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An approximate solution of the Fredholm type equation of the second kind for any $\lambda \neq 0$

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Abstract. Consider the following equation

$$\big((I-\lambda K)\varphi\big)(s) = \varphi(s) - \lambda \int_a^b K(s,t)\varphi(t)\,dt = f(s).$$

Assume that the complex-valued kernel $K(s,t)$ is defined on $(a-\varepsilon, b+\varepsilon)\times(a-\varepsilon, b+\varepsilon)$ for some $\varepsilon > 0$ and

$$\|K\|_2^2 = \int_a^b |K(s,t)|^2\,ds\,dt,$$

$$p(s,t) = \lambda K(s,t) + \overline{\lambda}\,\overline{K(t,s)} - |\lambda|^2 \int_a^b \overline{K(\xi,s)}K(\xi,t)\,d\xi.$$

Consider the following mapping

$$f : [a,b] \ni \xi \to p(s,\xi)p(\xi,t) \in L_2([a,b]\times[a,b]).$$

If the function f is integrable according to definition of the Riemann integral (as the function with values in the space $L_2([a,b]\times[a,b])$), then the kernel of the square of the integral operator

$$(P\varphi)(s) = \int_a^b p(s,t)\varphi(t)\,dt$$

can be approximated by a finite dimensional kernel. The formula $(I-P)^+ = \big(I-P^2\big)^+(I+P)$ and the persistency of the operator $\big(I-P^2\big)^+$ with respect to perturbations of a special type are proved. For any $\lambda \neq 0$ we find approximations of the function φ which minimizes functional $\|(I-\lambda K)\varphi - f\|_2$ and has the least norm in $L_2[a,b]$ among all functions minimizing the above mentioned functional. Simultaneously we find approximations of the kernel and orthocomplement to the image of the operator $I-\lambda K$ if $\lambda \neq 0$ is a characteristic number. The corresponding approximation errors are obtained.

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Introduction

The classical theory [1, 2, 3] deals with the solution of an equation of the type

$$\varphi(s) + \lambda \int_a^b K(s,t)\varphi(t)\, dt = f(s) \qquad (a \le s \le b), \tag{1}$$

where $K(s,t)$ is a given continuous scalar function of two real variables with domain $[a,b] \times [a,b]$, f is a given continuous scalar function of a single real variable with domain $[a,b]$, λ is a scalar, and the continuous scalar function of a single real variable with domain $[a,b]$ is to be determined. Such an equation is known as a Fredholm integral equation ('of the second kind').

The integral theory of the equations (1) is stated in the classical monographies devoted to integral equations [1, 2]. The extension of the application area of the integral equations (1) (see, for example, [4]) has stimulated an intensive development of approximate solution methods. This gave a stimulus to development of some classical methods (method of mechanical quadratures, iteration methods, projection methods etc.). The above-mentioned methods are good in the case when $\lambda \ne 0$ is not a characteristic number of the equation (1). There is a principle distinction of this case from that one when $\lambda \ne 0$ is a characteristic number. If the operator determined by the left part of equality (1) is invertible, then an approximate solution can be obtained by reducing (1) to a finite system of the algebraic equations, since the inversion operation is persistent with respect to small norm perturbations. Otherwise the reduction to the finite system of the algebraic equations requires an additional research. This reduction (in the case when $\lambda \ne 0$ is a characteristic number) is not discussed in the works known to the author (see, for example, [4, 5]).

There is one more reason for this work appearance. In my opinion, it is closely related to the difficult description of finite dimensional approximations of the kernel $K(s,t)$. Usually, one appeals to the Weierstrass theorem for two-dimensional case, which implies the following statement.

Theorem ([5]). *Any L_2-integrable on $[a,b] \times [a,b]$ function $K(s,t)$ can be approximated in L^2 with arbitrary large accuracy by the following sums:*

$$\sum_{j=0}^{m} a_j(s)t^j \quad \text{or} \quad \sum_{k=0}^{m} b_k(t)s^k$$

with continuous coefficients.

Such approximations are used for reduction of the equation (1) to a finite system of the algebraic equations. The realization of these approximations is difficult. In the paper the method of the quadrature sums is used instead of the Weierstrass theorem.

Throughout the article we assume that the kernel $K(s,t)$ is a measurable complex-valued function defined in some neighbourhood of the set $[a,b]\times[a,b]$ and

$$\|K\|_2^2 = \int_a^b |K(s,t)|^2 \, ds\, dt < +\infty.$$

We shall write the equation (1) in the operator form

$$(I-\lambda K)\varphi = f, \tag{2}$$

where

$$(K\varphi)(s) = \int_a^b K(s,t)\varphi(t)\, dt.$$

Denote $\mathcal{L}(L_2[a,b], L_2[a,b])$ the set of all linear continuous operators from $L_2[a,b]$ to $L_2[a,b]$ with uniform operator topology. The norm in this space will be denoted $|\!|\!|\cdot|\!|\!|_2$. The inner product in $L_2[a,b]$ will be denoted $\langle\cdot,\cdot\rangle$. For the norm of function from $L_2[a,b]$ we keep the notation $\|\cdot\|$.

Pseudo-inversion of the operator $(\mathbf{I}-\lambda\mathbf{K})$

Let A be a linear bounded operator in the separable Hilbert space H. Let $R(A)$ denote the range of values of an operator A, $N(A)$ be a kernel A, $R(A)^{\perp}$, $N(A)^{\perp}$ be the orthocomplements to the range of values and the kernel of an operator A, respectively.

If M is a closed subspace in H, then $A|_M$ means a restriction of an operator A on M.

Let the range of values of an operator A be closed. The pseudoinversed operator A^{+} is defined according to the formula

$$A^{+}h = \begin{cases} \left(A\Big|_{N(A)^{\perp}}\right)^{-1} h, & h\in R(A),\\ \theta, & h\in R(A)^{\perp}.\end{cases}$$

The operator A^{+} is continuous.

Denote

$$I-P = (I-\overline{\lambda}K^{*})(I-\lambda K),$$
$$I-Q = (I-\lambda K)(I-\overline{\lambda}K^{*}).$$

The kernels of the integral operators P and Q are calculated as follows:

$$p(s,t) = \overline{\lambda}\,\overline{K(t,s)} + \lambda K(s,t) - |\lambda|^2 \int_a^b \overline{K(\xi,s)}K(\xi,t)\, d\xi,$$

$$q(s,t) = \lambda K(s,t) + \overline{\lambda}\,\overline{K(t,s)} - |\lambda|^2 \int\limits_a^b K(s,\xi)\overline{K(t,\xi)}\,d\xi.$$

Obviously, the operators P and Q are selfadjoint. The following formulas are valid

$$(I - \lambda K)^+ = (I - P)^+(I - \overline{\lambda}K^*), \tag{3}$$
$$(I - \lambda K)^+ = (I - \overline{\lambda}K^*)(I - Q)^+. \tag{4}$$

The formulas (3) and (4) are obtained in [7]. It follows from results in [8, 9] that

$$(I - P)^+ = \left(I - P^2\right)^+ (I + P)$$

and

$$(I - Q)^+ = (I - Q^2)^+(I + Q).$$

Hence,

$$(I - \lambda K)^+ = \left(I - P^2\right)^+ (I + P)(I - \overline{\lambda}K^*), \tag{5}$$
$$(I - \lambda K)^+ = (I - \overline{\lambda}K^*)(I - Q^2)^+(I + Q). \tag{6}$$

The kernels of the integral operators P^2 and Q^2 are determined as follows:

$$p_2(s,t) = \int\limits_a^b p(s,\xi)p(\xi,t)\,dt,$$
$$q_2(s,t) = \int\limits_a^b q(s,\xi)q(\xi,t)\,dt.$$

According to (5), (6) the problem of the pseudo-inversion of the operator $(I - \lambda K)$ is reduced to the problem of the pseudo-inversion of the operator $\left(I - P^2\right)$ or the operator $(I - Q^2)$.

In this article the persistency of pseudoinverse operators $\left(I - P^2\right)^+$, $(I - Q^2)^+$ with respect to perturbations of special type is proved, and a method of the approximate solution of the equation (2) for any $\lambda \neq 0$ is stated. The concept of the pseudo-inversion is closely connected to the solution of the equation (2). An element φ_* of $L_2[a,b]$ is called pseudo-solution of the equation (2) if φ_* minimizes the discrepancy norm of this equation, i.e.

$$\|(I - \lambda K)\varphi_* - f\|_2 = \inf_{\varphi \in L_2[a,b]} \|(I - \lambda K)\varphi - f\|_2. \tag{7}$$

If in (7) the infimum is taken over a proper subset of the whole space, then φ_* is called a quasi-solution of the equation (2) on this set. The concept of quasi-solution was introduced by V. K. Ivanov in his works [10, 11]. An element φ_* of $L_2[a,b]$

is called normal pseudo-solution of the equation (2) if φ_* minimizes the norm of pseudo-solutions, i.e.

$$\|\varphi^*\|_2 = \inf\big\{\|\varphi_*\|_2 : \varphi_* \ satisfies\ (7)\big\}.$$

Since the range of values of an operator $(I - \lambda K)$ is closed, then

$$\varphi^* = (I - \lambda K)^+ f \in N(I - \lambda K)^\perp,$$

and any solution of the equation (2) is represented as follows:

$$\varphi = \varphi^* + h, \quad h \in N(I - \lambda K).$$

An important parameter characterizing an operator with closed range of values is the number

$$\nu = \inf_{\substack{\varphi \in L_2[a,b] \\ \varphi \in N(I-\lambda K)^\perp \\ \|\varphi\|_2 = 1}} \|(I - \lambda K)\varphi\|_2 > 0.$$

The following equality is true (see, for example, [12]):

$$\big\|(I - \lambda K)^+\big\|_2 = \frac{1}{\nu}.$$

The number ν is called the reduced minimum module.

Finite-dimensional approximations of the kernels $\mathbf{p_2}, \mathbf{q_2}$

Further consider only the kernel $p_2(s,t)$. The approximation of the kernel $q_2(s,t)$ can be obtained similarly. Consider the mapping

$$f : [a,b] \to p(s,\xi)p(\xi,t) \in L_2([a,b]\times[a,b]).$$

If the function f is Riemann-integrable, then the Cauchy–Riemann sums

$$S_n(s,t) = \frac{(b-a)}{n}\sum_{k=0}^{n-1} p\left(s, a + k\frac{(b-a)}{n}\right) p\left(a + k\frac{(b-a)}{n}, t\right)$$

converge to the function $p_2(s,t)$ in $L_2([a,b]\times[a,b])$. A sufficient condition of Riemann-integrability of a function f is its continuity everywhere, except for a finite number of points, and boundedness. The rate of convergence of the sums $S_n(s,t)$ can be determined if the mapping f has some additional smoothness properties. Here we consider the case, when the function f is defined on the set $(a-\varepsilon, b+\varepsilon)$ for some $\varepsilon > 0$, continuously differentiable and is bounded together with its derivative. Assume that

$$\omega_0(t) = \begin{cases} t, & 0 \le t \le 1, \\ 2 - t, & 1 \le t \le 2, \\ 0, & t \notin [0,2]. \end{cases}$$

Put $h = \dfrac{b-a}{2n}$. Consider the sums

$$S_h(\xi) = \sum_{j=-1}^{2n-1} \omega_0\left(\frac{\xi - a}{h} - j\right) p\big(s, h(j+1)+a\big)\, p\big(h(j+1)+a, t\big).$$

For any h the sum $S_h(\xi)$ is a mapping of the closed interval $[a,b]$ to $L_2([a,b]\times[a,b])$.

Theorem. *If $h \to 0$, then*

$$\int_a^b S_h(\xi)\, d\xi \longrightarrow \int_a^b p(s,\xi)p(\xi,t)\, d\xi = p_2(s,t)$$

in the space $L_2([a,b]\times[a,b])$ and the following estimate is valid

$$\left\| \int_a^b S_h(\xi)\, d\xi - p_2(s,t) \right\|_2 \le \frac{(b-a)^2}{2n} \operatorname*{vrai\,sup}_{\xi\in[a,b]} \|f'(\xi)\|_2.$$

Here norm in the spaces $L_2[a,b]$ and $L_2([a,b]) \times L_2([a,b])$ is denoted $\|\cdot\|_2$.

Proof. We have

$$\int_a^b \|f(\xi) - S_h(\xi)\|_2\, d\xi = \sum_{k=0}^{2n-1} \int_{a+kh}^{a+(k+1)h} \|f(\xi) - S_h(\xi)\|_2\, d\xi.$$

On the closed interval $[a+kh, a+(k+1)h]$ the following equality holds

$$S_h(\xi) = f(a+kh) + \frac{(\xi - (a+kh))}{h}\big(f(a+(k+1)h) - f(a+kh)\big).$$

Denote

$$E(s,t) = \begin{cases} 0, & s < t, \\ 1, & s \ge t. \end{cases}$$

Then

$$f(s) - f(a+kh) = \int_{a+kh}^{a+(k+1)h} E(s,t) f'(t)\, dt.$$

Hence we obtain $(a+kh \le s \le a+(k+1)h)$:

$$f(s) - S_h(s) = \int_{a+kh}^{a+(k+1)h} \left(E(s,t) - \frac{(s-(a+kh))}{h}\right) f'(t)\, dt.$$

Note that if $a + kh \leq s \leq a + (k+1)h$, then the value

$$E(s,t) - \frac{(s-(a+kh))}{h}$$

is between -1 and $+1$. So the following estimate holds

$$\|f(s) - S_h(s)\|_2 \leq h \cdot \operatorname*{vrai\ sup}_{\xi \in [a,b]} \|f'(\xi)\|_2. \tag{8}$$

Therefore

$$\int\limits_{a+kh}^{a+(k+1)h} \|f(s) - S_h(s)\|_2 \, ds \leq h^2 \cdot \operatorname*{vrai\ sup}_{\xi \in [a,b]} \|f'(\xi)\|_2.$$

Summing (8) we obtain

$$\sum_{k=0}^{2n-1} \int\limits_a^b \|f(s) - S_h(\xi)\|_2 \, ds \leq \frac{(b-a)^2}{2n} \operatorname*{vrai\ sup}_{\xi \in [a,b]} \|f'(\xi)\|_2,$$

since $2n \cdot h^2 = \dfrac{(b-a)^2}{2n}$. In particular,

$$\int\limits_a^b S_h(\xi)\, d\xi \longrightarrow \int\limits_a^b f(\xi)\, d\xi = h_2(s,t)$$

in the space $L_2([a,b] \times [a,b])$ and the following estimate holds

$$\left\| \int\limits_a^b f(\xi)\, d\xi - \int\limits_a^b S_h(\xi)\, d\xi \right\|_2 \leq \frac{(b-a)^2}{2n} \operatorname*{vrai\ sup}_{\xi \in [a,b]} \|f'(\xi)\|_2.$$

The theorem is proved.

Denote

$$c_j - \int\limits_a^b \omega_0 \left(\frac{\xi - a}{h} - j \right) d\xi \geq 0.$$

It follows from the prove of the theorem that the sequence

$$P_n(s,t) = \sum_{j=-1}^{2n-1} c_j p(s, h(j+1)+a) \cdot p(h(j+1)+a, t) \tag{9}$$

converges to $p_2(s,t)$ in the space $L_2([a,b] \times [a,b])$. The similar statement is also valid for the sequence

$$Q_n(s,t) = \sum_{j=-1}^{2n-1} c_j q(s, h(j+1)+a) \cdot q(h(j+1)+a, t). \tag{10}$$

Further we use the following simplified notation

$$P_n(s,t) = \sum_{i=1}^{n} c_j p(s,\tau_i) \cdot p(\tau_i,t),$$

where $\tau_i,\ i = 1,\dots,n$ are some points of the closed interval $[a,b]$. Formulas (9), (10) determine the structure of finite-dimensional approximations of the kernel $p_2(s,t)$ (respectively, the kernel $q_2(s,t)$) that will be used in the this article. Without loss of generality, we assume that the sequences of functions $p(s,\tau_1),\dots,p(s,\tau_n)$ and $q(s,\tau_1),\dots,q(s,\tau_n)$ are linearly independent over the field of complex numbers. The linear hulls of the sequences of functions $p(s,\tau_1),\dots,p(s,\tau_n)$ and $q(s,\tau_1),\dots,q(s,\tau_n)$ are denoted $M_p^{(n)}$ and $M_q^{(n)}$, respectively. A basis in the finite-dimensional subspaces $M_p^{(n)}$ and $M_q^{(n)}$ is found by means of the Schmidt process.

Construction of the orthonormal basis in $\mathbf{M_p^{(n)}}$

Consider only the subspace $M_p^{(n)}$. For simplification of the subsequent formulas we use the following notation:

$$U(s,\tau_i) = \sqrt{c_j}\, p(s,\tau_i), \quad i = 1,\dots,n,\ s \in [a,b].$$

We use the Schmidt process for the construction of an orthonormal basis in the subspace $M_p^{(n)}$. Recall the basic steps of this construction. Suppose $a_{ik} = 0$ for $k > i$ and select $a_{ik} = 0$ for $k \le i$ so that the functions

$$\begin{aligned}
&\varphi_1(s) = a_{11}U(s,\tau_1),\\
&\varphi_2(s) = a_{21}U(s,\tau_1) + a_{22}U(s,\tau_2),\\
&\quad\vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots\\
&\varphi_n(s) = a_{n1}U(s,\tau_1) + a_{n2}U(s,\tau_2) + \ldots + a_{nn}U(s,\tau_n).
\end{aligned} \tag{11}$$

are orthogonal and normalized. Obviously, $a_{ii} \neq 0$ for any $1 \le i \le n$. The reversion of the equalities (11) shows that the functions $\varphi_i(s)$ are orthogonal to the functions $U(s,\tau_k)$ if $k < i$. If we require the orthogonality of the functions $\varphi_i(s)$ to the functions $U(s,\tau_1), U(s,\tau_2),\dots,U(s,\tau_{i-1})$, then according to (11) they are orthogonal to each other. For $k = 1,2,\dots,i-1$ we have

$$0 = \int_a^b U(s,\tau_k)\overline{\varphi_i(s)}\,ds = \int_a^b U(s,\tau_k)\left(\sum_{j=1}^{i} \overline{a_{ij}}\,\overline{U(s,\tau_j)}\right) ds =$$

$$= \sum_{j=1}^{i} \overline{a_{ij}} \int_a^b U(s,\tau_k)\overline{U(s,\tau_j)}\,ds = \sum_{j=1}^{i} \overline{a_{ij}}\, g_{kj},$$

where

$$g_{kj} = \int_a^b U(s,\tau_k)\overline{U(s,\tau_j)}\,ds$$

are the elements of the Gramm matrix for the system of functions $U(s,\tau_1),\ldots,U(s,\tau_n)$. Hence, the collection of numbers $\left\{\overline{a_{ij}}\right\}$, $j=1,2,\ldots,i-1$, is the solution $\{x_j\}$ of the following system of $(i-1)$ equations with i unknowns:

$$\sum_{j=1}^{i} g_{kj}x_j = 0 \quad (k=1,2,\ldots,i-1). \tag{12}$$

The system of equations (12) has the rank $i-1$ according to the linear independence of the functions $U(s,\tau_1),\ldots,U(s,\tau_n)$. Therefore one can write

$$x_j = \lambda\mu_j, \quad j=1,2,\ldots,i.$$

Hence

$$a_{ij}x_j = \overline{\lambda}\cdot\overline{\mu}_j \quad j=1,2,\ldots,i,$$

where λ is an arbitrary complex number, and μ_j is the coefficient at the variable ξ_j in the expansion of the determinant

$$\begin{vmatrix} \xi_1 & \xi_2 & \ldots & \xi_i \\ g_{11} & g_{12} & \ldots & g_{1i} \\ \vdots & \vdots & & \vdots \\ g_{i-1,1} & g_{i-1,2} & \ldots & g_{i-1,i} \end{vmatrix}$$

by the first row. If we write the same for all $i=1,2,\ldots,n$, then we add the upper index i, and obtain

$$\begin{aligned} \varphi_1(s) &= \overline{\lambda^{(1)}}\,\overline{\mu_1^{(1)}}U(s,\tau_1), \\ \varphi_2(s) &= \overline{\lambda^{(2)}}\,\overline{\mu_1^{(2)}}U(s,\tau_1) + \overline{\lambda^{(2)}}\,\overline{\mu_2^{(2)}}U(s,\tau_2), \\ &\vdots \\ \varphi_n(s) &= \overline{\lambda^{(n)}}\,\overline{\mu_1^{(n)}}U(s,\tau_1) + \overline{\lambda^{(n)}}\,\overline{\mu_2^{(n)}}U(s,\tau_2) + \ldots + \overline{\lambda^{(n)}}\,\overline{\mu_n^{(n)}}U(s,\tau_n). \end{aligned}$$

Obviously, $\mu_i^{(i)} \neq 0$, $i=1,2,\ldots,n$, and the functions $\varphi_1(s),\ldots,\varphi_n(s)$ are pairwise orthogonal. If it is necessary, the functions can be normalized. In this case the numbers $\lambda^{(i)}$, $i=1,2,\ldots,n$, should be chosen according to the following conditions

$$\left|\lambda^{(i)}\right|^2 \int_a^b \left|\sum_{k=1}^{i} \overline{\mu_k^{(i)}}U(s,\tau_k)\right|^2 ds = 1.$$

They may be arbitrary numbers laying on the circle with the center at the point of origin and of the radius

$$\frac{1}{\left(\int\limits_a^b \left|\sum\limits_{k=1}^{i} \overline{\mu_k^{(i)}}U(s,\tau_k)\right|^2 ds\right)^{\frac{1}{2}}}.$$

We choose

$$\lambda^{(i)} = \frac{(-1)^{i+1}}{\left(\int\limits_a^b \left| \sum\limits_{k=1}^{i} \overline{\mu_k^{(i)}} U(s,\tau_k) \right|^2 ds \right)^{\frac{1}{2}}} \qquad (i = 1, 2, \ldots, n).$$

Taking into account the equalities

$$\mu_i^{(i)} = (-1)^{i+1} \begin{vmatrix} g_{11} & g_{11} & \cdots & g_{1,i-1} \\ g_{21} & g_{22} & \cdots & g_{2,i-1} \\ \vdots & \vdots & & \vdots \\ g_{i-1,1} & g_{i-1,2} & \cdots & g_{i-1,i-1} \end{vmatrix}$$

we obtain

$$a_{ii} = \frac{\begin{vmatrix} g_{11} & g_{11} & \cdots & g_{1,i-1} \\ g_{21} & g_{22} & \cdots & g_{2,i-1} \\ \vdots & \vdots & & \vdots \\ g_{i-1,1} & g_{i-1,2} & \cdots & g_{i-1,i-1} \end{vmatrix}}{\left(\int\limits_a^b \left| \sum\limits_{k=1}^{i} \overline{\mu_k^{(i)}} U(s,\tau_k) \right|^2 ds \right)^{\frac{1}{2}}} > 0 \qquad (i = 1, 2, \ldots, n).$$

Let A_n be the matrix with elements $\{a_{ij}\}$. It is a lower triangular matrix. Denote the inverse matrix $B_n = \{b_{ij}\}$. The matrix B_n is lower triangular also.

Persistent approximations of the operators $(\mathbf{I} - \mathbf{P^2})^+$ and $(\mathbf{I} - \mathbf{Q^2})^+$

Consider the operator $I - P^2$. Represent it in the form

$$I - P^2 = (I - P_n^{(2)}) + \Delta_n,$$

where

$$(P_n^{(2)}\varphi)(s) = \int\limits_a^b P_n(s,t)\varphi(t)\,dt$$

and

$$(\Delta_n\varphi)(s) = \int\limits_a^b \big(P_n(s,t) - p_2(s,t)\big)\varphi(t)\,dt.$$

The representation of the operator $I - Q^2$ can be written similarly. All results which are obtained below remain valid for the operator $I - Q^2$ also.

In the representation $I - P^2 = (I - P_n^{(2)}) + \Delta_n$ the operator Δ_n can be considered as small value since the norm of this operator can be made less than any given number $\varepsilon > 0$ if the number n is sufficiently large. It is known that for sufficiently large n the invertibility of the operator $I - P^2$ is equivalent to the invertibility of the operator $I - P_n^{(2)}$. This property is called the persistency of the operator $\left(I - P^2\right)^{-1}$ with respect to perturbations with small norm. If $\lambda \neq 0$ is not a characteristic number of the operator K, then this property is used for reducing the equation (2) to the system of algebraic equations of finite order. If $\lambda \neq 0$ is a characteristic number of the operator K, then $I - \lambda K$ is not invertible and, hence, the operator $I - P^2$ is not invertible too. In this case one should investigate the persistency of the operator $\left(I - P^2\right)^+$ with respect to perturbations with small norm.

The operator V^+ is said to be persistent with respect to perturbations $W \in \mathcal{L}(L_2[a,b], L_2[a,b])$ with small norm if V^+ is bounded and $(V+W)^+ \in D(V^+, \varepsilon)$ provided $W \in D(0,\delta)$. Here $D(V,r) = \left\{C \in \mathcal{L}(L_2[a,b], L_2[a,b]) : |||V - C|||_2 < r\right\}$.

Generally a pseudo-inversed operator can not be persistent (see [13]). It is also obvious that the operator $\left(I - P^2\right)^+$ is not persistent in sense of the above definition. This negative result means that one should reduce the class of possible perturbations. For obtaining some concrete results we use the concept of the spread of two subspaces of Hilbert space H (see, for example, [14]).

The spread of two closed subspaces R_1 and R_2 is the value

$$\theta(R_1, R_2) = \max\left\{|||(1 - P_2)P_1|||_H\,, |||(1 - P_1)P_2|||_H\right\}, \tag{13}$$

where P_1 and P_2 are the orthogonal projectors on R_1 and R_2, respectively. The spread is called acute if $\theta(R_1, R_2) < 1$. It is obvious that always $0 \leq \theta(R_1, R_2) \leq 1$, and $\theta(R_1, R_2) = 0$ if and only if $R_1 = R_2$. The spread can be defined in a different way also as follows:

$$\theta(R_1, R_2) = |||P_1 - P_2|||_H\,.$$

The spread of the operators $V,\ U \in L(H,H)$ is said to be acute if the spread of the subspaces $\overline{R(V)},\ \overline{R(U)}$ and $\overline{R(V^*)},\ \overline{R(U^*)}$ is acute.

Denote

$$M_V = \left\{W \in \mathcal{L}(L_2[a,b], L_2[a,b]) : V + W \text{ and } V \text{ form an acute spread}\right\}.$$

It is known that the operator U^+ is persistent with respect to perturbations from M_U if and only if U^+ is normally solvable (see, for example, [13]).

By definition the operator U^+ is persistent with respect to perturbations from M_U if U^+ is bounded and $(U+W)^+ \in D(U^+, W)$ provided $W \in D(0,\delta) \bigcap M_U$.

The operators $I - P^2$ and $I - P_n^{(2)}$ are normally solvable and selfadjoint. In order to determine if the operators form an acute angle it is sufficient to estimate the norm of the difference of the orthogonal projectors $P_{R(I-P^2)}$ and $P_{R\left(I-P_n^{(2)}\right)}$ onto the range of values of the operators $I - P^2$ and $I - P_n^{(2)}$, respectively. As it follows from [15], the following estimate is valid

$$\left\| P_{R(I-P^2)} - P_{R\left(I-P_n^{(2)}\right)} \right\|_2 \leq$$
$$\leq \|\Delta_n\|_2 \cdot \max\left\{ \left\| \left(I-P^2\right)^+ \right\|_2, \left\| \left(I-P_n^{(2)}\right)^+ \right\|_2 \right\}. \tag{14}$$

Let $\nu_{(1-P^2)}$ and $\nu_{\left(1-P_n^{(2)}\right)}$ be the reduced minimum modules of the operators $I-P^2$ and $I-P_n^{(2)}$, respectively, then the estimate (14) can be rewritten in the form

$$\left\| P_{R(I-P^2)} - P_{R\left(I-P_n^{(2)}\right)} \right\|_2 \leq$$
$$\leq \|\Delta_n\|_2 \cdot \max\left\{ \frac{1}{\nu_{(I-P^2)}}, \frac{1}{\nu_{\left(I-P_n^{(2)}\right)}} \right\}. \tag{15}$$

Lemma 1. *Let* $\|\Delta_n\|_2 < \eta,\ 0 < \eta < \frac{1}{2}\nu_{(I-P^2)}$, *then*

$$\left| \nu_{\left(I-P_n^{(2)}\right)} - \nu_{(I-P^2)} \right| \leq \eta.$$

Proof. It can be obtained from [12] immediately. The lemma is proved.

Lemma 2. *For sufficiently large n the operators $I-P^2$ and $I-P_n^{(2)}$ form an acute spread.*

Proof. By Lemma 1 there exists a constant $c > 0$ not depending on n such that

$$\frac{1}{\nu_{(I-P^2)}} \leq c.$$

It follows from the estimate (15) that there exists some constant $c_1 > 0$ such that

$$\left\| P_{R(I-P^2)} - P_{R\left(I-P_n^{(2)}\right)} \right\|_2 \leq c_1 \|\Delta_n\|_2 .$$

For all n such that $c_1 \|\Delta_n\|_2 < 1$, the operators $I-P^2$ and $I-P_n^{(2)}$ form an acute spread. The lemma is proved.

It follows from the formula (15) that the operators $I-P^2$ and $I-P_n^{(2)}$ form an acute spread if

$$\|\Delta_n\|_2 < \min\left\{ \nu_{(I-P^2)}, \nu_{\left(I-P_n^{(2)}\right)} \right\}.$$

If the operators $I-P^2$ and $I-P_n^{(2)}$ form an acute spread, then (see, for example, [13]):

$$\left|\left|\left|\left(I-P^2\right)^+ - \left(I-P_n^{(2)}\right)^+\right|\right|\right|_2 \le \frac{1+\sqrt{5}}{2}\cdot\frac{1}{\nu_{(I-P^2)}}\cdot\frac{1}{\nu_{\left(I-P_n^{(2)}\right)}}\cdot|||\Delta_n|||_2\,. \qquad (16)$$

Within the discrepancy given in the inequality (16), the operator $\left(I-P^2\right)^+$ can be replaced by the operator $\left(I-P_n^{(2)}\right)^+$. Below the construction of the operator $\left(I-P_n^{(2)}\right)^+$ and the corresponding approximations are realized.

Constructions

By definition

$$\left(P_n^{(2)}\varphi\right)(s) = \int_a^b P_n(s,t)\varphi(t)\,dt = \sum_{i=1}^n U(s,\tau_i)\int_a^b U(\tau_i,t)\varphi(t)\,dt.$$

Therefore

$$\left(\left(I-P_n^{(2)}\right)(\varphi)\right)(s) = \varphi(s) - \sum_{i=1}^n U(s,\tau_i)\int_a^b U(\tau_i,t)\varphi(t)\,dt.$$

Denote P_{M_P} and $P_{M_P^\perp}$ the orthogonal projectors on the subspace $M_p^{(n)}$ and the orthogonal complement to $M_P^{(n)}$, respectively, then

$$1-P_n^{(2)} = \left(P_{M_P}-P_n^{(2)}\right) + P_{M_P^\perp}.$$

We have

$$\left(1-P_n^{(2)}\right)^+ = \left(P_{M_P}-P_n^{(2)}\right)^+ + P_{M_P^\perp}.$$

Thus the problem of pseudo-inversion of the operator $1-P_n^{(2)}$ is reduced to the problem of pseudo-inversion of the operator $P_{M_P}-P_n^{(2)}$ acting in the finite-dimensional subspace $M_p^{(n)}$. We use the notation M_p instead of $M_p^{(n)}$. In the constructed above basis $\varphi_1(s),\ldots,\varphi_n(s)$ we obtain the representation

$$\left(\left(P_{M_P}-P_n^{(2)}\right)(\varphi)\right)(s) = \left(P_{M_P}(\varphi)\right)(s) -$$
$$-\sum_{i=1}^n\sum_{j=1}^n\left(\sum_{k=1}^n \overline{b_{ki}}\,b_{kj}\right)\varphi_j(s)\int_a^b \overline{\varphi_i(t)}\varphi(t)\,dt.$$

It is necessary to find the matrix pseudo-inversed to the matrix B_n (we accept the shortened notation $B_n = B$):

$$\left\{ \delta_{ij} - \sum_{k=1}^{n} \overline{b_{ki}} b_{kj} \right\},$$

where δ_{ij} is the Kronecker delta. This matrix can be written in the form $E - B^*B$, where E is the identity. The eigenvalues of the matrix B^*B can be found by the formula

$$\lambda_i(B^*B) = \frac{\int\limits_a^b \left| \sum\limits_{k=1}^{i} \overline{\mu_k^{(i)}} U(s,\tau_k) \right|^2 ds}{\begin{vmatrix} g_{11} & g_{11} & \cdots & g_{1,i-1} \\ g_{21} & g_{22} & \cdots & g_{2,i-1} \\ \vdots & \vdots & & \vdots \\ g_{i-1,1} & g_{i-1,2} & \cdots & g_{i-1,i-1} \end{vmatrix}^2}, \qquad 1 \le i \le n.$$

In particular, it follows that

$$\nu_{\left(I - P_n^{(2)}\right)} = \min_{\substack{\lambda_i(B^*B) \neq 1, \\ 1 \le i \le n}} \left| I - \lambda_i(B^*B) \right|.$$

by the spectral theorem for self-adjoint operators.

The following formula is valid

$$\left(\left(P_{M_P} - P_n^{(2)} \right)^+ (\varphi) \right)(s) = \sum_{i=1}^{n} \left\langle \varphi, \sum_{k=1}^{n} (E - BB^*)^+_{ik} \, \varphi_k \right\rangle \varphi_i(s), \tag{17}$$

where $(E - B^*B)^+$ is the matrix pseudo-inversed to the matrix $(E - B^*B)$. In the formula (17) one can use the system of functions $p(s,\tau_1), \ldots, p(s,\tau_n)$.

The formula for the operator $\left(P_{M_Q} - Q_n^{(2)} \right)^+$ is obtained analogously to the formula (17).

Denote $(I - \lambda K)^+_{approxim}$ the operator which is obtained by replacing in formula (5) the operator $\left(1 - P^2\right)^+$ by the operator $\left(P_{M_P} - P_n^{(2)} \right)^+ + P_{M_P^\perp}$. The following estimate of the approximation is valid

$$\left|\left|\left| (I - \lambda K)^+ - (I - \lambda K)^+_{approxim} \right|\right|\right|_2 \le \left|\left|\left| I - \overline{\lambda} K^+ \right|\right|\right|_2 \cdot |||I + P|||_2 \, \delta_n,$$

where

$$\delta_n = \left|\left|\left| \left(I - P^2\right)^+ - \left(I - P_n^{(2)}\right)^+ \right|\right|\right|_2 .$$

The estimate of the value δ_n follows from (16).

By definition of pseudo-inversed operator and the formula (5) it follows that the operator

$$I-\left(I-P^2\right)^+\left(I-P^2\right) \tag{18}$$

is the orthogonal projection onto the kernel of the operator $I-\lambda K$, and by the formula (6) it follows that the operator

$$I-\left(I-Q^2\right)\left(I-Q^2\right)^+ \tag{19}$$

is the orthogonal projection onto the orthogonal complement to the range of values of the operator $I-\lambda K$.

So, the approximate normal pseudo-solution of the equation (2) can be found by the formula

$$\varphi^* = (I-\lambda K)^+_{approxim}(f).$$

Take the operator

$$I-\left(I-P_n^{(2)}\right)^+\left(I-P_n^{(2)}\right). \tag{20}$$

as an approximation to the projector (18).

The norm of the difference between (18) and (20) can be estimated as follows:

$$\begin{gathered}\left|\!\left|\!\left|\left(I-P^2\right)^+\left(I-P^2\right)-\left(I-P_n^{(2)}\right)^+\left(I-P_n^{(2)}\right)\right|\!\right|\!\right|_2 \le \frac{|\!|\!|\Delta_n|\!|\!|_2}{\nu_{\left(I-P_n^{(2)}\right)}}+\\ +\max_{1\le i\le n}|I-\lambda_i(B^*B)|\,\frac{(1+\sqrt{5})}{2}\cdot\frac{1}{\nu_{(I-P^2)}}\cdot\frac{1}{\nu_{\left(I-P_n^{(2)}\right)}}\cdot|\!|\!|\Delta_n|\!|\!|_2+\\ +\frac{(1+\sqrt{5})}{2}\cdot\frac{1}{\nu_{(I-P^2)}}\cdot\frac{1}{\nu_{\left(I-P_n^{(2)}\right)}}\cdot|\!|\!|\Delta_n|\!|\!|_2\,.\end{gathered} \tag{21}$$

The estimate (21) shows that the dimension of the kernel of the operator $I-\lambda K$ coincides with the dimension of a basis of the fundamental system of solutions of the equation

$$\left(P_{M_P}^{(2)}-P_n^{(2)}\right)^+\left(P_{M_P}^{(2)}-P_n^{(2)}\right)(\varphi)=P_{M_P}^{(2)}(\varphi), \tag{22}$$

if the number n is sufficiently large. The basis of the fundamental system of solutions of the equation (22) can be considered as the basis of the kernel of the operator $I-\lambda K$. In the formula (22) $\varphi=\sum\limits_{i=1}^{n}c_i\varphi_i$. All said above is immediately extended to the orthogonal projector (19).

Final remarks

Different methods of computer-algebraic procedures for the pseudo-inversion of matrices are discussed in the article [17]. Here one can also find a list of literature on this problem. In case of the matrix $E_n-B_n^*B_n$ one can additionally take into account that its eigenvalues are explicitly calculated. Therefore in order to find the operator $(E_n - B_n^* B_n)^+$ it is sufficient to obtain a singular basis of this operator (see, for example, [16]).

For finding approximations to the kernel of the operator $I - \lambda K$ one can find a basis of the homogeneous system of the equations

$$B_n^*(\varphi) = B_n^{-1}(\varphi), \qquad \varphi = \sum_{i=1}^{n} c_i \varphi_i.$$

This system should be solved only in the case when there exists at least one index j, $1 \leq j \leq n$ such that

$$\lambda_j(B_n^* B) = 1.$$

Otherwise the matrix $E_n - B_n^* B_n$ is invertible.

If the operators $I - P^2$ and $I - P_n^{(2)}$ form an acute spread and the operator $I - P_n^{(2)}$ is invertible, then the constant $\dfrac{1+\sqrt{5}}{2}$ in the inequality (16) can be replaced by 1.

The results of this work were reported at the international conference [18].

The description of different methods for solving the equations of the form (1) can be found in [19–26].

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