BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(48), 2005, Pages 43–50
ISSN 1024–7696

On the lattice of closed classes of modules

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Abstract. The family of closed classes of left R-modules R-cl (i.e. of classes which can be described by sets of left ideals of R) is transformed in a lattice and its properties are studied. The lattice R-cl is a frame (or Brouwerian lattice, or Heyting algebra). For every class $\mathcal{K} \in R$-cl its pseudocomplement $\mathcal{K}^*$ in R-cl is characterized. The skeleton of R-cl (i.e. the set of classes of the form $\mathcal{K}^*$, $\mathcal{K} \in R$-cl) coincides with the boolean lattice R-nat of natural classes of R-Mod. In parallels the isomorphic with R-cl lattice R-Cl of closed sets of left ideals of R is investigated, exposing some similar properties.

Mathematics subject classification: 16D80, 16D90, 16D20.
Keywords and phrases: Closed class of modules, natural class, frame (Brouwerian lattice), pseudocomplement, boolean lattice.

1 Preliminary notions and results

Every torsion r of the category R-Mod can be described by the set of left ideals (radical filter)

$$\mathcal{E}_r = \{I \in \mathbb{L}({}_R R) \,|\, I = (0:m), m \in M, M \in \mathcal{R}(r)\},$$

where $\mathcal{R}(r) = \{M \in$R-Mod$\,|\, r(M) = M\}$ [1–3]. Generalizing this fact, the notions of closed class of R-modules and closed set of left ideals of R were introduced in [2, 4, 5].

All studied classes $\mathcal{K} \subseteq R$-Mod are considered *abstract*, i.e. if $M \in \mathcal{K}$ and $M \cong N$, then $N \in \mathcal{K}$. The relations between the classes of R-modules and sets of left ideals of R are established by the following mappings (operators):

$\mathbf{\Gamma}(\mathcal{K}) = \{(0:m) \,|\, m \in M, M \in \mathcal{K}\}$ for $\mathcal{K} \subseteq R$-Mod;

$\mathbf{\Delta}(\mathcal{E}) = \{M \in R\text{-Mod} \,|\, (0:m) \in \mathcal{E} \;\forall\, m \in M\}$ for $\mathcal{E} \subseteq \mathbb{L}({}_R R)$,

where $\mathbb{L}({}_R R)$ is the lattice of left ideals of R.

Definition 1. *A class of modules $\mathcal{K} \subseteq R$-Mod is called* **closed** *if $\mathcal{K} = \mathbf{\Delta\Gamma}(\mathcal{K})$. A set $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ is called* **closed** *if $\mathcal{E} = \mathbf{\Gamma\Delta}(\mathcal{E})$.*

Lemma 1.1 [2, 4, 5]. *(a) A class $\mathcal{K} \subseteq R$-Mod is closed if and only if $\mathcal{K}$ satisfied the condition:*

(A_1) $M \in \mathcal{K} \Leftrightarrow Rm \in \mathcal{K} \;\; \forall\, m \in M$, *i.e.*

$$\begin{cases} (A_1') \;\; M \in \mathcal{K} \Rightarrow Rm \in \mathcal{K} \;\; \forall\, m \in M; \\ (A_1'') \;\; Rm \in \mathcal{K} \;\; \forall\, m \in M \Rightarrow M \in \mathcal{K}. \end{cases}$$

(b) *A set $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ is closed if and only if $\mathcal{E}$ satisfied the condition:*
(a_1) If $I \in \mathcal{E}$, then $(I : a) \in \mathcal{E}$ $\forall a \in R$,
where $(I : a) = \{a' \in R \,|\, a'a \in I\}$.

It is obvious that every closed class $\mathcal{K} \subseteq R$-Mod is hereditary: if $M \in \mathcal{K}$ and $N \subseteq M$, then by (A_1') $Rn \in \mathcal{K}$ for every $n \in N$ and (A_1'') now implies $N \in \mathcal{K}$.

Proposition 1.2 [2, 4, 5]. *The operators $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ define a preserving order bijection between the closed classes of R-Mod and closed sets of $\mathbb{L}({}_R R)$. If $\mathcal{K}$ and $\mathcal{E}$ correspond each other in this bijection, then:*

$$I \in \mathcal{E} \Leftrightarrow R/I \in \mathcal{K}.$$

2 The closed classes of modules

In continuation we will use the following notations:
R-cl is the family of all closed classes of R-Mod;
R-Cl is the family of all closed sets of $\mathbb{L}({}_R R)$.

It is clear that since R-Cl is a set, the bijection of Proposition 1.2 implies that R-cl also is a set. We will transform the sets R-cl and R-Cl in the complete lattices and will investigate their properties. The lattice operations are obtained by the relations of partial order, which in both cases are the inclusions (of classes or of sets of left ideals).

Lemma 2.1. *If $\{\mathcal{K}_\alpha \,|\, \alpha \in \mathfrak{A}\}$ is a family of closed classes of R-Mod, then the intersection $\mathcal{K} = \bigcap\limits_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha$ also is a closed class.*

Proof. (A_1') If $M \in \mathcal{K}$ then for every $\alpha \in \mathfrak{A}$ we have $Rm \in \mathcal{K}_\alpha$ for all $m \in M$, so $Rm \in \bigcap\limits_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha = \mathcal{K}$.

(A'') Let $M \in R$-Mod and $Rm \in \mathcal{K}$ for every $m \in M$. Then for all $\alpha \in \mathfrak{A}$ we have $Rm \in \mathcal{K}_\alpha$ and appling (A_1'') for $\mathcal{K}_\alpha$ we obtain $M \in \mathcal{K}_\alpha$, therefore $M \in \bigcap\limits_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha = \mathcal{K}$. $\square$

Now we define on the set R-cl the operation „$\bigwedge$" by the rule:

$$\bigwedge_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha = \bigcap_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha,$$

obtaining a complete semilattice, which in a standard way can be transformed in a complete lattice with the dual operation of join „$\bigwedge$", defined by the rule:

$$\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha = \bigcap \Big\{ \mathcal{L} \in R\text{-cl} \,\Big|\, \mathcal{L} \supseteq \bigcup_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha \Big\},$$

where $\bigcup\limits_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha$ is the ordinary union of $\mathcal{K}_\alpha$ in the sense of set theory. So we have a complete lattice R-cl $(\subseteq, \bigwedge, \bigvee)$ of all closed classes of R-Mod with the smallest element $\{0\}$ and the greatest element R-Mod.

From Lemma 2.1 it follows that every class $\mathcal{K} \subseteq R$-Mod can be imbedded in the smallest closed class $\langle \mathcal{K} \rangle$ which contains $\mathcal{K}$:

$$\langle \mathcal{K} \rangle = \bigcap \{\mathcal{L} \in R\text{-cl} \,|\, \mathcal{L} \supseteq \mathcal{K}\}.$$

This class will be called the closed class *generated* by $\mathcal{K}$. Now we indicate a simple description of this class. For that we denote:

$$\overline{\mathcal{K}} = \{M \in R\text{-Mod} \mid \exists\, N \in \mathcal{K},\ M \cong N' \subseteq N\},$$

i.e. $\overline{\mathcal{K}}$ is the closure of $\mathcal{K}$ with respect to submodules and isomorphic images.

Lemma 2.2. *For every class $\mathcal{K} \subseteq R$-Mod the following relation is true*:

$$\langle \mathcal{K} \rangle = \{M \in R\text{-Mod} \mid Rm \in \overline{\mathcal{K}} \ \ \forall\, m \in M\}.$$

Proof. We denote the right part of this relation by $\mathcal{L}$. It is obvious that $\mathcal{L} \supseteq \mathcal{K}$. Now we verify that $\mathcal{L}$ is a closed class.

(A_1') If $M \in \mathcal{L}$ and $m \in M$, then $Rm \in \overline{\mathcal{K}}$ and since $\overline{\mathcal{K}}$ is hereditary we obtain that every cyclic submodule $Ram \subseteq Rm$ also belongs to $\overline{\mathcal{K}}$, so by definition $Rm \in \mathcal{L}$.

(A_1'') Let $M \in R$-Mod and $Rm \in \mathcal{L}$ for every $m \in M$. Then by definition of $\mathcal{L}$ every cyclic submodule $Ram \subseteq Rm$ belongs to $\overline{\mathcal{K}}$. But for $a = 1$ we have $rm \in \overline{\mathcal{K}}$ for every $m \in M$ and this means that $M \in \mathcal{L}$.

So $\mathcal{L}$ is a closed class containing $\mathcal{K}$ and now by definition of $\langle \mathcal{K} \rangle$ we obtain $\langle \mathcal{K} \rangle \subseteq \mathcal{L}$.

It remains to verify the inclusion $\langle \mathcal{K} \rangle \supseteq \mathcal{L}$. Let $M \in \mathcal{L}$ and $\mathcal{N}$ be an arbitrary closed class, containing $\mathcal{K}$. Since $\mathcal{N}$ is hereditary, from $\mathcal{K} \subseteq \mathcal{N}$ follows $\overline{\mathcal{K}} \subseteq \mathcal{N}$. From $M \in \mathcal{L}$ we have $Rm \in \overline{\mathcal{K}} \subseteq \mathcal{N}$ for every $m \in M$. But $\mathcal{N}$ is closed and now by (A_1'') we obtain $M \in \mathcal{N}$.

Therefore $\mathcal{L} \subseteq \mathcal{N}$ for every closed class $\mathcal{N}$ with $\mathcal{N} \supseteq \mathcal{K}$ and this means that $\mathcal{L} \subseteq \langle \mathcal{K} \rangle$. $\square$

In continuation we will use this lemma for characterization of the join $\bigvee\limits_{\alpha \in \mathfrak{A}} \mathcal{K}_\alpha$ of classes $\mathcal{K}_\alpha$ in the lattice R-cl. At first we remark that since the classes $\mathcal{K}_\alpha$ are abstract and hereditary, we have $\mathcal{K}_\alpha = \overline{\mathcal{K}}_\alpha$ for every $\alpha \in \mathfrak{A}$ and $\bigcup\limits_{\alpha \in \mathfrak{A}} \mathcal{K}_\alpha = \overline{\bigcup\limits_{\alpha \in \mathfrak{A}} \mathcal{K}_\alpha}$. So by Lemma 2.2 follows

Corollary 2.3. *For every family $\{\mathcal{K}_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq R$-cl of closed classes of R-Mod the following relation is true:*

$$\bigvee_{\alpha \in \mathfrak{A}} \mathcal{K}_\alpha = \Big\langle \bigcup_{\alpha \in \mathfrak{A}} \mathcal{K}_\alpha \Big\rangle = \Big\{ M \in R\text{-Mod} \;\Big|\; Rm \in \bigcup_{\alpha \in \mathfrak{A}} \mathcal{K}_\alpha \ \ \forall m \in M \Big\},$$

i.e. $\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha$ *consists of modules* M *with property: for every* $m \in M$ *there exists* $\alpha_m \in \mathfrak{A}$ *such that* $Rm \in \mathcal{K}_{\alpha_m}$.

Now we can prove the basic property of the lattice R-cl. Remember that a lattice L is called a *frame* (or Brouwerian lattice, or Heyting algebra) if $a \bigwedge \Big(\bigvee_{\alpha\in\mathfrak{A}} b_\alpha\Big) = \bigvee_{\alpha\in\mathfrak{A}} \Big(a \bigwedge b_\alpha\Big)$ for arbitrary $a \in L$ and $\{b_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq L$ [3, 6]. If L is a frame then every element a in L has the *pseudocomplement* a^* of $a \in L$, which is the greatest between elements $b \in L$ with $a \bigwedge b = 0$.

Theorem 2.4. *The lattice R-cl of closed classes of R-Mod is a frame.*

Proof. We will verify that for every class $\mathcal{K} \in R$-cl and every family $\{\mathcal{K}_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq R$-cl the following relation is true:

$$\mathcal{K} \bigwedge \Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big) = \bigvee_{\alpha\in\mathfrak{A}} \Big(\mathcal{K} \bigwedge \mathcal{K}_\alpha\Big).$$

The inclusion ($\supseteq$) is true in general, since the left part contains $\mathcal{K} \bigwedge \mathcal{K}_\alpha$ for every $\alpha \in \mathfrak{A}$.

To verify the inclusion ($\subseteq$), let $M \in \mathcal{K} \bigwedge \Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)$. We must show that $M \in \bigvee_{\alpha\in\mathfrak{A}} \Big(\mathcal{K} \bigwedge \mathcal{K}_\alpha\Big)$, i.e. that for every $m \in M$ there exists $\alpha_m \in \mathfrak{A}$ such that $Rm \in \mathcal{K} \bigwedge \mathcal{K}_{\alpha_m}$.

Let $m \in M$. Since $M \in \mathcal{K}$ and $\mathcal{K}$ is a closed class we have $Rm \in \mathcal{K}$. But from assumption $M \in \bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha$ and using Corollary 2.3, it follows that there exists $\alpha_m \in \mathfrak{A}$ such that $Rm \in \mathcal{K}_{\alpha_m}$. So we obtain $Rm \in \mathcal{K} \bigwedge \mathcal{K}_{\alpha_m}$ and this means that $M \in \bigvee_{\alpha\in\mathfrak{A}} \Big(\mathcal{K} \bigwedge \mathcal{K}_\alpha\Big)$. □

In the subsequent investigation the following operator, which acts on the classes of modules $\mathcal{K} \subseteq R$-Mod, plays an important role:

$$\mathcal{K}^\perp = \{M \in R\text{-Mod} \mid M \text{ has no nonzero submodules from } \mathcal{K}\} \quad [7].$$

From definition is clear that $\mathcal{K} \cap \mathcal{K}^\perp = \{0\}$.

Lemma 2.5. *If* $\mathcal{K}$ *is a closed (or only hereditary) class, then* $\mathcal{K}^\perp$ *is a closed class.*

Proof. From definition follows that $\mathcal{K}^\perp$ is hereditary, so it satisfied (A_1').

(A_1'') Let $M \in R$-Mod and $Rm \in \mathcal{K}^\perp$ for every $m \in M$. If $M \notin \mathcal{K}^\perp$, then M has a submodule $0 \neq N \subseteq M$ such that $N \in \mathcal{K}$. Since $\mathcal{K}$ is hereditary, M has a cyclic submodule $0 \neq Rm \in \mathcal{K}$. Then $0 \neq Rm \notin \mathcal{K}^{\perp}$, in contradiction with the choice of M. This shows that $M \in \mathcal{K}^{\perp}$. □

Since R-cl is a frame (Theorem 2.4), every element $\mathcal{K} \in R$-cl has the pseudocomplement $\mathcal{K}^*$ in the lattice R-cl, i.e. the greatest between the classes $\mathcal{L} \in R$-cl with $\mathcal{K} \bigwedge \mathcal{L} = \{0\}$:

$$\mathcal{K}^* = \bigvee_{\alpha \in \mathfrak{A}} \Big\{ \mathcal{L}_\alpha \in R\text{-cl} \;\Big|\; \mathcal{L}_\alpha \bigwedge \mathcal{K} = \{0\} \Big\}.$$

For any $\mathcal{K} \in R$-cl the class $\mathcal{K}^*$ can be described by operator $(\ )^{\perp}$.

Lemma 2.6. *For every $\mathcal{K} \in R$-cl the pseudocomplement $\mathcal{K}^*$ of $\mathcal{K}$ in R-cl coincides with the class $\mathcal{K}^{\perp}$, i.e. $\mathcal{K}^* = \mathcal{K}^{\perp}$.*

Proof. By definition $\mathcal{K} \bigwedge \mathcal{K}^{\perp} = \{0\}$ and by Lemma 2.5 the class $\mathcal{K}^{\perp}$ is closed. Now we verify that $\mathcal{K}^{\perp}$ is the greatest between such $\mathcal{L} \in R$-cl that $\mathcal{K} \bigwedge \mathcal{L} = \{0\}$.

Let $\mathcal{L} \in R$-cl and $\mathcal{K} \bigwedge \mathcal{L} = \{0\}$. If $0 \neq M \in \mathcal{L}$, then since $\mathcal{K} \bigvee \mathcal{L} = \{0\}$ and $\mathcal{L}$ is hereditary, every submodule $0 \neq N \subseteq M$ belongs to $\mathcal{L}$, therefore $N \notin \mathcal{K}$. By definition of $\mathcal{K}^{\perp}$, this means that $M \in \mathcal{K}^{\perp}$. So $\mathcal{L} \subseteq \mathcal{K}^{\perp}$, therefore $\mathcal{K}^{\perp}$ is the pseudocomplement of $\mathcal{K}$ in R-cl. □

Definition 2. *The* **skeleton** *of the lattice R-cl is the set of all pseudocomplements of closed classes:*

$$Sk(R\text{-cl}) = \{\mathcal{K}^{\perp} \mid \mathcal{K} \in R\text{-cl}\}.$$

To describe the skeleton of R-cl we remember some notions and facts on natural classes and their properties.

Definition 3. *A class $\mathcal{K} \subseteq R$-Mod is called* **natural** *(or saturated) ([8–11]) if it is closed with respect to submodules, direct sums and injective envelopes (or essential extensions).*

Every natural class is closed also with respect to extensions. Moreover, if $\mathcal{K}$ is a natural class, then $\mathcal{K} \in R$-cl [7]. The set R-nat of all natural classes of R-Mod can be transformated in a lattice, which is boolean [8, 9]. For every $\mathcal{K} \in R$-nat the class $\mathcal{K}^{\perp}$ is the complement of $\mathcal{K}$ in R-nat, therefore $\mathcal{K} = \mathcal{K}^{\perp\perp}$.

Proposition 2.7. *If $\mathcal{K} \in R$-cl, then the pseudocomplement $\mathcal{K}^{\perp}$ of $\mathcal{K}$ in R-cl is a natural class.*

Proof. Let $\mathcal{K} \in R$-cl. Since $\mathcal{K}^{\perp}$ is closed (Lemma 2.5), this class is hereditary. Now we verify that $\mathcal{K}^{\perp}$ is closed with respect to direct sums.

Let $\{M_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathcal{K}^{\perp}$ and $M = \dot{\sum} M_\alpha$. If $M \notin \mathcal{K}^{\perp}$, then there exists a cyclic submodule $0 \neq Rm \subseteq M$ such that $Rm \in \mathcal{K}$.

Let $m = m_{\alpha_1} + \ldots + m_{\alpha_n}$, where $m_{\alpha_k} \in M_{\alpha_k}$ for $k = 1, \ldots, n$, and $m_{\alpha_i} \neq 0$. Then $0 \neq Rm_{\alpha_i} \subseteq Rm \in \mathcal{K}$, therefore $0 \neq Rm_{\alpha_i} \in \mathcal{K}$, in contradiction with $M_{\alpha_i} \in \mathcal{K}^{\perp}$. This proves that $M \in \mathcal{K}^{\perp}$, i.e. $\mathcal{K}^{\perp}$ is closed with respect to direct sums.

Now we will convince that the class $\mathcal{K}^{\perp}$ is stable, i.e. $M \in \mathcal{K}^{\perp}$ implies $E(M) \in \mathcal{K}^{\perp}$, where $E(M)$ is the injective envelope of M. Let $M \in \mathcal{K}^{\perp}$. If $E(M) \notin \mathcal{K}^{\perp}$, then there exists a submodule $0 \neq N \subseteq E(M)$ such that $N \in \mathcal{K}$. Since M is essential in $E(M)$, we have $N' = N \cap M \neq 0$. The class $\mathcal{K}$ is hereditary and $N \in \mathcal{K}$, therefore $0 \neq N' \in \mathcal{K}$, where $N' \subseteq M$, in contradiction with $M \in \mathcal{K}^{\perp}$. So from $M \in \mathcal{K}^{\perp}$ follows $E(M) \in \mathcal{K}^{\perp}$.

The previous arguments show that $\mathcal{K}^{\perp}$ is a natural class. □

Remark. In [11, Theorem 6] it is proved that if $\mathcal{K}$ is hereditary then the pseudocomplement of $\mathcal{K}$ in the lattice R-her of all hereditary classes of R-Mod is a natural class.

Corollary 2.8. *If $\mathcal{K} \in R$-cl, then the class $\mathcal{K}^{\perp\perp}$ is the smallest natural class, containing $\mathcal{K}$, i.e. $\mathcal{K}^{\perp\perp}$ is the natural class generated by $\mathcal{K}$.*

Proof. By definition $\mathcal{K}^{\perp\perp}$ consists of such $M \in R$-Mod that every nonzero submodule of M contains a nonzero submodule from $\mathcal{K}$. Since $\mathcal{K}$ is hereditary, we have $\mathcal{K} \subseteq \mathcal{K}^{\perp\perp}$ and by Proposition 2.7 $\mathcal{K}^{\perp\perp}$ is a natural class.

Let $\mathcal{L} \in R$-nat and $\mathcal{L} \supseteq \mathcal{K}$. Then $\mathcal{L}^{\perp} \subseteq \mathcal{K}^{\perp}$ and $\mathcal{L}^{\perp\perp} \supseteq \mathcal{K}^{\perp\perp}$. Since $\mathcal{L}$ is natural, we have $\mathcal{L}^{\perp\perp} = \mathcal{L}$, so $\mathcal{L} \supseteq \mathcal{K}^{\perp\perp}$. Therefore $\mathcal{K}^{\perp\perp}$ is the smallest natural class containing $\mathcal{K}$. □

Now we can describe the skeleton of the lattice R-cl.

Theorem 2.9. *The skeleton of the lattice R-cl coincides with the lattice of natural classes: $Sk(R\text{-}cl) = R\text{-}nat$.*

Proof. If $\mathcal{K} \in R$-cl then $\mathcal{K}^{\perp}$ is natural (Proposition 2.7), where $\mathcal{K}^{\perp}$ is the pseudocomplement of $\mathcal{K}$ in R-cl (Lemma 2.6), therefore $Sk(R\text{-cl}) \subseteq R$-nat. From the other hand, every natural class $\mathcal{K} \in R$-nat has the property $\mathcal{K} = \mathcal{K}^{\perp\perp}$, therefore $\mathcal{K} \in Sk(R\text{-cl})$. □

We remark that in the lattices R-cl and R-nat (where R-nat $\subseteq R$-cl) the partial order is the same (inclusion of classes), as well as the operations „$\bigwedge$" (intersections of classes), however the operations „$\bigvee$" in general are different. It is a well known fact that for every frame the skeleton forms a boolean lattice [6]. So from Theorem 2.9 we obtain one more confirmation of the result that R-nat is a boolean lattice [8, 9]. We mention also that in the paper [11] it is proved that R-nat is the skeleton of the lattice R-her of all hereditary classes of R-Mod.

3 The closed sets of left ideals

In this section we pass from the closed classes of R-Mod to the closed sets of left ideals of R. The bijection of Proposition 1.2 guarantees us that in the set R-Cl of all closed sets of $\mathbb{L}({}_RR)$ the similar results as in R-cl will be true.

By Definition 1 the set $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ is closed if $\mathcal{E} = \mathbf{\Gamma\Delta}(\mathcal{E})$ and such sets are characterized (Lemma 1.1 (b)) by the condition:

(a_1) *If* $I \in \mathcal{E}$, then $(I : a) \in \mathcal{E}$ for every $a \in R$.

Using this condition it is easy to verify

Lemma 3.1. *If* $\{\mathcal{E}_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq R$*-Cl, then* $\bigcap_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha \in R$*-Cl and* $\bigcup_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha \in R$-Cl.

We consider the set R-Cl as a complete lattice with the operations:

$$\bigwedge_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha = \bigcap_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha, \quad \bigvee_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha = \bigcup_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha.$$

From the definition it is clear that the lattice $(R\text{-Cl}, \subseteq, \bigwedge, \bigvee)$ satisfies both infinite distributive laws. The greatest element of R-Cl is $\mathbb{L}({}_RR)$ and the smallest element is $\{R\}$.

Since the operators $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ define a bijection between R-cl and R-Cl, which preserves the order, we obtain

Corollary 3.2. *The lattices R-cl and R-Cl are isomorphic.*

Corollary 3.3. *The lattice R-Cl is a frame.*

Now we define an operator in R-Cl, which is the analogue of the operator $(\ )^\perp$ in R-cl, by the rule:

$$\mathcal{E}^\perp = \{I \in \mathbb{L}({}_RR) \,|\, (I : r) \notin \mathcal{E} \ \ \forall\, r \in R \setminus I\},$$

where $\mathcal{E} \subseteq \mathbb{L}({}_RR)$. For $I = R$ we have $R \setminus I = \varnothing$ and we can consider that $R \in \mathcal{E}^\perp$.

Proposition 3.4. *For every (non-empty) set of left ideals* $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ *the set* $\mathcal{E}^\perp$ *is closed.*

Proof. Let $I \in \mathcal{E}^\perp$ and $r \in R$. If $r \in I$, then $(I : r) = R \in \mathcal{E}^\perp$. If $r \notin I$, then by definition $(I : r) \notin \mathcal{E}$. Moreover, every proper quotient of $(I : r)$ does not belong to $\mathcal{E}$: for every $r' \notin (I : r)$ we have $r'r \notin I$ and $(I : r'r) \notin \mathcal{E}$, since $I \in \mathcal{E}^\perp$. This means that $(I : r) \in \mathcal{E}^\perp$ for every $r \in R$, so by Lemma 1.1 (b) the set $\mathcal{E}^\perp$ is closed. $\square$

Proposition 3.5. *For every closed set* $\mathcal{E} \in R$*-Cl the set* $\mathcal{E}^\perp$ *is the pseudocomplement of* $\mathcal{E}$ *in the lattice R-Cl.*

Proof. By the definition of $\mathcal{E}^\perp$ is clear that $\mathcal{E} \cap \mathcal{E}^\perp = \{R\}$. Let $\mathcal{E}' \in R$-Cl and $\mathcal{E} \cap \mathcal{E}' = \{R\}$. If $I \in \mathcal{E}'$ and $I \neq R$, then $I \notin \mathcal{E}$ and for every $r \notin I$ we have $(I : r) \in \mathcal{E}'$, $(I : r) \neq R$, therefore $(I : r) \notin \mathcal{E}$. This means that $I \in \mathcal{E}^\perp$, so $\mathcal{E}' \subseteq \mathcal{E}^\perp$ and $\mathcal{E}^\perp$ is the greatest closed set with $\mathcal{E} \cap \mathcal{E}^\perp = \{R\}$. $\square$

The operators $(\ )^\perp$ of pseudocomplementation in the lattices R-cl and R-Cl are compatible with the mappings $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ in the following sense.

Proposition 3.6 [7, Prop. 2.1]. *Let* $\mathcal{K} \in R$*-cl and* $\mathcal{E} \in R$*-Cl correspond each other in the bijection defined by* $\mathbf{\Gamma}$ *and* $\mathbf{\Delta}$*, i.e.* $\mathcal{E} = \mathbf{\Gamma}(\mathcal{K})$ *and* $\mathcal{K} = \mathbf{\Delta}(\mathcal{E})$*. Then the following relations are true:*

$$\mathbf{\Gamma}(\mathcal{K}^\perp) = \mathcal{E}^\perp, \quad \mathbf{\Delta}(\mathcal{E}^\perp) = \mathcal{K}^\perp.$$

This means that the bijection between R-cl and R-Cl defined by $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ preserves not only the lattice operations, but also preserves the pseudocomplements.

In the papers [10] and [7] the *natural sets* of left ideals are described, i.e. the sets which correspond to the natural classes. In particular, $\mathcal{E} \subseteq \mathbb{L}(_R R)$ is natural if it satisfies the conditions (a_1), (a_3) and (a_6) from [7], where:

(a_3) $I, J \in \mathcal{E} \Rightarrow I \cap J \in \mathcal{E}$;

(a_6) $J \subseteq I,\ (J:i) \in \mathcal{E}\ \ \forall\, i \in I, \quad I/J \subseteq^* \ R/J \Rightarrow J \in \mathcal{E}$.

Some variants of (a_6) are indicated in [7], using the relation $\mathcal{E} = \mathcal{E}^{\perp\perp}$.

Proposition 3.7. *For every closed set $\mathcal{E} \in R$-Cl the set $\mathcal{E}^{\perp}$ is natural.*

Proof. If $\mathcal{E} \in R$-Cl, then $\mathbf{\Delta}(\mathcal{E})$ is a closed class, therefore $\big(\mathbf{\Delta}(\mathcal{E})\big)^{\perp}$ is a natural class (Lemma 2.7). Then the set $\mathbf{\Gamma}\Big(\big(\mathbf{\Delta}(\mathcal{E})\big)^{\perp}\Big)$ is natural [7, Theorem 1.7]. By Proposition 3.6 $\big($for $\mathcal{K} = \mathbf{\Delta}(\mathcal{E})\big)$ we have $\mathbf{\Gamma}\Big(\big(\mathbf{\Delta}(\mathcal{E})\big)^{\perp}\Big) = \mathcal{E}^{\perp}$, therefore $\mathcal{E}^{\perp}$ is a natural set. $\square$

Now is obvious that for every natural set $\mathcal{E} \in R$-Nat we have $\mathcal{E} = \mathcal{E}^{\perp\perp}$.

Corollary 3.8. *The skeleton of the lattice R-Cl coincides with the lattice R-Nat of natural sets of left ideals of R.*

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Received June 21, 2005