BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(48), 2005, Pages 72–76
ISSN 1024–7696

Left exact radicals in module categories over principal ideal domains

O. Horbachuk, M. Komarnytskyi, Yu. Maturin

Abstract. In the paper left exact radicals in the category of right modules over a principal ideal domain are described.

Mathematics subject classification: 16S90, 16D90.
Keywords and phrases: Principal ideal domain, left exact radicals, ring.

Throughout the whole text, all rings are considered to be associative with $1 \neq 0$. All modules are unitary right modules. Let A be a ring. The set of all invertible elements of A will be detoted by $U(A)$. The set of all right ideals of A will be denoted by $L_r(A)$.

A right ideal $\mathcal{U}$ of A is similar to a right ideal $\mathcal{G}$ of A if $A/\mathcal{U} \cong \mathcal{A}/\mathcal{G}$. In this case we shall write $\mathcal{U} \sim \mathcal{G}$.

An element $a_1 \in A$ is similar to an element $a_2 \in A$ if $a_1 A \sim a_2 A$. In this case we shall write $a_1 \sim a_2$.

Set for a right ideal I of A and for an element $q \in A$

$$(I : q) := \{a \in A \mid qa \in I\}.$$

Then $(I : q)$ is a right ideal of A.

A radical filter of A is a set $\mathcal{E}$ of right ideals satisfying the following conditions [Mishina A.P, Skorniakov L.A.,1]:

G1. $I \in \mathcal{E},\ J \in L_r(A),\ I \subseteq J \Rightarrow J \in \mathcal{E}.$

G2. $I \in \mathcal{E},\ a \in R \Rightarrow (I : a) \in \mathcal{E}.$

G3. $I \in \mathcal{E},\ J \in L_r(A),\ J \subseteq I \wedge \forall u \in I : (J : u) \in \mathcal{E} \Rightarrow J \in \mathcal{E}.$

Proposition 1 (Cohn P.M., 3). *Let A be a ring and $\mathcal{U}$, $\mathcal{G}$ be right ideals of A. Then $\mathcal{U} \sim \mathcal{G} \Leftrightarrow \exists a \in A : aA + \mathcal{G} = R \wedge \mathcal{U} = (\mathcal{G} : a)$.*

A ring R is said to be a principal ideal domain in case it is an integral domain such that every its right ideal is a right principal ideal and every its left ideal is a left principal ideal.

Let R be a principal ideal domain.

An element $p \in R$ is said to be an atom in case

$$p \neq 0 \wedge p \notin U(R) \wedge (\forall a, b \in R : (p = ab \Rightarrow a \in U(R) \vee b \in U(R))).$$

The set of all atoms of R will be denoted by Ω_R.

A set $\mathcal{P} \subseteq R$ is said to be similarly-closed in case

$$\forall p \in \mathcal{P}\ \forall r \in R: \ r \sim p \Rightarrow r \in \mathcal{P}.$$

Proposition 2. *Let R be an integral domain, $\{a, b\} \subseteq R$. If $aR \cap bR = cR$ then there exists only one element $d \in R$ such that $bd = c$. Moreover, in this case $(aR : b) = dR$.*

Proof. Since $aR \cap bR = cR$, there exist elements $d, e \in R : ae = c, bd = c$. The uniqueness of the element $d \in R$ for which $bd = c$ follows from the fact that R is an integral domain.

We shall prove that $(aR : b) = dR$. It is clear that $\forall r \in R : b(dr) = (bd)r = cr = (ae)r = a(er) \in aR$. Therefore $dR \subseteq (aR : b)$.

Conversely, $\forall r \in (aR : b) : br \in aR \cap bR = cR = (bd)R$. Hence $\exists t \in R : br = bdt$. Since R is an integral domain, from the last equality we obtain $r = dt \in dR$. Therefore $(aR : b) \subseteq dR$. □

We think that the following Proposition is well-known:

Proposition 3. *Let A be a ring and I be a right ideal of A. Then $\forall w, g \in A : (0 : w) = 0 \Rightarrow (I : g) = (wI : wg)$.*

Proof. $\forall x \in A : x \in (I : g) \Leftrightarrow gx \in I \Leftrightarrow \exists i \in I : gx = i \Leftrightarrow \exists i \in I : wgx = wi \Leftrightarrow (wg)x \in wI \Leftrightarrow x \in (wI : wg)$. □

Proposition 4. *Let R be an integral domain. If $\forall i \in \{1, 2, \ldots, n\} : a_i \in R \backslash \{0\} \wedge a_i R$ is a maximal right ideal of R, $\forall i \in \{1, 2, \ldots, k\} : b_i \in R \setminus \{0\} \wedge b_i R$ is a maximal right ideal of R and $a_1 a_2 \ldots a_n \sim b_1 b_2 \ldots b_k$ then $n = k$ and there exists a bijection $z : \{1, 2, \ldots, n\} \to \{1, 2, ..., k\}$ such that $a_i \sim b_{z(i)}, \forall i \in \{1, 2, ..., n\}$.*

Proof. Set

$$L_s = a_1 a_2 \ldots a_{n-s} R / a_1 a_2 \ldots a_n R, \ s \in \{1, 2, \ldots, n-1\}, \ L_n = R / a_1 a_2 \ldots a_n R,$$

$$M_i = b_1 b_2 \ldots b_{k-l} R / b_1 b_2 \ldots b_k R, \ i \in \{1, 2, \ldots, k-1\}, \ M_k = R / b_1 b_2 \ldots b_k R.$$

It is clear that $L_n \cong M_k$. Since

$$L_{s+1}/L_s \cong R/a_{n-s}R, \ s \in \{1, 2, \ldots, n\}, \ M_{i+1}/M_i \cong R/b_{k-1}R, i \in \{1, 2, \ldots, k\},$$

$L_0 \subseteq L_1 \subseteq \ldots \subseteq L_{n-1} \subseteq L_n$, $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_{k-1} \subseteq M_k$ are composition series for $L_n \cong M_k$. By the Jordan-Hölder Theorem, $n = k$ and there exists a bijection $z : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ such that $L_{s+1}/L_s \cong M_{z(s+1)}/M_{z(s)}$, $\forall s \in \{1, 2, \ldots, n\}$.

Hence $R/a_{n-s}R \cong R/b_{z(n-s)}R$, $\forall s \in \{1, 2, \ldots, n\}$, i.e. $a_{n-s} \sim b_{z(n-s)}$, $\forall s \in \{1, 2, \ldots, n\} J$. □

Proposition 5 (Jacobson N., 2). *Let R be a principal ideal domain. If $a \in R \ \wedge \ a \notin U(R) \ \wedge \ a \neq 0$ then $a = a_1 a_2 \ldots a_n$ for some set $\{a_1, a_2, \ldots, a_n\} \subseteq \Omega_R$.*

Moreover, if $b_1b_2\dots b_n = c_1, c_2\dots c_k$ *and* $\{b_1, b_2, \dots, b_n, c_1, c_2, \dots c_k\} \subseteq \Omega_R$ *then* $n = k$ *and there exists a bijection* $z : \{1, 2, \dots, n\} \to \{1, 2, \dots, k\}$ *such that* $b_i \sim c_{z(i)}$, $\forall i \in \{1, 2, \dots, n\}$.

Now set $\mathcal{E}_{\mathcal{P}} := \{I \in L_{r(R)} \mid (\exists n \in \mathbb{N}\ \exists a_1, a_2, \dots a_n \in \mathcal{P} :\ I = a_1a_2\dots a_nR \vee I = R\}$.

Lemma 1. *Let* R *be a principal ideal domain and let* $\mathcal{P} \subseteq \Omega_R$ *be a similarly-closed set. Then* $\mathcal{E}_{\mathcal{P}}$ *is a radical filter.*

Proof. We shall verify conditions **G1**–**G3** [1].

Gl. Let $I \in \mathcal{E}_{\mathcal{P}}$, $J \in L_r(R)$, $I \subseteq J$. We shall show that $J \in \mathcal{E}_{\mathcal{P}}$.

If $I = R$ then $R = I \subseteq J \subseteq R$. Hence $J = R \in \mathcal{E}_{\mathcal{P}}$. So, consider the case when $I \neq R$. It is clear that $\exists a, b \in R :\ I = aR\ \wedge\ J = bR$, where $a = a_1a_2\dots a_n$, $\{a_1, a_2, ..., a_n\} \subseteq \mathcal{P}$ (see Proposition 5). Since $(a_1a_2\dots, a_nR = aR \subseteq bR$, $\exists r \in R :$ $a_1a_2\dots a_n = br$. Two cases are possible $b \in U(R)\ \vee\ b \notin U(R)$.

If $b \in U(R)$, then $J = bR = R \in \mathcal{E}_{\mathcal{P}}$. So, let $b \notin U(R)$. Taking into consideration that $a_1a_2\dots a_n = br$, by Proposition 5, we have $b = a'_1a'_2\dots a'_m$, where $a'_i \sim a_{k(i)}$, $\forall i \in \{1, 2, \dots, m\}$, $1 \le m \le n$, $k : \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$ is an injection. Since $\mathcal{P}$ is a similarly-closed set, it follows from $a'_i \sim a_{k(i)}$, $\forall i \in \{1, 2, \dots, m\}$ that $\{a'_1, a'_2, \dots, a'_m\} \in \mathcal{P}$. Hence $J = bR = (a'_1a'_2\dots a'_m)R \in \mathcal{E}_{\mathcal{P}}$.

G2. Let $I \in \mathcal{E}_{\mathcal{P}}$, $a \in R$. We shall show that $(I : a) \in \mathcal{E}_{\mathcal{P}}$.

Two cases are possible: $I = R\ \vee\ I \neq R$. In the first case $(I : a) = (R : a) = R \in \mathcal{E}_{\mathcal{P}}$. Consider the second case. It is clear that $I = bR$ for some $b = b_1b_2\dots b_n$, where $\{b_1, b_2, \dots, b_n\} \subseteq \mathcal{P}$. Let $aR + bR = uR$, where $u \in R$. It is obvious that $u \neq 0$, $a = u\alpha$, $b = u\beta$, where $\{\alpha, \beta\} \subseteq R$. Hence $\alpha R + \beta R = R$. By Proposition 3, $(\beta R : \alpha) = (u\beta R : u\alpha) = (bR : a)$. By Proposition 1, $\beta R \sim (\beta R : \alpha)$. It follows from this that $\beta R \sim (bR : a)$. Let $(bR : a) = hR$, where $h \in R$. Hence $\beta \sim h$. Taking into account that $b = b_1, b_2, \dots, b_n$ and $b = u\beta$, by Proposition 5, we have that $\beta = b'_1b'_2\dots b'_q$, where $b'_i \sim b_{z(i)}$, $\forall i \in \{1, 2, \dots, q\}$, $1 \le q \le n$, $z : \{1, 2, \dots, q\} \to \{1, 2, \dots, n\}$ is an injection. Taking into consideration that $\beta \sim h$, by Proposition 4, we have that $h = b''_1b''_2\dots b''_q$, where $b''_i \sim b'_{p(i)}$, $\forall i \in \{1, 2, \dots, q\}$, $p : \{1, 2, \dots, q\} \to \{1, 2, \dots, q\}$ is a bijection.

Hence $b''_i \sim b'_{p(i)} \sim b_{zop(i)}$, $\forall i \in \{1, 2, \dots, q\}$, i.e. $b''_i \sim b_{zop(i)} \in \mathcal{P}$, $\forall i \in \{1, 2, \dots, q\}$. It is clear that $z \circ p$ is an injection. Since $\mathcal{P}$ is a similarly-closed set, $b''_i \in \mathcal{P}$, $\forall i \in \{1, 2, \dots, q\}$. Hence $(bR : a) = hR \in \mathcal{E}_{\mathcal{P}}$, i.e. $(I : a) \in \mathcal{E}_{\mathcal{P}}$.

G3. Let $I \in \mathcal{E}_{\mathcal{P}}$, $J \in L_r(R)$, $J \subseteq I\ \wedge\ \forall u \in I : (J : u) \in \mathcal{E}_{\mathcal{P}}$. We shall show that $J \in e\mathcal{E}_{\mathcal{P}}$.

It is clear that $J = aR$, $I = bR$, where $a \in R$, $b = b_1b_2\dots b_m$, $\{b_1, b_2, \dots, b_m\} \subseteq \mathcal{P}$. It is obvious that $J \cap I = aR \cap bR = aR$. It follows from $aR \subseteq bR$ that $a = bd$ for some $d \in R$. Hence, by Proposition 2 $(J : b) = dR$. Since $(J : b) \in \mathcal{E}_{\mathcal{P}}$, $(J : b) = (d_1d_2\dots d_nR$ and $\{d_1, d_2, \dots, d_n\} \subseteq \mathcal{P}$. Then $dR = (d_1d_2\dots d_n)R$. Hence $d = d_1d_2\dots d_nv$, where $v \in U(R)$. Then $a = bd = b_1b_2\dots b_md_1d_2\dots d_nv$, where

$$\{b_1, b_2, \dots, b_m, d_1, d_2, \dots, d_{n-1}, d_n, v\} \subseteq \mathcal{P}.$$

Hence $J = aR \in \mathcal{E}_{\mathcal{P}}$. □

Lemma 2. *Let R be a principal ideal domain. If $0 \notin \mathcal{E}$ is a radical filter of R then there exists a similarly-closed set $\mathcal{P} \subseteq \Omega_R$ such that $\mathcal{P} = \mathcal{E}_{\mathcal{P}}$.*

Proof. Let $0 \notin \mathcal{E}$ be a radical filter of R . Set $\mathcal{P} := \{a \mid aR \in \mathcal{E} \ \wedge \ a \in \Omega_R\}$. Let $a \sim b$, $a \in \mathcal{P}$. It is obvious that $b \in \Omega_R$. By Proposition 1, $\exists c \in R : bR = (aR : c)$. Thus, taking into consideration $aR \in \mathcal{E}$, by G2, we have $bR \in \mathcal{E}$. Hence $b \in \mathcal{P}$. Therefore $\mathcal{P}$ is a similarly-closed set. We shall show that $\mathcal{E} = \mathcal{E}_{\mathcal{P}}$.

First let us show that $\mathcal{E} \subseteq \mathcal{E}_{\mathcal{P}}$.

Let $I = a_1 a_2 \ldots a_n R \in \mathcal{E}$, where $a_1, a_2, \ldots, a_n \in \Omega_R$. It is clear that $I = a_1 a_2 \ldots a_n R \subseteq a_1 a_2 \ldots a_i R$, $\forall i \in \{1, 2, \ldots, n\}$. By **Gl**, $a_1 a_2 \ldots a_i R \in \mathcal{E}$, $\forall i \in \{1, 2, \ldots, n\}$. It is obvious that $a_1 a_2 \ldots a_i R \subseteq a_1 a_2 \ldots a_{i-1} R$, $\forall i \in \{1, 2, \ldots, n\}$ and $(a_1 a_2 \ldots a_{i-1}) a_i = a_1 a_2 \ldots a_i$, $\forall i \in \{1, 2, \ldots, n\}$. Then by Proposition 2, $(a_1 a_2 \ldots a_i R : a_1 a_2 \ldots a_{i-1}) = a_i R$, $\forall i \in \{1, 2, \ldots, n\}$. By **G2**, $a_i R = (a_1 a_2 \ldots a_i R : a_1 a_2 \ldots a_{i-1}) \in \mathcal{E}$, $\forall i \in \{1, 2, \ldots, n\}$. Hence $a_i \in \mathcal{P}$, $\forall i \in \{1, 2, \ldots, n\}$. Therefore $I = a_1 a_2 \ldots a_n R \in \mathcal{E}_{\mathcal{P}}$.

Now we shall show that $\mathcal{E}_{\mathcal{P}} \subseteq \mathcal{E}$.

We shall prove by induction in n that $a_1 a_2 \ldots a_n R \in \mathcal{E}$ for every set $\{a_1, a_2, \ldots a_n\} \subseteq \mathcal{P}$.

At $n = 1$ it is clear by the definition of $\mathcal{P}$.

Suppose $\forall n \leq k : a_1 a_2 \ldots a_n R \in \mathcal{E}$, where $k \geq 1$.

We shall prove the statement for $n = k + 1$.

Since $a_{k+1} R \in \mathcal{E}$, by Proposition 3 and by **G2**,

$$\forall r \in R : (a_1 a_2 \ldots a_{k+1} R : \ a_1, a_2 \ldots a_k r) = (a_{k+1} R : r) \in \mathcal{E}.$$

By the induction hypothesis, $a_1 a_2 \ldots a_k R \in \mathcal{E}$. Since $\forall r \in R : \ (a_1 a_2 \ldots a_{k+1} R : a_1 a_2 \ldots a_k r) \in \mathcal{E}$ and $a_1 a_2 \ldots a_{k+1} R \subseteq a_1 a_2 \ldots a_k R \in \mathcal{E}$, by **G3**, $a_1 a_2 \ldots a_{k+1} R \in \mathcal{E}$. □

A left exact radical t in the category of right modules over a ring A assigns to each A-module C a submodule $t(C)$ in such a way that the following conditions are fulfilled:

- for every A - homomorphism $\mu : \ C \to D \quad \mu(t(C) \subset t(D)$,
- for every right A-module $M \quad t(M/t(M)) = 0$,
- for every right A-module M and for every its submodule $N \quad t(N) = t(M) \cap N$.

By Lemmas 1-2, from the Gabriel-Maranda Theorem [1] we obtain the following result:

Theorem 1. *Over a principal ideal domain R there exists a bijective correspondence between left exact radicals $\neq 1$ in the category of right R - modules and similarly-closed subsets of Ω_R.*

Corollary 1. *There is only one non-trivial left exact radical in the category of right R-modules over a principal ideal domain R if and only if $|\Omega_R| \neq 0$ and all atoms in R are similar.*

By Corollary 1, taking into account the results obtained in [5] we have

Corollary 2. *Let $R = k[y, D]$, where k is a universal differential field with derivation D. Then there is only one non-trivial left exact radical in the category of right R-modules. Moreover, it coincides with the right socle and for every right R-module M there exists a submodule H of M such that $M = \mathrm{soc}(M) \oplus H$.*

References

[1] Mishina A.P., Skorniakov L.A. *Abelian groups and modules.* Providence, Rhode Island: Amer. Math. Soc. Trans., series 2, vol. 107, 1976.

[2] Jacobson N. *The theory of rings.* Surveys of the Amer. Math. Soc., 1942, **2**.

[3] Cohn P.M. *Free rings and their relations.* Academic Press, London–New York, 1971.

[4] Horbachuk O., Komarnytskyi M. *Radical filters in principal ideal domains.* Dopovidi Akademii Nauk URSR, Fiz.-Mat. Ta Tehn. Nauky, 1977, **2**, A, p. 103–104.

[5] Cozzens J.H. *Homological properties of the ring of differential polynomials.* Bull. Amer. Math. Soc., 1970, **76**, N 1, p. 15–19.

[6] Kashu A.I. *Radicals and torsions in modules.* Kishinev, Stiintsa, 1983 (in Russian)

Received May 27, 2005

O. Horbachuk, M. Komarnytskyi
Department of Mechanics and Mathematics
Lviv National University
Universitetska str. 1
Lviv 79000, Ukraine

Yu. Maturin
Department of Physics, Mathematics
and Computer Sciences
Drohobych State Pedagogical University
Stryjska str. 3
Drohobych 82100, Lvivska oblast, Ukraine