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Lie algebras of the operators and three-dimensional polynomial differential systems

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Abstract. The defining equations are built for the representation of continuous groups in the space of variables and coefficients of multi-dimensional polynomial differential systems of the first order. Lie theorem on integrating factor is obtained for three-dimensional polynomial differential systems and the invariant $GL(3,\mathbb{R})-$integrals are constructed for three-dimensional affine differential system.

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1 The system of defining equations

Consider the multi-dimensional polynomial differential system written in the tensorial form as follows [1]:

$$\frac{dx^j}{dt} = \sum_{k\in A} a^j_{j_1 j_2 \dots j_k} x^{j_1} x^{j_2} \dots x^{j_k} \equiv P^j(x) \quad (j, j_1, j_2, \dots, j_k = \overline{1,n}), \tag{1}$$

where the coefficient tensor $a^j_{j_1 j_2 \dots j_k} (k \in A)$ is symmetrical in lower indices, in which the complete convolution takes place, A is a finite set of the different positive integers, and $x = (x^1, x^2, \dots, x^n)$ is the vector of phase variables of system (1).

Let in the space $E^n(x)$ the continuous group of transformations G_1 acts defined by formulas

$$\overline{x}^j = f^j(x,\alpha) \quad (j = \overline{1,n}), \tag{2}$$

where α is the real parameter taking values in some interval in $\mathbb{R}$, for which holds $\overline{x}^j|_{\alpha=0} = x^j \quad (j = \overline{1,n})$, and the value $\alpha = 0$ with this property is the unique on this interval.

Denote by $E^N(a)$ the space of coefficients of system (1), where N is the dimension of this space, and by a the set of coefficients of the right-hand sides of system (1) is denoted. Assume that after transformation (2) the system (1) does not change its form and new coefficients $\overline{a} \in E^N(a)$ of the system

$$\frac{d\overline{x}^j}{dt} = \sum_{k\in A} \overline{a}^j_{j_1 j_2 \dots j_k} \overline{x}^{j_1} \overline{x}^{j_2} \dots \overline{x}^{j_k} \quad (j, j_1, j_2, \dots, j_k = \overline{1,n})$$

are linear functions on coefficients a and functions on α, i. e.

$$\overline{a}^j_{j_1 j_2 \dots j_k} = b^j_{j_1 j_2 \dots j_k}(a, \alpha) \quad (j, j_1, j_2, ..., j_k = \overline{1, n}). \tag{3}$$

Then the last equalities define the linear group of transformations in the space of coefficients $E^N(a)$ of system (1), homomorphic to the group G_1, or, as they say, (3) defines the linear representation of the group G_1 in the space $E^N(a)$.

According to [2], Lie operator corresponding to the group G_1 and acting in the space $E^{n+N}(x, a) = E^n(x) \oplus E^N(a)$ takes the form

$$X = \xi^j(x) \frac{\partial}{\partial x^j} + D, \quad (j = \overline{1, n}), \tag{4}$$

where

$$D = \sum_{k \in A} \eta^j_{j_1 j_2 \dots j_k}(a) \frac{\partial}{\partial a^j_{j_1 j_2 \dots j_k}} \quad (j, j_1, j_2, ..., j_k = \overline{1, n}; k \in A), \tag{5}$$

for which

$$\xi^j(x) = \frac{\partial \overline{x}^j(x, a)}{\partial \alpha}|_{\alpha=0}, \quad \eta^j_{j_1 j_2 \dots j_k}(a) = \frac{\partial \overline{a}^j_{j_1 j_2 \dots j_k}}{\partial \alpha}|_{\alpha=0} \quad (j, j_1, j_2, ..., j_k = \overline{1, n}; k \in A). \tag{6}$$

The operator X (respectively D) is called the operator of the representation of the group G_1 in the space $E^{n+N}(x, a)$ (respectively $E^N(a)$).

Consider system (1) written in the form

$$U_i(x, a, \dot{x}) = 0 \quad (i = \overline{1, n}), \tag{7}$$

where

$$U_i(x, a, \dot{x}) = \dot{x}^i - P^i(x) \quad (i = \overline{1, n}) \tag{8}$$

and $\dot{x}^i = \dfrac{dx^i}{dt}$.

In this case we will say that (7) forms a manifold in the space $\widetilde{E}^{2n+N}(x, a, \dot{x})$ (for ex., see this definition in [3]). Then the extended group of transformations $\widetilde{G}_1$ in the space $\widetilde{E}^{2n+N}(x, a, \dot{x})$ corresponds to the representation of the group G_1 defined by formulas (2) and (3) in the space $E^{n+N}(x, a)$.

Let the operator of the representation of the group G_1 in the space $E^{n+N}(x, a)$ has the form (4)–(5). Then the operator of the group $\widetilde{G}_1$, called the extended operator of the representation of the group G_1, can be written as follows

$$\widetilde{X} = X + \zeta^i(x, \dot{x}) \frac{\partial}{\partial \dot{x}^i} \quad (i = \overline{1, n}), \tag{9}$$

where

$$\zeta^i(x, \dot{x}) = \frac{\partial \dot{\overline{x}}^i}{\partial \alpha}|_{\alpha=0} = \dot{x}^k \frac{\partial \xi^i}{\partial x^k} \quad (i, k = \overline{1, n}). \tag{10}$$

According to [3] we will say that the system of differential equations admits the group G_1 if its equations define an invariant manifold with respect to the extended group $\widetilde{G}_1$.

For the definition of the group G_1 admitted by system (1) consider, according to [3], necessary and sufficient conditions of invariance of the manifolds $U_i = 0$ $(i = \overline{1,n})$, which have the form

$$\widetilde{X}(U_i)|_{U_i=0} = 0 \quad (i = \overline{1,n}). \tag{11}$$

The equations (11) are called defining equations of the group (of Lie operator) admitted by system (7), or, what is the same, admitted by system (1).

Taking into consideration formulas of the operators X and $\widetilde{X}$ from (4)–(5) and (7)–(10), respectively, with the aid of (11) we obtain

$$\Big[\xi^i(x)\frac{\partial}{\partial x^i} + D + \dot{x}^k\frac{\partial}{\partial x^k}\frac{\partial \xi^i}{\partial x^k}\frac{\partial}{\partial \dot{x}^i}\Big]\Big[\dot{x}^i - P^i(x)\Big]\Big|_{\dot{x}^i=P^i(x)} = 0 \quad (i,k = \overline{1,n}).$$

After the simplification in the last expressions with the aid of convolution by i and j we obtain finally system of defining equations as follows:

$$\xi^i_{x^k}P^k = \xi^j P^i_{x^j} + D(P^i) \quad (i,j,k = \overline{1,n}), \tag{12}$$

where $\xi^i_{x^k} = \dfrac{\partial \xi^i}{\partial x^k}$ and $P^i_{x^j} = \dfrac{\partial P^i}{\partial x^j}$.

The defining equations (12) are differential equations with respect to the function ξ^i. Defining the general solution of this equations we obtain the explicit form and number of Lie operators admitted by the differential system. Hence, by Lie equations we define the widest group and corresponding Lie algebra, admitted by this system.

Remark 1. *The defining equations for the operators of the representation of linear group for system (1) in the case $n = 2$ were obtained in [2].*

2 Lie theorem on integrating factor for system (1) for n = 3

Consider system (1) for $n = 3$ and write it as follows:

$$\frac{dx^j}{dt} = P^j(x) \quad (j = \overline{1,3}). \tag{13}$$

It is well known that $F(x) = C$ is the first integrals of this system iff $\Lambda(F) = 0$, where

$$\Lambda = P^j\frac{\partial}{\partial x^j} \quad (j = \overline{1,3}) \tag{14}$$

and the complete convolution takes place in index j.

The system consisting of two functional-independent first integrals is called *general integral* of system (13).

To obtain the first integrals of system (13) with the aid of equation $\Lambda(F) = 0$ it is useful to write characteristic system of differential equations in the symmetrical form

$$\frac{dx^1}{P^1} = \frac{dx^2}{P^2} = \frac{dx^3}{P^3}. \tag{15}$$

In some cases to find first integrals of system (13) it is useful to obtain from (15) integrating equations of Pfaff [4, 5], which can be written in the form

$$P_1^1 dx^1 + P_1^2 dx^2 + P_1^3 dx^3 = 0, \tag{16}$$

where $P_1^j \ \ (j = \overline{1,3})$ is the function in $P^j \ \ (j = \overline{1,3})$.

Assume that the system (13) admits two-dimensional commutative Lie algebra with operators

$$X_\alpha = \xi_\alpha^j(x)\frac{\partial}{\partial x^j} \quad (\alpha = 1,2; \ \ j = \overline{1,3}), \tag{17}$$

where $\xi_\alpha^j(x) \ \ (j = \overline{1,3})$ are polynomials on coordinates of vector $x = (x^1, x^2, x^3)$.

By action of group, generated by operators (17), any integral $F(x) = C$ transforms into integral $F(\overline{x}) = C'$. So by virtue of the representation of the one-parameter group with exponent reflection [3] the functions $X_\alpha(F) = C_\alpha \ \ (\alpha = 1,2)$ will be integrals too. Hence, beside $F(x)$, the functions $X_\alpha(F)$, where $X_\alpha \ \ (\alpha = 1,2)$ are taken from (17), are also solutions of the system (13).

Then we have

$$X_\alpha(F) = \Psi_\alpha(F) \ \ (\alpha = 1,2), \tag{18}$$

$$\Lambda F = 0, \tag{19}$$

where Λ is from (14). So, we obtain that the integral F from general integral of the system (13) has to satisfy the system of three equations (18)-(19).

Denote by

$$\Delta = \begin{vmatrix} \xi_1^1 & \xi_1^2 & \xi_1^3 \\ \xi_2^1 & \xi_2^2 & \xi_2^3 \\ P^1 & P^2 & P^3 \end{vmatrix} \tag{20}$$

the determinant of system (18)-(19) and assume that it is not equal to zero.

$$\frac{\partial F}{\partial x^1} = \frac{1}{\Delta}\Big[(\xi_2^2 P^3 - \xi_2^3 P^2)\Psi_1 + (\xi_1^3 P^2 - \xi_1^2 P^3)\Psi_2\Big],$$

$$\frac{\partial F}{\partial x^2} = \frac{1}{\Delta}\Big[(\xi_2^3 P^1 - \xi_2^1 P^3)\Psi_1 + (\xi_1^1 P^3 - \xi_1^3 P^1)\Psi_2\Big],$$

$$\frac{\partial F}{\partial x^3} = \frac{1}{\Delta}\Big[(\xi_2^1 P^2 - \xi_2^2 P^1)\Psi_1 + (\xi_1^2 P^1 - \xi_1^1 P^2)\Psi_2\Big].$$

With the aid of last equalities we obtain

$$dF = \frac{\Psi_1\Big[(\xi_2^2 P^3 - \xi_2^3 P^2)dx^1 + (\xi_2^3 P^1 - \xi_2^1 P^3)dx^2 + (\xi_2^1 P^2 - \xi_2^2 P^1)dx^3\Big]}{\Delta} +$$

$$+\frac{\Psi_2\Big[(\xi_1^3 P^2 - \xi_1^2 P^3)dx^1 + (\xi_1^1 P^3 - \xi_1^3 P^1)dx^2 + (\xi_1^2 P^1 - \xi_1^1 P^2)dx^3\Big]}{\Delta}, \tag{21}$$

where Δ is from (20).

As the operators (17) form a commutative Lie algebra, we have $[X_1, X_2](F) = 0$, or $X_1(X_2(F)) - X_2(X_1(F)) = 0$. Hereby taking into account equalities (18) we obtain $X_1(\Psi_2(F)) - X_2(\Psi_1(F)) = 0$, whence taking into account (18) again, two independent solutions of this equation can be written as follows:

1) $\Psi_1(F) \not\equiv 0, \Psi_2 \equiv 0$;

2) $\Psi_1(F) \equiv 0, \Psi_2(F) \not\equiv 0$.

According to this for each case we obtain

$$\frac{dF}{\Psi_1(F)} = \frac{\Big[(\xi_2^2 P^3 - \xi_2^3 P^2)dx^1 + (\xi_2^3 P^1 - \xi_2^1 P^3)dx^2 + (\xi_2^1 P^2 - \xi_2^2 P^1)dx^3\Big]}{\Delta},$$

$$\frac{dF}{\Psi_2(F)} = \frac{\Big[(\xi_1^3 P^2 - \xi_1^2 P^3)dx^1 + (\xi_1^1 P^3 - \xi_1^3 P^1)dx^2 + (\xi_1^2 P^1 - \xi_1^1 P^2)dx^3\Big]}{\Delta}. \qquad (22)$$

As the expressions $\dfrac{dF}{\Psi_1(F)}$ and $\dfrac{dF}{\Psi_2(F)}$ are total differentials then we obtain with the aid of (22) Lie theorem on integrating factor as follows:

Theorem 1. *If three-dimensional differential system (13) admits the two-dimensional commutative Lie algebra with operators (17), then the function* $\mu = \frac{1}{\Delta}$, *where* $\Delta \not\equiv 0$ *has form (20), is an integrating factor for the equations of Pfaff*

$$(\xi_\alpha^3 P^2 - \xi_\alpha^2 P^3)dx^1 + (\xi_\alpha^1 P^3 - \xi_\alpha^3 P^1)dx^2 + (\xi_\alpha^2 P^1 - \xi_\alpha^1 P^2)dx^3 = 0 \quad (\alpha = 1, 2), \quad (23)$$

which define general integral of system (13).

3 First integrals of the affine differential system for $\delta_4 \not\equiv 0$

Consider three-dimensional affine differential system

$$\frac{dx^j}{dt} = a^j + a_\alpha^j x^\alpha \quad (j, \alpha = \overline{1,3}) \qquad (24)$$

and the group of centro-affine transformations $GL(3, \mathbb{R})$ given by the equalities $\overline{x}^r = q_i^r x^i \quad (\Delta = \det(q_i^r) \neq 0; \quad r, i = \overline{1,3})$. In [6] the functional base of centro-affine comitants of system (24) is given as follows:

$$\delta_1 = a^\alpha u_\alpha, \quad \delta_2 = a_\beta^\alpha a^\beta u_\alpha, \quad \delta_3 = a_\gamma^\alpha a_\alpha^\beta a^\gamma u_\beta, \quad \delta_4 = a_\gamma^\alpha a_p^\beta a_q^\gamma u_\alpha u_\beta u_r \varepsilon^{pqr},$$

$$\varkappa_1 = x^\alpha u_\alpha, \quad \varkappa_2 = a_\beta^\alpha x^\beta u_\alpha, \quad \varkappa_3 = a_\gamma^\alpha a_\alpha^\beta x^\gamma u_\beta,$$

$$\theta_1 = a_\alpha^\alpha, \quad \theta_2 = a_\beta^\alpha a_\alpha^\beta, \quad \theta_3 = a_\gamma^\alpha a_\alpha^\beta a_\beta^\gamma, \qquad (25)$$

where coordinates of the vector $u = (u_1, u_2, u_3)$ are varying by the low of covariant vector [7], and ε^{pqr} is unit three-vector with coordinates $\varepsilon^{123} = -\varepsilon^{132} = \varepsilon^{312} = -\varepsilon^{321} = \varepsilon^{231} = -\varepsilon^{213} = 1$ and $\varepsilon^{pqr} = 0 \;\; (p, q, r = \overline{1,3})$ in other cases.

In [6] it is shown that for $\delta_4 \not\equiv 0$ by a centro-affine transformation the system (24) can be transformed into the system

$$\frac{dx^1}{dt} = a + x^2, \quad \frac{dx^2}{dt} = b + x^3, \quad \frac{dx^3}{dt} = c + lx^1 + mx^2 + nx^3, \tag{26}$$

where a, b, c, l, m, n are some parameters. One can verify that for some values of the parameters the system (26) is at the $GL(3, \mathbb{R})-$orbit of maximal dimension [2].

Remark 2. *We will not consider systems (26) if one of the right-hand sides of this system is equal to zero, as in this case system (26) becomes a two-dimensional affine system, which is investigated in [2].*

One can verify with the aid of defining equations (14) that the following assertion holds:

Corollary 1. *The system (26) admits six operators of the representation of one-parameter groups of transformation as follows*

$$Y_1 = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + D_1, \quad Y_2 = x^2\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + (lx^1 + mx^2 + nx^3)\frac{\partial}{\partial x^3} + D_2,$$

$$Y_3 = x^3\frac{\partial}{\partial x^1} + (lx^1 + mx^2 + nx^3)\frac{\partial}{\partial x^2} + [lnx^1 + (l + mn)x^2 + (m + n^2)x^3]\frac{\partial}{\partial x^3} + D_3,$$

$$Y_4 = \frac{\partial}{\partial x^1} + D_4, \quad Y_5 = \frac{\partial}{\partial x^2} + D_5, \quad Y_6 = \frac{\partial}{\partial x^3} + D_6, \tag{27}$$

where

$$D_1 = a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}, \quad D_2 = b\frac{\partial}{\partial a} + c\frac{\partial}{\partial b} + (al + bm + cn)\frac{\partial}{\partial c},$$

$$D_3 = c\frac{\partial}{\partial a} + (la + mb + nc)\frac{\partial}{\partial b} + [aln + b(l + mn) + c(m + n^2)]\frac{\partial}{\partial c},$$

$$D_4 = -l\frac{\partial}{\partial c}, \quad D_5 = -\frac{\partial}{\partial a} - m\frac{\partial}{\partial c}, \quad D_6 = -\frac{\partial}{\partial b} - n\frac{\partial}{\partial c}. \tag{28}$$

The nonzero structure equations which connect mentioned operators up to anti-symmetry are the following

$$[Y_1, Y_4] = -Y_4, \quad [Y_1, Y_5] = -Y_5, \quad [Y_1, Y_6] = -Y_6, \quad [Y_2, Y_4] = -lY_6,$$

$$[Y_2, Y_5] = -Y_4 - mY_6, \quad [Y_2, Y_6] = -Y_5 - nY_6, \quad [Y_3, Y_4] = -lY_5 - lnY_6,$$

$$[Y_3, Y_5] = -mY_5 - (l + mn)Y_6, \; [Y_3, Y_6] = -Y_4 - nY_5 - (m + n^2)Y_6.$$

Calculating the comitants (25) for the system (26) we obtain:

$$\delta_1 = au_1 + bu_2 + cu_3, \quad \delta_2 = bu_1 + cu_2 + (al + bm + cn)u_3,$$

$$\delta_3 = cu_1 + (al + bm + cn)u_2 + [aln + b(l + mn) + c(m + n^2)]u_3,$$

$$\delta_4 = u_1^3 + nu_1^2u_2 + (2m + n^2)u_1^2u_3 - mu_1u_2^2 - (3l + mn)u_1u_2u_3 + (m^2 - 2ln)u_1u_3^2 +$$

$$+ lu_2^3 + lnu_2^2u_3 - lmu_2u_3^2 + l^2u_3^3,$$

$$\varkappa_1 = u_1x^1 + u_2x^2 + u_3x^3, \quad \varkappa_2 = u_1x^2 + u_2x^3 + u_3(lx^1 + mx^2 + nx^3),$$

$$\varkappa_3 = u_1x^3 + u_2(lx^1 + mx^2 + nx^3) + u_3[lnx^1 + (l + mn)x^2 + (m + n^2)x^3],$$

$$\theta_1 = n, \quad \theta_2 = 2m + n^2, \quad \theta_3 = 3l + 3mn + n^3. \tag{29}$$

Consider the first integrals of system (26) for $l = m = n = 0$ and in all cases when $(l^2 + m^2)(l^2 + n^2)(m^2 + n^2) = 0, \quad l^2 + m^2 + n^2 \neq 0$. With the aid of (29) these conditions can be easily written as the following cases:

3.1 Let the conditions $\theta_1 = \theta_2 = \theta_3 = 0$ hold

Then with the aid of (29) we obtain that the system (26) takes the form

$$\frac{dx^1}{dt} = a + x^2, \quad \frac{dx^2}{dt} = b + x^3, \quad \frac{dx^3}{dt} = c. \tag{30}$$

It is easily to verify that for

$$c \neq 0 \tag{31}$$

the system (30) has two functional-independent first integrals as follows

$$F_1 \equiv 2cx^2 - 2bx^3 - (x^3)^2 = C_1, \;\; F_2 \equiv 6c^2x^1 - 6c(a + x^2)x^3 + (3b + x^3)(x^3)^2 = C_2. \tag{32}$$

3.2 Let the conditions $\theta_1 \neq 0, \quad \theta_2 = \theta_1^2, \quad \theta_3 = \theta_1^3$ hold

Then from (29) we have $l = m = 0, \quad n \neq 0$ and the system (26) takes the form

$$\frac{dx^1}{dt} = a + x^2, \quad \frac{dx^2}{dt} = b + x^3, \quad \frac{dx^3}{dt} = c + nx^3. \tag{33}$$

Remark that for

$$c + nx^3 \not\equiv 0 \tag{34}$$

the system (33) has two functional-independent first integrals as follows:

$$F_1 \equiv n^2x^2 - nx^3 + (c - bn)\ln|c + nx^3| = C_1, \quad F_2 \equiv 2n(n^2x^1 - x^3) +$$

$$+ 2(c - an^2 - n^2x^2 + nx^3)\ln|c + nx^3| + (bn - c)\ln^2|c + nx^3| = C_2. \tag{35}$$

If $c + nx^3 \equiv 0$ then the system (33) transforms into the two-dimensional case which we will not consider.

3.3 Let the conditions $\theta_3 = \theta_1 = 0, \quad \theta_2 \neq 0$ hold

Then from (29) we find $l = n = 0, \quad m \neq 0$ and the system (26) takes the form

$$\frac{dx^1}{dt} = a + x^2, \quad \frac{dx^2}{dt} = b + x^3, \quad \frac{dx^3}{dt} = c + mx^2. \tag{36}$$

With the aid of group of transformations corresponding to the operators $Y_5, \quad Y_6$ from (27)–(28) by the change of variables

$$\overline{x}^1 = x^1, \quad \overline{x}^2 = x^2 + \frac{c}{m}, \quad \overline{x}^3 = x^3 + b \tag{37}$$

the system (36) takes the form

$$\frac{d\overline{x}^1}{dt} = \overline{a} + \overline{x}^2, \quad \frac{d\overline{x}^2}{dt} = \overline{x}^3, \quad \frac{d\overline{x}^3}{dt} = m\overline{x}^2 \quad (m \neq 0), \tag{38}$$

where

$$\overline{a} = a - \frac{c}{m}. \tag{39}$$

One can verify that from the second and third equations of (38) we obtain the first integral

$$\overline{F}_1 \equiv (\overline{x}^3)^2 - m(\overline{x}^2)^2 = \overline{C}_1. \tag{40}$$

Proposition 1. *If the condition $(\overline{x}^3)^2 - m(\overline{x}^2)^2 \neq 0$ holds, then the system (38) has, besides (40), first integrals as follows*

$$\overline{F}_2^{(1)} \equiv m\overline{x}^1 - \overline{x}^3 - \overline{a}\sqrt{m}\ln|\overline{x}^3 + \sqrt{m}\overline{x}^2| = \overline{C}_2 \quad \textit{for} \quad m > 0, \quad \overline{x}^3 > 0; \tag{41}$$

$$\overline{F}_2^{(2)} \equiv m\overline{x}^1 - \overline{x}^3 + \overline{a}\sqrt{m}\ln|\overline{x}^3 - \sqrt{m}\overline{x}^2| = \overline{C}_2 \quad \textit{for} \quad m > 0, \quad \overline{x}^3 < 0; \tag{42}$$

$$\overline{F}_2^{(3)} \equiv m\overline{x}^1 - \overline{x}^3 + \overline{a}\sqrt{-m}\arcsin\frac{\overline{x}^2\sqrt{-m}}{\sqrt{(\overline{x}^3)^2 - m(\overline{x}^2)^2}} = \overline{C}_2 \quad \textit{for} \quad m < 0, \quad \overline{x}^3 > 0; \tag{43}$$

$$\overline{F}_2^{(4)} \equiv m\overline{x}^1 - \overline{x}^3 - \overline{a}\sqrt{-m}\arcsin\frac{\overline{x}^2\sqrt{-m}}{\sqrt{(\overline{x}^3)^2 - m(\overline{x}^2)^2}} = \overline{C}_2 \quad \textit{for} \quad m < 0, \quad \overline{x}^3 < 0. \tag{44}$$

Proof. From (40) it follows

$$|\overline{x}^3| = \sqrt{m(\overline{x}^2)^2 + \overline{C}_1}. \tag{45}$$

1) Let $m > 0, \quad \overline{x}^3 > 0$. Then from the first and second equations of (38) we obtain

$$\frac{d\overline{x}^2}{d\overline{x}^1} = \frac{\sqrt{m(\overline{x}^2)^2 + \overline{C}_1}}{\overline{a} + \overline{x}^2}, \tag{46}$$

taking into consideration (38), we obtain the first integral (41).

2) Let $m > 0,\ \ \overline{x}^3 < 0$. Then from the first and second equations of (44) we obtain

$$\frac{d\overline{x}^2}{d\overline{x}^1} = -\frac{\sqrt{m(\overline{x}^2)^2 + \overline{C}_1}}{\overline{a} + \overline{x}^2}, \tag{47}$$

hereby, taking into consideration (40), we obtain the first integral (42).

3) Let $m < 0,\ \ \overline{x}^3 > 0$. Then with the aid of (46) and taking into consideration (40) we obtain the first integral (43).

4) Let $m < 0,\ \ \overline{x}^3 < 0$. Then with the aid of (47) and taking into consideration (40) we obtain the first integral (44). Proposition 1 is proved.

With the aid of expressions (37), (39), (40) and Proposition 1 is proved

Lemma 1. *For $m \neq 0$ the system (36) has the first integral*

$$F_1 \equiv m(x^3 + b)^2 - (mx^2 + c)^2 = C_1, \tag{48}$$

and for $m(x^3 + b)^2 - (mx^2 + c)^2 \not\equiv 0$ beside (48) it has another one first integral among the following four:

$$F_2^{(1)} \equiv m^2x^1 - mx^3 - (am - c)\sqrt{m}\ln|bm + mx^3 + (mx^2 + c)\sqrt{m}| = C_2,$$

$$\textit{for}\ \ m > 0,\ \ x^3 + b > 0; \tag{49}$$

$$F_2^{(2)} \equiv m^2x^1 - mx^3 + (am - c)\sqrt{m}\ln|bm + mx^3 - (mx^2 + c)\sqrt{m}| = C_2,$$

$$\textit{for}\ \ m > 0,\ \ x^3 + b < 0; \tag{50}$$

$$F_2^{(3)} \equiv m^2x^1 - mx^3 - (am - c)\sqrt{-m}\arcsin\frac{mx^2 + c}{\sqrt{(mx^2 + c)^2 - m(x^3 + b)^2}} = C_2,$$

$$\textit{for}\ \ m < 0,\ \ x^3 + b > 0; \tag{51}$$

$$F_2^{(4)} \equiv m^2x^1 - mx^3 + (am - c)\sqrt{-m}\arcsin\frac{mx^2 + c}{\sqrt{(mx^2 + c)^2 - m(x^3 + b)^2}} = C_2,$$

$$\textit{for}\ \ m < 0,\ \ x^3 + b < 0. \tag{52}$$

3.4 Let the conditions $\theta_2 = \theta_1 = 0,\ \ \theta_3 \neq 0$ hold

Then from (29) we find $l \neq 0,\ \ m = n = 0$ and the system (26) takes the form

$$\frac{dx^1}{dt} = a + x^2,\ \ \frac{dx^2}{dt} = b + x^3,\ \ \frac{dx^3}{dt} = c + lx^1. \tag{53}$$

Lemma 2. *The general integral of the system*

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = z, \quad \frac{dz}{dt} = x \tag{54}$$

consists of two first integrals

$$F_1 \equiv x^3 + y^3 + z^3 - 3xyz = C_1, \tag{55}$$

$$F_2 \equiv 2\ln|x+y+z| - \ln|x^2+y^2+z^2-xy-xz-yz| - 2\sqrt{3}\arctan\frac{2x-y-z}{\sqrt{3}(y-z)} = C_2. \tag{56}$$

Proof. Remark that the system (54) admits two-dimensional commutative Lie algebra with operators

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad X_2 = z\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}. \tag{57}$$

Then according to Theorem 1 we obtain the following integrating Pfaff equations:

$$(xy - z^2)dx + (yz - x^2)dy + (xz - y^2)dz = 0, \tag{58}$$

$$(x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz = 0 \tag{59}$$

with integrating factor

$$\mu^{-1} = x^3 + y^3 + z^3 - 3xyz. \tag{60}$$

Making with the aid of (60) some elementary calculation on integrating equations (58)–(59) we obtain the functional-independent first integrals (55)–(56). Lemma 2 is proved.

Consider the system (53). Assume that $a^2 + b^2 + c^2 \neq 0$ and $l \neq 0$, then with the aid of the transformation

$$\overline{x}^1 = x^1 + \frac{c}{l}, \quad \overline{x}^2 = x^2 + a, \quad \overline{x}^3 = x^3 + b \tag{61}$$

we find

$$\frac{d\overline{x}^1}{dt} = \overline{x}^2, \quad \frac{d\overline{x}^2}{dt} = \overline{x}^3, \quad \frac{d\overline{x}^3}{dt} = l\overline{x}^1. \tag{62}$$

It is easy to verify that the system (62) admits the two-dimensional commutative Lie algebra of operators

$$Z_1 = \overline{x}^1\frac{\partial}{\partial\overline{x}^1} + \overline{x}^2\frac{\partial}{\partial\overline{x}^2} + \overline{x}^3\frac{\partial}{\partial\overline{x}^3}, \quad Z_2 = \overline{x}^3\frac{\partial}{\partial\overline{x}^1} + l\overline{x}^1\frac{\partial}{\partial\overline{x}^2} + l\overline{x}^2\frac{\partial}{\partial\overline{x}^3}. \tag{63}$$

Then according to Theorem 1 we obtain the following integrating Pfaff equations:

$$\left[l\overline{x}^1\overline{x}^2 - (\overline{x}^3)^2\right]d\overline{x}^1 + \left[\overline{x}^2\overline{x}^3 - l(\overline{x}^1)^2\right]d\overline{x}^2 + \left[(\overline{x}^1\overline{x}^3 - (\overline{x}^2)^2\right]d\overline{x}^3 = 0, \tag{64}$$

$$\Big[l^2(\overline{x}^1)^2 - l\overline{x}^2\overline{x}^3\Big]d\overline{x}^1 + \Big[l(\overline{x}^2)^2 - l\overline{x}^1\overline{x}^3\Big]d\overline{x}^2 + \Big[(\overline{x}^3)^2 - l\overline{x}^1\overline{x}^2\Big]d\overline{x}^3 = 0. \qquad (65)$$

with the integrating factor

$$\mu^{-1} = l^2(\overline{x}^1)^3 + l(\overline{x}^2)^3 + (\overline{x}^3)^3 - 3l\overline{x}^1\overline{x}^2\overline{x}^3. \qquad (66)$$

Setting in (64)–(65) the notations

$$x = \overline{x}^1\sqrt[3]{l^2}, \quad y = \overline{x}^2\sqrt[3]{l}, \quad z = \overline{x}^3 \qquad (67)$$

we obtain integrating Pfaff equations (58)-(59) with integrating factor (60) and, hence, the first integrals (55)-(56). After inverse change of variables (67) and (61) in the last expressions we obtain

Lemma 3. *The general integral of the system (53) for $l \neq 0$ consists of the first integrals*

$$F_1 \equiv l^2(x^1 + \frac{c}{l})^3 + l(x^2 + a)^3 + (x^3 + b)^3 - 3(lx^1 + c)(x^2 + a)(x^3 + b) = C_1, \quad (68)$$

$$F_2 \equiv 2\ln|(x^1+\frac{c}{l})\sqrt[3]{l^2}+(x^2+a)\sqrt[3]{l}+x^3+b| - \ln|l(x^1+\frac{c}{l})^2\sqrt[3]{l}+(x^2+a)^2\sqrt[3]{l^2}+(x^3+b)^2-$$
$$-(lx^1 + c)(x^2 + a) - (x^1 + \frac{c}{l})(x^3 + b)\sqrt[3]{l^2} - (x^2 + a)(x^3 + b)\sqrt[3]{l}|-$$
$$-2\sqrt{3}\arctan\frac{2(x^1 + \frac{c}{l})\sqrt[3]{l^2} - (x^2 + a)\sqrt[3]{l} - x^3 - b}{\sqrt{3}\Big[(x^2 + a)\sqrt[3]{l} - x^3 - b\Big]} = C_2. \qquad (69)$$

4 Invariant expressions for the first $\mathbf{GL(3,\mathbb{R})}-$integrals of the system (24)

Theorem 2. *If the conditions $\delta_3\delta_4 \not\equiv 0$ and $\theta_1 = \theta_2 = \theta_3 = 0$ hold then the system (24) has an invariant $GL(3,\mathbb{R})-$integral as follows*

$$F \equiv 2(\delta_3\varkappa_2 - \delta_2\varkappa_3) - \varkappa_3^2 = C, \qquad (70)$$

where $\delta_2, \delta_3, \delta_4, \varkappa_2, \varkappa_3, \theta_1, \theta_2, \theta_3$ are taken from (25).

Proof. For the system (30) the values of the comitants (29) are the follows:

$$\delta_1 = au_1 + bu_2 + cu_3, \quad \delta_2 = bu_1 + cu_2, \quad \delta_3 = cu_1, \quad \delta_4 = u_1^3,$$

$$\varkappa_1 = u_1x^1 + u_2x^2 + u_3x^3, \quad \varkappa_2 = u_1x^2 + u_2x^3, \quad \varkappa_3 = u_1x^3.$$

Hereby for $\delta_4 \not\equiv 0$ we find

$$a = \frac{\delta_1u_1^2 - \delta_2u_1u_2 + \delta_3u_2^2 - \delta_3u_1u_3}{u_1^3}, \quad b = \frac{\delta_2u_1 - \delta_3u_2}{u_1^2}, \quad c = \frac{\delta_3}{u_1},$$

$$x^1 = \frac{\varkappa_1 u_1^2 - \varkappa_2 u_1 u_2 + \varkappa_3 u_2^2 - \varkappa_3 u_1 u_3}{u_1^3}, \quad x^2 = \frac{\varkappa_2 u_1 - \varkappa_3 u_2}{u_1^2}, \quad x^3 = \frac{\varkappa_3}{u_1}.$$

After the substitution of these expressions in F_1 from (32) we obtain the invariant $GL(3,\mathbb{R})-$integral for system (24) of the form (70). Theorem 2 is proved.

Theorem 3. *If the conditions* $\delta_4 \not\equiv 0, \quad \theta_1 \neq 0, \quad \theta_2 = \theta_1^2, \quad \theta_3 = \theta_1^3$ *hold then the system (24) has an invariant* $GL(3,\mathbb{R})-$*integral as follows*

$$F \equiv \theta_1^2 \varkappa_2 - \theta_1 \varkappa_3 + \delta_3 \ln|\delta_3 + \theta_1 \varkappa_3| - \delta_2 \theta_1 \ln|\delta_3 + \theta_1 \varkappa_3| = C, \tag{71}$$

where $\delta_3 + \theta_1 \varkappa_3 \not\equiv 0$ *and* $\delta_3, \delta_4, \varkappa_2, \varkappa_3, \theta_1, \theta_2, \theta_3$ *are taken from (25).*

Proof. For system (33) the values of the comitants (29) are the follows:

$$\delta_1 = au_1 + bu_2 + cu_3, \quad \delta_2 = bu_1 + cu_2 + cnu_3, \quad \delta_3 = cu_1 + cnu_2 + cn^2 u_3,$$

$$\delta_4 = u_1^3 + nu_1^2 u_2 + n^2 u_1^2 u_3, \quad \varkappa_1 = u_1 x^1 + u_2 x^2 + u_3 x^3, \quad \varkappa_2 = u_1 x^2 + u_2 x^3 + u_3 n x^3,$$

$$\varkappa_3 = u_1 x^3 + nu_2 x^3 + n^2 u_3 x^3.$$

Hereby for $\delta_4 \not\equiv 0$ we find

$$a = \frac{1}{u_1^2(u_1 + nu_2 + n^2 u_3)} \left[\delta_1 u_1^2 + n\delta_1 u_1 u_2 + n^2 \delta_1 u_1 u_3 - \delta_2 u_1 u_2 - n\delta_2 u_2^2 - \right.$$

$$\left. -n^2 \delta_2 u_2 u_3 + \delta_3 u_2^2 - \delta_3 u_1 u_3 + n\delta_3 u_2 u_3\right],$$

$$b = \frac{\delta_2 u_1 + n\delta_2 u_2 + n^2 \delta_2 u_3 - \delta_3 u_2 - n\delta_3 u_3}{u_1(u_1 + nu_2 + n^2 u_3)}, \quad c = \frac{\delta_3}{u_1 + nu_2 + n^2 u_3},$$

$$x^1 = \frac{1}{u_1^2(u_1 + nu_2 + n^2 u_3)} \left[\varkappa_1 u_1^2 + n\varkappa_1 u_1 u_2 + n^2 \varkappa_1 u_1 u_3 - \varkappa_2 u_1 u_2 - \right.$$

$$\left. -n\varkappa_2 u_2^2 - n^2 \varkappa_2 u_2 u_3 + \varkappa_3 u_2^2 - \varkappa_3 u_1 u_3 + n\varkappa_3 u_2 u_3\right],$$

$$x^2 = \frac{\varkappa_2 u_1 + n\varkappa_2 u_2 + n^2 \varkappa_2 u_3 - \varkappa_3 u_2 - n\varkappa_3 u_3}{u_1(u_1 + nu_2 + n^2 u_3)}, \quad x^3 = \frac{\varkappa_3}{u_1 + nu_2 + n^2 u_3}.$$

After the substitution of these expressions in F_1 from (35), taking into consideration $n = \theta_1$, we obtain the invariant $GL(3,\mathbb{R})-$integral of system (24) of the form (71). Theorem 3 is proved.

Theorem 4. *If the conditions* $\delta_4 \not\equiv 0, \quad \theta_3 = \theta_1 = 0, \quad \theta_2 \neq 0$ *hold then the system (24) has an invariant* $GL(3,\mathbb{R})-$*integral as follows*

$$F \equiv 2\theta_2(\delta_2 + \varkappa_3)^2 - (2\delta_3 + \theta_2 \varkappa_2)^2 = C, \tag{72}$$

where $\delta_2, \delta_3, \delta_4, \varkappa_2, \varkappa_3, \theta_1, \theta_2, \theta_3$ *are taken from (25).*

Proof. For the system (36) the values of the comitants (29) are the follows:

$$\delta_1 = au_1 + bu_2 + cu_3, \quad \delta_2 = bu_1 + cu_2 + bmu_3, \quad \delta_3 = cu_1 + bmu_2 + cmu_3,$$

$$\delta_4 = u_1^3 + 2mu_1^2u_3 - mu_1u_2^2 + m^2u_1u_3^2,$$

$$\varkappa_1 = u_1x^1 + u_2x^2 + u_3x^3, \quad \varkappa_2 = u_1x^2 + u_2x^3 + mu_3x^2, \quad \varkappa_3 = u_1x^3 + mu_2x^2 + mu_3x^3.$$

Hereby for $\delta_4 \not\equiv 0$ we find

$$a = \frac{\delta_1u_1^2 - m\delta_1u_2^2 + 2m\delta_1u_1u_3 + m^2\delta_1u_3^2 - \delta_2u_1u_2 + \delta_3u_2^2 - \delta_3u_1u_3 - m\delta_3u_3^2}{u_1(u_1^2 - mu_2^2 + 2mu_1u_3 + m^2u_3^2)},$$

$$b = \frac{\delta_2u_1 + m\delta_2u_3 - \delta_3u_2}{u_1^2 - mu_2^2 + 2mu_1u_3 + m^2u_3^2}, \quad c = \frac{-m\delta_2u_2 + \delta_3u_1 + m\delta_3u_3}{u_1^2 - mu_2^2 + 2mu_1u_3 + m^2u_3^2},$$

$$x^1 = \frac{\varkappa_1u_1^2 - m\varkappa_1u_2^2 + 2m\varkappa_1u_1u_3 + m^2\varkappa_1u_3^2 - \varkappa_2u_1u_2 + \varkappa_3u_2^2 - \varkappa_3u_1u_3 - m\varkappa_3u_3^2}{u_1(u_1^2 - mu_2^2 + 2mu_1u_3 + m^2u_3^2)},$$

$$x^2 = \frac{\varkappa_2u_1 + m\varkappa_2u_3 - \varkappa_3u_2}{u_1^2 - mu_2^2 + 2mu_1u_3 + m^2u_3^2}, \quad x^3 = \frac{-m\varkappa_2u_2 + \varkappa_3u_1 + m\varkappa_3u_3}{u_1^2 - mu_2^2 + 2mu_1u_3 + m^2u_3^2}.$$

After the substitution of these expressions in F_1 from (48), taking into consideration $m = \frac{\theta_2}{2}$, we obtain invariant $GL(3,\mathbb{R})-$integral of system (24) of the form (72). Theorem 4 is proved.

Theorem 5. *If the conditions $\delta_4 \not\equiv 0, \quad \theta_2 = \theta_1 = 0, \quad \theta_3 \neq 0$ hold then the system (24) has invariant $GL(3,\mathbb{R})-$integral as follows*

$$F \equiv (3\delta_3+\theta_3\varkappa_1)^3+3\theta_3^2(\delta_1+\varkappa_2)^3-9\theta_3(3\delta_3+\theta_3\varkappa_1)(\delta_1+\varkappa_2)(\delta_2+\varkappa_3)+9\theta_3(\delta_2+\varkappa_3)^3 = C, \tag{73}$$

where $\delta_1, \delta_2, \delta_3, \delta_4, \varkappa_1, \varkappa_2, \varkappa_3, \theta_1, \theta_2, \theta_3$ are taken from (25).

Proof. For the system (53) the values of the comitants (29) are the follows:

$$\delta_1 = au_1 + bu_2 + cu_3, \quad \delta_2 = bu_1 + cu_2 + alu_3, \quad \delta_3 = cu_1 + alu_2 + blu_3,$$

$$\delta_4 = u_1^3 + lu_2^3 + l^2u_3^3 - 3lu_1u_2u_3,$$

$$\varkappa_1 = u_1x^1 + u_2x^2 + u_3x^3, \quad \varkappa_2 = u_1x^2 + u_2x^3 + lu_3x^1, \quad \varkappa_3 = u_1x^3 + lu_2x^1 + lu_3x^2.$$

Hereby for $\delta_4 \not\equiv 0$ we find

$$a = \frac{\delta_1u_1^2 - l\delta_1u_2u_3 - \delta_2u_1u_2 + l\delta_2u_3^2 + \delta_3u_2^2 - \delta_3u_1u_3}{u_1^3 + lu_2^3 + l^2u_3^3 - 3lu_1u_2u_3},$$

$$b = \frac{l\delta_1u_2^2 - l\delta_1u_1u_3 + \delta_2u_1^2 - l\delta_2u_2u_3 - \delta_3u_1u_2 + l\delta_3u_3^2}{u_1^3 + lu_2^3 + l^2u_3^3 - 3lu_1u_2u_3},$$

$$c = \frac{-l\delta_1u_1u_2 + l^2\delta_1u_3^2 + l\delta_2u_2^2 - l\delta_2u_1u_3 + \delta_3u_1^2 - l\delta_3u_2u_3}{u_1^3 + lu_2^3 + l^2u_3^3 - 3lu_1u_2u_3},$$

$$x^1 = \frac{\varkappa_1 u_1^2 - l\varkappa_1 u_2 u_3 - \varkappa_2 u_1 u_2 + l\varkappa_2 u_3^2 + \varkappa_3 u_2^2 - \varkappa_3 u_1 u_3}{u_1^3 + l u_2^3 + l^2 u_3^3 - 3l u_1 u_2 u_3},$$

$$x^2 = \frac{l\varkappa_1 u_2^2 - l\varkappa_1 u_1 u_3 + \varkappa_2 u_1^2 - l\varkappa_2 u_2 u_3 - \varkappa_3 u_1 u_2 + l\varkappa_3 u_3^2}{u_1^2 + l u_2^3 + l^2 u_3^3 - 3l u_1 u_2 u_3},$$

$$x^3 = \frac{-l\varkappa_1 u_1 u_2 + l^2\varkappa_1 u_3^2 + l\varkappa_2 u_2^2 - l\varkappa_2 u_1 u_3 + \varkappa_3 u_1^2 - l\varkappa_3 u_2 u_3}{u_1^3 + l u_2^3 + l^2 u_3^3 - 3l u_1 u_2 u_3}.$$

After the substitution of these expressions in F_1 from (68), taking into consideration $l = \dfrac{\theta_3}{3}$, we obtain invariant $GL(3,\mathbb{R})-$integral for system (24) of the form (73). Theorem 5 is proved.

There is an open question: Is it possible to write the first integrals $F_2, F_2^{(i)}$ $(i = \overline{1,4})$ from (32), (35), (49)–(52) and (69) through $GL(3,\mathbb{R})-$invariants, contravariants and comitants of system (24)?

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