved the following criterion of \( K \)-normality of mappings in the arbitrary domains in \( \mathbb{C}^n \).

**Proposition 1.** A mapping \( f \in H(D, \overline{U}) \) is \( K \)-normal in the domain \( D \subset \mathbb{C}^n \) if and only if the family \( \mathcal{F} = \{ f \circ h, h \in H(U, D) \} \) is normal.

Any mapping \( g \in H(U, M) \) such that \( g(0) = p \) and \( K_M(p, v) = \frac{v}{g'(0)} \) is called extremal for \( K_M(p, v) \).

A complex manifold \( M \) is called taut if the family of mappings \( H(U, M) \) is normal. If the manifold \( M \) is taut then the infimum in (1) is always attained for some mapping \( g \).

From now on we shall consider \( M \) such a complex manifold that for every point \( p \in M \) and for every tangent vector \( v \in T_p M \) there exists an extremal mapping for \( K_M(p, v) \). Let’s denote the set of all extremal mappings by \( E(M) \) and give the following definition of normal mapping.

**Definition 2.** Let \( Y \) be a relatively compact complex subspace of a Hermitian manifold \( N \). A mapping \( f \in H(M, Y) \) holomorphic on the manifold \( M \) is called normal if the family \( \mathcal{F} = \{ f \circ g, g \in E(M) \} \) is normal.

If \( D = U \), the above definition is equivalent with the classical one, since every mapping \( \phi \in Aut U \) is an extremal in \( U \).

**Remarks.**

1) Let \( D = \{ z \in \mathbb{C}^n : \phi(z) < 0 \} \) be a bounded domain where the function \( \phi \) is plurisubharmonic in \( D \) and continuous in \( \overline{D} \) or is obtained from such a domain by eliminating an analytic subset \( A \) of codimension 1. Then the infimum in (1) is always attained [7].
2) Lempert [8] proved that the extremal mapping exists and is unique for all strictly linear convex domains.

3) It is well known that domains $D \subset \mathbb{C}^n$, whose every boundary point is a peak point (see [9, definition 15.2.1]), are taut and then for this type of domains extremal mappings exist.

**Theorem 1.** A mapping $f \in H(M, N)$ is $K$-normal on the manifold $M$ if and only if the family $\mathcal{F} = \{f \circ g, g \in E(M)\}$ is normal.

**Proof.** Let $\mathcal{F}$ be a normal family on $U$. Let’s take as compact $K \subset U$ the point 0. According to Marty’s criterion there is a constant $L > 0$ such that for all $g \in E(M)$

$$(f \circ g)^* ds_N^2(0, 1) \leq L \cdot (K_U(0, 1))^2. \quad (5)$$

For every point $p \in M$ and for every vector $v \in T_pM$ there is an extremal mapping $g : U \to M$ such that

$$g(0) = p \text{ and } K_M(p, v) = \frac{v}{|g'(0)|}.$$ 

Therefore

$$f^* ds_N^2(p, v) < L'(K_M(p, v))^2.$$ 

Since

$$(f \circ g)^* ds_N^2(0, 1) \equiv f^* ds_N^2(g(0), g'(0) \cdot 1) = f^* ds_N^2 \left(p, \frac{v}{K_M(p, v)}\right) =$$

$$= \frac{1}{K^2_M(p, v)} f^* ds_N^2(p, v),$$

then from the previous inequality, taking into account that $K_U(0, 1) = 1$, we obtain

$$\frac{1}{K^2_M(p, v)} f^* ds_N^2(p, v) \leq L.$$ 

Consequently

$$f^* ds_N^2(p, v) \leq L \cdot (K_M(p, v))^2.$$ 

We conclude that the mapping $f$ is $K$-normal on the manifold $M$ from the above inequality and on the basis of Definition 1.

Conversely, let $f$ be a $K$-normal mapping on the manifold $M$. Then by Definition 1 there is a constant $L > 0$ such that

$$f^* ds_N^2(p, v) \leq L \cdot (K_M(p, v))^2 \quad (6)$$

for all $p \in M$ and all $v \in T_pM$. Let’s fix a point $p \in M$, a vector $v \in T_pM$ and a point $\lambda \in U$ and let $g : U \to M$ be such an extremal mapping that $g(\lambda) = p$ and $g'(\lambda) = v$. Since

$$f^* ds_N^2(p, v) = f^* ds_N^2(g(\lambda), g'(\lambda)) = (f \circ g)^* ds_N^2(\lambda, 1),$$
then from (6) we obtain
\[ f^*ds_N^2(p, v) \leq L \cdot (K_M(p, v))^2. \]

Hence, by the contracting property of the Kobayashi pseudometric
\[ K_M(\lambda, g, g') \leq K_U(\lambda, 1) \]

it follows that
\[ (f \circ g)^*ds_N^2(\lambda, 1) \leq L \cdot (K_U(\lambda, 1))^2. \]

Thus, taking into consideration Lemma 1 we conclude that the family
\[ F = \{ f \circ g, g \in E(M) \} \]

is normal. \qed

References


