

On geometrical properties of the spaces defined by the Pfaff equations

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Abstract. Geometrical properties of holonomic and non holonomic varieties defined by the Pfaff equations connected with the first order system of equations are studied. The Riemann extensions of affine connected spaces for investigations of geodesics and asymptotic lines are used.

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1 Introduction

There is the connection between of the Pfaff equation

$$P(x, y, z) \frac{dx}{ds} + Q(x, y, z) \frac{dy}{ds} + R(x, y, z) \frac{dz}{ds} = 0 \quad (1)$$

and the first order system differential equations in form

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)}. \quad (2)$$

For example the equation (1) is exactly integrable at the conditions

$$\frac{\partial P(x, y, z)}{\partial y} - \frac{\partial Q(x, y, z)}{\partial x} = 0, \quad \frac{\partial Q(x, y, z)}{\partial z} - \frac{\partial R(x, y, z)}{\partial y},$$

$$\frac{\partial R(x, y, z)}{\partial x} - \frac{\partial P(x, y, z)}{\partial z} = 0$$

and its general integral determines the family of the surfaces in R^3 space

$$V(x, y, z) = \text{constant},$$

which are orthogonal to the lines of the vector field

$$\vec{N} = (P(x, y, z), Q(x, y, z), R(x, y, z)). \quad (3)$$

In more general case

$$P(x, y, z) \left(\frac{\partial Q(x, y, z)}{\partial z} - \frac{\partial R(x, y, z)}{\partial y} \right) + Q(x, y, z) \left(\frac{\partial R(x, y, z)}{\partial x} - \frac{\partial P(x, y, z)}{\partial z} \right) +$$

$$+R(x, y, z) \left(\frac{\partial R(x, y, z)}{\partial y} - \frac{\partial Q(x, y, z)}{\partial x} \right) = 0$$

the equation (1)

$$\mu(P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz) = dU(x, y, z)$$

is also integrable by means of integrating multiplier μ and determines the family of the surfaces $U(x, y, z) = \text{const}$ passing through the each point of the space and orthogonal to the vector field (3).

The vector field (3) in the space R^3 with conditions

$$\left(\vec{N}, \text{rot} \vec{N} \right) = 0$$

is called holonomic.

In general case the Pfaff equation (1) is not integrable and then corresponds the system of the integral curves (Pfaff variety) passing through each point (x, y, z) with tangent lines lying on the plane

$$P(x, y, z)(X - x) + Q(x, y, z)(Y - y) + R(x, y, z)(Z - z) = 0. \quad (4)$$

The set of the planes and points (4) defined by the equation (1) in general case forms two-dimensional non holonomic variety M^2 which is generalization of the surface.

For the variety M^2 may be extended many of the results of classical differential geometry of surfaces. For example the notion of the asymptotic lines, curvature lines and geodesic has analog for the variety M^2 [1-3].

In fact the solutions of the system

$$Pdx + Qdy + Rdz = 0, \quad dPdx + dQdy + dRdz = 0$$

or

$$P\dot{x} + \dot{y} + R\dot{z} = 0,$$

$$P_x(\dot{x})^2 + Q_y(\dot{y})^2 + R_z(\dot{z})^2 + (P_y + Q_x)\dot{x}\dot{y} + (P_z + R_x)\dot{x}\dot{z} + (Q_z + R_y)\dot{y}\dot{z} = 0$$

give us the curve lines of the variety M^2 which are the analog of the of asymptotic lines on the holonomic surface.

The notion of the curvature lines also can be generalized on the variety M^2 .

They may be of the two kinds and one of them is defined by the solutions of the system of equations

$$Pdx + Qdy + Rdz = 0, \quad \begin{vmatrix} 2P_x dx + (Q_x + P_y)dy + (P_x + R_x)dz & P & dx \\ (Q_x + P_y)dx + 2Q_y dy + (Q_z + R_y)dz & Q & dy \\ (P_z + R_x)dx + (Q_z + R_y)dy + 2R_z dz & R & dz \end{vmatrix} = 0.$$

The notion of the geodesics also can be extended on the variety V^2 (1) and they may be of two types.

The first type is determined from the condition

$$\begin{bmatrix} P(x, y, z) & Q(x, y, z) & R(x, y, z) \\ \frac{d}{ds}x(s) & \frac{d}{ds}y(s) & \frac{d}{ds}z(s) \\ \frac{d^2}{ds^2}x(s) & \frac{d^2}{ds^2}y(s) & \frac{d^2}{ds^2}z(s) \end{bmatrix} = 0,$$

or

$$\begin{aligned} & \left(P(x, y, z) \frac{d}{ds}y(s) - \left(\frac{d}{ds}x(s) \right) Q(x, y, z) \right) \frac{d^2}{ds^2}z(s) + \\ & + \left(-P(x, y, z) \frac{d}{ds}z(s) + \left(\frac{d}{ds}x(s) \right) R(x, y, z) \right) \frac{d^2}{ds^2}y(s) + \\ & + \left(\frac{d^2}{ds^2}x(s) \right) \left(Q(x, y, z) \frac{d}{ds}z(s) - R(x, y, z) \frac{d}{ds}y(s) \right) = 0. \end{aligned}$$

This relation is equivalent the system of equations

$$\frac{d^2x}{ds^2} + \frac{P}{P^2 + Q^2 + R^2} \left(\frac{dx}{ds} \frac{dP}{ds} + \frac{dy}{ds} \frac{dQ}{ds} + \frac{dz}{ds} \frac{dR}{ds} \right) = 0, \quad (5)$$

$$\frac{d^2y}{ds^2} + \frac{Q}{P^2 + Q^2 + R^2} \left(\frac{dx}{ds} \frac{dP}{ds} + \frac{dy}{ds} \frac{dQ}{ds} + \frac{dz}{ds} \frac{dR}{ds} \right) = 0, \quad (6)$$

$$\frac{d^2z}{ds^2} + \frac{R}{P^2 + Q^2 + R^2} \left(\frac{dx}{ds} \frac{dP}{ds} + \frac{dy}{ds} \frac{dQ}{ds} + \frac{dz}{ds} \frac{dR}{ds} \right) = 0. \quad (7)$$

Remark that after substitution of the corresponding expressions for the second derivatives on coordinates from the (5)–(7) into the relations

$$\begin{aligned} & \frac{dP(x, y, z)}{ds} \frac{dx}{ds} + \frac{dQ(x, y, z)}{ds} \frac{dy}{ds} + \frac{dR(x, y, z)}{ds} \frac{dz}{ds} + P(x, y, z) \frac{d^2x}{ds^2} + \\ & + Q(x, y, z) \frac{d^2y}{ds^2} + R(x, y, z) \frac{d^2z}{ds^2} = 0 \end{aligned}$$

one get the identity.

The definition of the second type of geodesic in the system of integral curves of the Pfaff equation (1) is more complicated and does not be used hereinafter.

2 The Riemann extension of the affine connected space

The formulas (5)–(7) can be rewritten in form of geodesic of the R^3 -space equipped with symmetrical affine connection $\Pi_{jk}^i(x^l) = \Pi_{kj}^i(x^l)$

$$\frac{d^2x^i}{ds^2} + \Pi_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$

In our case we get the system of equations

$$\begin{aligned} \frac{d^2x}{ds^2} + \frac{P}{\Delta} \left(P_x \left(\frac{dx}{ds} \right)^2 + (P_y + Q_x) \frac{dx}{ds} \frac{dy}{ds} + (P_z + R_x) \frac{dx}{ds} \frac{dz}{ds} + Q_y \left(\frac{dy}{ds} \right)^2 + \right. \\ \left. + (Q_z + R_y) \frac{dy}{ds} \frac{dz}{ds} + R_z \left(\frac{dz}{ds} \right)^2 \right) = 0, \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{ds^2} + \frac{Q}{\Delta} \left(P_x \left(\frac{dx}{ds} \right)^2 + (P_y + Q_x) \frac{dx}{ds} \frac{dy}{ds} + (P_z + R_x) \frac{dx}{ds} \frac{dz}{ds} + Q_y \left(\frac{dy}{ds} \right)^2 + \right. \\ \left. + (Q_z + R_y) \frac{dy}{ds} \frac{dz}{ds} + R_z \left(\frac{dz}{ds} \right)^2 \right) = 0, \end{aligned}$$

$$\begin{aligned} \frac{d^2z}{ds^2} + \frac{R}{\Delta} \left(P_x \left(\frac{dx}{ds} \right)^2 + (P_y + Q_x) \frac{dx}{ds} \frac{dy}{ds} + (P_z + R_x) \frac{dx}{ds} \frac{dz}{ds} + Q_y \left(\frac{dy}{ds} \right)^2 + \right. \\ \left. + (Q_z + R_y) \frac{dy}{ds} \frac{dz}{ds} + R_z \left(\frac{dz}{ds} \right)^2 \right) = 0, \end{aligned}$$

where

$$\Delta = P^2 + Q^2 + R^2,$$

from which we get the expressions for the coefficients of affine connection. They are

$$\Pi_{11}^1 = \frac{PP_x}{\Delta}, \quad \Pi_{22}^1 = \frac{PQ_y}{\Delta}, \quad \Pi_{33}^1 = \frac{PR_z}{\Delta},$$

$$\Pi_{12}^1 = \frac{P(P_y + Q_x)}{2\Delta}, \quad \Pi_{13}^1 = \frac{P(P_z + R_x)}{2\Delta}, \quad \Pi_{23}^1 = \frac{P(Q_z + R_y)}{2\Delta},$$

$$\Pi_{11}^2 = \frac{QP_x}{\Delta}, \quad \Pi_{22}^2 = \frac{QQ_y}{\Delta}, \quad \Pi_{33}^2 = \frac{QR_z}{\Delta},$$

$$\begin{aligned}\Pi_{12}^2 &= \frac{Q(P_y + Q_x)}{2\Delta}, & \Pi_{13}^2 &= \frac{Q(P_z + R_x)}{2\Delta}, & \Pi_{23}^2 &= \frac{Q(Q_z + R_y)}{2\Delta}, \\ \Pi_{11}^3 &= \frac{RP_x}{\Delta}, & \Pi_{22}^3 &= \frac{RQ_y}{\Delta}, & \Pi_{33}^3 &= \frac{RR_z}{\Delta}, \\ \Pi_{12}^3 &= \frac{R(P_y + Q_x)}{2\Delta}, & \Pi_{13}^3 &= \frac{R(P_z + R_x)}{2\Delta}, & \Pi_{23}^3 &= \frac{R(Q_z + R_y)}{2\Delta}.\end{aligned}$$

So with any equation (1) can be associated 3-dimensional affine connected space and its properties will be dependent from the coefficients of connection Π_{ij}^k which are determined by the functions P, Q, R .

In general case such type of connection is not metrizable and corresponding space is not a Riemannian.

Further we apply the notion of the Riemann extension of nonriemannian space which were used earlier in [4–6].

Remind basic properties of this construction.

With help of the coefficients of affine connection of a given n -dimensional space can be introduced a $2n$ -dimensional Riemann space D^{2n} with local coordinates (x^i, Ψ_i) having the metric in form

$${}^6 ds^2 = -2\Pi_{ij}^k(x^l)\Psi_k dx^i dx^j + 2d\Psi_k dx^k \quad (8)$$

where Ψ_k are the additional coordinates.

The important property of such type metric is that the geodesic equations of metric (8) decomposes into two parts

$$\ddot{x}^k + \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0, \quad (9)$$

and

$$\frac{\delta^2 \Psi_k}{ds^2} + R_{kji}^l \dot{x}^j \dot{x}^i \Psi_l = 0, \quad (10)$$

where

$$\frac{\delta \Psi_k}{ds} = \frac{d\Psi_k}{ds} - \Pi_{jk}^l \Psi_l \frac{dx^j}{ds}$$

and R_{kji}^l are the curvature tensor of 3-dimensional space with a given affine connection.

The first part (9) is the system of equations for geodesic of basic space with local coordinates x^i and they does not contains the supplementary coordinates Ψ_k .

The second part (10) of the full system of geodesics has the form of linear 3×3 matrix system of second order ODE's for supplementary coordinates Ψ_k

$$\frac{d^2 \vec{\Psi}}{ds^2} + A(s) \frac{d\vec{\Psi}}{ds} + B(s) \vec{\Psi} = 0. \quad (11)$$

It is important to note that the geometry of extended space connects with geometry of basic space. For example the property of the space to be Ricci-flat keeps also for the extended space.

This fact give us the possibility to use the linear system of equation (11) for the studying of geometrical properties of the basic space.

In particular the invariants of $k \times k$ matrix-function

$$E = B - \frac{1}{2} \frac{dA}{ds} - \frac{1}{4} A^2$$

under change of the coordinates Ψ_k can be of used for that.

The first applications the notion of extended spaces for the studying of nonlinear second order differential equations connected with nonlinear dynamical systems have been considered in articles of author [4-6].

Here we consider some properties of the space defined by the Pfaff equations connected with a nonlinear dynamical systems.

3 The Lorenz dynamical system

The equations of Lorenz dynamical system are

$$\frac{dx}{ds} = \sigma(y - x), \quad \frac{dy}{ds} = rx - y - zx, \quad \frac{dz}{ds} = xy - bz, \quad (12)$$

where σ, b, r are the parameters.

These equations describe the behaviour of the flow lines along the vector field

$$\vec{N} = [\sigma(y - x), rx - y - zx, xy - bz]$$

depending from the parameters σ, b, r .

The properties of non holonomic variety L^2 in this case are determined by the following Pfaff equation

$$\sigma(y - x) \frac{dx}{ds} + (rx - y - zx) \frac{dy}{ds} + (xy - bz) \frac{dz}{ds} = 0. \quad (13)$$

The object of holonomicity for the Lorenz vector field is

$$\left(\vec{N}, \text{rot} \vec{N} \right) = \sigma xy - 2\sigma x^2 + y^2 - bzx + bz^2 + bz\sigma.$$

The properties of asymptotic lines of the corresponding variety M^2 are defined by the system of equations

$$\begin{aligned} & \left(\frac{\partial}{\partial x} P(x, y, z) \right) \left(\frac{d}{ds} x(s) \right)^2 + \left(\frac{\partial}{\partial y} Q(x, y, z) \right) \left(\frac{d}{ds} y(s) \right)^2 + \left(\frac{\partial}{\partial z} R(x, y, z) \right) \times \\ & \times \left(\frac{d}{ds} z(s) \right)^2 + \left(\frac{\partial}{\partial y} P(x, y, z) + \frac{\partial}{\partial x} Q(x, y, z) \right) \left(\frac{d}{ds} x(s) \right) \frac{d}{ds} y(s) + \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{\partial}{\partial z} P(x, y, z) + \frac{\partial}{\partial x} R(x, y, z) \right) \left(\frac{d}{ds} x(s) \right) \frac{d}{ds} z(s) + \\
& + \left(\frac{\partial}{\partial y} R(x, y, z) + \frac{\partial}{\partial z} Q(x, y, z) \right) \left(\frac{d}{ds} z(s) \right) \frac{d}{ds} y(s) = 0, \tag{14}
\end{aligned}$$

$$P(x, y, z) \frac{d}{ds} x(s) + Q(x, y, z) \frac{d}{ds} y(s) + R(x, y, z) \frac{d}{ds} z(s) = 0, \tag{15}$$

which take the form

$$\begin{aligned}
& - \left(\frac{d}{ds} y(s) \right)^2 + (r + \sigma - z) \left(\frac{d}{ds} x(s) \right) \frac{d}{ds} y(s) - \sigma \left(\frac{d}{ds} x(s) \right)^2 + \\
& + y \left(\frac{d}{ds} x(s) \right) \frac{d}{ds} z(s) - b \left(\frac{d}{ds} z(s) \right)^2 = 0, \tag{16}
\end{aligned}$$

$$(-bz + xy) \frac{d}{ds} z(s) + (rx - y - zx) \frac{d}{ds} y(s) + (\sigma y - \sigma x) \frac{d}{ds} x(s) = 0. \tag{17}$$

We find from these equations the

$$\begin{aligned}
& \frac{d}{ds} x(s) = -\frac{1}{\sigma(-y+x)} \times \\
& \times \left\{ - \left(\frac{d}{ds} y(s) \right) rx + \left(\frac{d}{ds} y(s) \right) y + \left(\frac{d}{ds} y(s) \right) zx - \left(\frac{d}{ds} z(s) \right) xy + \left(\frac{d}{ds} z(s) \right) bz \right\}, \tag{18}
\end{aligned}$$

and

$$A(x(s))^2 + Bx(s) + C = 0, \tag{19}$$

where

$$\begin{aligned}
A & = (-\sigma z(s) + \sigma r - \sigma) \left(\frac{d}{ds} y(s) \right)^2 + \left(\frac{d}{ds} y(s) \right) \sigma \left(\frac{d}{ds} z(s) \right) y(s) - b \left(\frac{d}{ds} z(s) \right)^2 \sigma, \\
B & = \left(- (y(s))^3 + 2b\sigma y(s) + y(s)bz(s) \right) \left(\frac{d}{ds} z(s) \right)^2 + \\
& + \left(rbz(s) + 2z(s)(y(s))^2 - \sigma bz(s) - (z(s))^2 b - 2r(y(s))^2 - \sigma(y(s))^2 + (y(s))^2 \right) \times
\end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{d}{ds} y(s) \right) \frac{d}{ds} z(s) + (-z(s)y(s) + \sigma z(s)y(s) - r^2 y(s) - \sigma r y(s) + 2 r z(s)y(s) + \\
& \quad + \sigma y(s) + r y(s) - (z(s))^2 y(s)) \left(\frac{d}{ds} y(s) \right)^2, \\
& C = \left(-b\sigma (y(s))^2 - b^2 (z(s))^2 + (y(s))^2 b z(s) \right) \left(\frac{d}{ds} z(s) \right)^2 + \\
& + \left(-(z(s))^2 b y(s) + r b z(s)y(s) - 2 y(s) b z(s) + (y(s))^3 + \sigma b z(s)y(s) \right) \times \\
& \times \left(\frac{d}{ds} y(s) \right) \frac{d}{ds} z(s) + \left(r (y(s))^2 - (y(s))^2 - z(s) (y(s))^2 \right) \left(\frac{d}{ds} y(s) \right)^2.
\end{aligned}$$

After differentiating the equation (19) on the variable s and taking into account the expression (18) for $\frac{dx(s)}{ds}$ we get the relation

$$E(x(s))^3 + F(x(s))^2 + Hx(s) + K = 0 \quad (20)$$

with some functions E, F, H, K which does not contains the variable $x(s)$.

The resultant of the equations (19) and (20) with regard of the variable

$$\frac{d}{ds} z(s) = \frac{\frac{d}{ds} y(s)}{\frac{d}{dz} y(z)}$$

give us a following conditions

$$(r - 1 - z) y(z) \frac{d}{dz} y(z) - b z + (y(z))^2 = 0, \quad (21)$$

$$L \left(\frac{d}{dz} y(z) \right)^2 + M \frac{d}{dz} y(z) + N = 0, \quad (22)$$

where the coefficients of equation are

$$\begin{aligned}
\frac{L}{1 - r + z} &= -z^4 b^2 + \left(2\sigma b^2 + b (y(z))^2 + 2 r b^2 \right) z^3 + \\
&+ \left((-2 r b - 2\sigma b) (y(z))^2 + 4\sigma b^2 - r^2 b^2 - \sigma^2 b^2 - 2\sigma b^2 r \right) z^2 + \\
&+ \left((y(z))^4 + (\sigma^2 b + 2\sigma r b + r^2 b - 4\sigma b) (y(z))^2 \right) z + (1 - r) (y(z))^4,
\end{aligned}$$

$$\begin{aligned}
-1/2 \frac{M}{y(z)(1-r+z)} &= -2z^3b^2 + \left(3\sigma b^2 + 3rb^2 + b(y(z))^2\right)z^2 + \\
&+ \left((-2\sigma b - rb - 2b)(y(z))^2 - 2\sigma b^2r - \sigma^2b^2 + 4\sigma b^2 - r^2b^2\right)z + \\
&+ (y(z))^4 + (-3\sigma b + \sigma rb + rb + \sigma^2b)(y(z))^2,
\end{aligned}$$

$$\begin{aligned}
N &= b^3z^4 + \left(-3(y(z))^2b^2 - 2rb^3 + 2\sigma b^3\right)z^3 + \\
&+ \left(b(y(z))^4 + (5rb^2 + 4\sigma b^2)(y(z))^2 + \sigma^2b^3 + r^2b^3 - 2\sigma b^3r\right)z^2 + \\
&+ \left((-rb - 2\sigma b - 3b)(y(z))^4 + (-8\sigma b^2r - 2r^2b^2 + 12\sigma b^2 - 2\sigma^2b^2)(y(z))^2\right)z + \\
&+ (y(z))^6 + (2rb + \sigma^2b + 2\sigma rb - b - 4\sigma b)(y(z))^4 + (4\sigma b^2 + 4\sigma r^2b^2 - 8\sigma b^2r)(y(z))^2
\end{aligned}$$

The solutions of the first order differential equations (21), (22) together with conditions (16)-(17) allow us to get the expressions for asymptotic line of the variety L^2 .

Let us consider some examples.

The equation (22) has a set of singular solutions $y(z)$.

One of them determines from the relation

$$z^2b + (-2\sigma b - 2rb)z + y^2 + 2br\sigma + br^2 - 4\sigma b + b\sigma^2 = 0$$

which presents the second order curve in the plane (z,y) .

Remark that the equation (22) presents the algebraic curve

$$\Phi(y', y, z) = 0$$

of genus $g = 1$ with respect to variables y', y in case $r \neq 1$ and genus $g = 0$ when $r = 1$.

According with general theory some of such type equations can be integrated with help of elliptic functions or can be brought to integration of the Rikkati equation.

In both cases the properties of asymptotic lines should be dependent from the parameters of model.

4 The Rössler dynamical system

Differential equations of the Rössler dynamical model are

$$\frac{dx}{ds} = -y - z, \quad \frac{dy}{ds} = x + ay, \quad \frac{dz}{ds} = b + xz - cz, \quad (23)$$

where a, b, c are the parameters.

For corresponding non holonomic variety V^2 we have a following Pfaff equation

$$(-y - z) \frac{dx}{ds} + (x + ay) \frac{dy}{ds} + (b + xz - cz) \frac{dz}{ds} = 0. \quad (24)$$

The object of holonomicity for the Rössler vector field is

$$\left(\vec{N}, \text{rot} \vec{N} \right) = -x + xz - ay - ayz + 2b - 2cz.$$

In this case the properties of the system (16)-(17) for asymptotic lines are determined by the equation

$$A \frac{d^2}{dz^2} y(z) + B = 0 \quad (25)$$

where

$$\begin{aligned} A = & (-y(z)az + az - az^2 + y(z)a) \left(\frac{d}{dz} y(z) \right)^2 + \\ & + \left(-2az^3 - 2y(z)az - 2az^2y(z) - 2(y(z))^2 a \right) \frac{d}{dz} y(z) - \\ & - ay(z)z^3 - 2bz - z^3c + cz^2 + y(z)c + bz^2 - cy(z)z - a(y(z))^2 z + \\ & + (y(z))^2 a + b + az^2y(z) \end{aligned}$$

and

$$\begin{aligned} B = & 2(z + y(z)) \left(\left(\frac{d}{dz} y(z) \right) ay(z) + \left(\frac{d}{dz} y(z) \right) c - \right. \\ & \left. - \left(\frac{d}{dz} y(z) \right)^3 a - az \left(\frac{d}{dz} y(z) \right)^2 + b \right). \end{aligned}$$

This is equation of the second range at the condition $a \neq 0$.

In the case $a = 0$ it takes a form of the first range equation

$$\begin{aligned} & (2bz - y(z)c + z^3c - cz^2 - b + cy(z)z - bz^2) \frac{d^2}{dz^2} y(z) + 2cz \frac{d}{dz} y(z) + \\ & + 2y(z)c \frac{d}{dz} y(z) + 2bz + 2y(z)b = 0. \end{aligned}$$

At the conditions $a = 0$, $b = 0$, $c \neq 0$ the properties of asymptotic lines of corresponding non holonomic variety are dependent from the solutions of the equation

$$(z^3 - z^2 - y(z) + y(z)z) \frac{d^2}{dz^2} y(z) + 2z \frac{d}{dz} y(z) + 2y(z) \frac{d}{dz} y(z) = 0.$$

5 Quadratic systems

Here we consider the properties of the Pfaff equations connected with a polynomial differential systems in R^2 defined by

$$\frac{dx}{ds} = p(x, y), \quad \frac{dy}{ds} = q(x, y), \quad (26)$$

where $p(x, y)$ and $q(x, y)$ are polynomials of degree 2.

The system (26) takes the form of the Pfaff equation after extension on a projective plane

$$(xQ(x, y, z) - yP(x, y, z)) dz - zQ(x, y, z) dx + zP(x, y, z) dy = 0, \quad (27)$$

where

$$P(x, y, z), \quad Q(x, y, z), \quad R(x, y, z)$$

are the homogeneous functions constructed from the functions $p(x, y)$ and $q(x, y)$.

As example for the system

$$\frac{dx}{ds} = kx + ly + ax^2 + bxy + cy^2, \quad \frac{dy}{ds} = mx + ny + ex^2 + fxy + hy^2,$$

with a ten parameters one get a Pfaff equation after a projective extension

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0,$$

or

$$\begin{aligned} &((-ny - mx)z^2 + (-hy^2 - fxy - ex^2)z) dx + ((ly + kx)z^2 + \\ &+ (cy^2 + bxy + ax^2)z) dy + ((-ly^2 + (n - k)xy + mx^2)z - cy^3 + \\ &+ (h - b)xy^2 + (f - a)x^2y + ex^3) dz = 0. \end{aligned} \quad (28)$$

This equation corresponds the system

$$\frac{dx}{ds} = P(x, y, z), \quad \frac{dy}{ds} = Q(x, y, z), \quad \frac{dz}{ds} = R(x, y, z), \quad (29)$$

where

$$P(x, y, z) = (-ny - mx)z^2 + (-hy^2 - fxy - ex^2)z,$$

$$Q(x, y, z) = (ly + kx)z^2 + (cy^2 + bxy + ax^2)z,$$

$$R(x, y, z) = (-ly^2 + (n - k)xy + mx^2)z - cy^3 + (h - b)xy^2 + (f - a)x^2y + ex^3$$

It is important to note that the vector field $\vec{N} = (P, Q, R)$ connected with a system (32) is holonomic due the condition

$$(\vec{N}, \text{rot}\vec{N}) = 0.$$

Corresponding Pfaff (31) equation is integrable and determines the family of developing surfaces $U(x, y, z) = C$ in the R^3 -space.

The investigation of the asymptotic lines of the surfaces which corresponds the equation (31) may be useful in applications.

Let us consider some of examples.

The system of equations

$$\frac{dx}{ds} = 5x + 6x^2 + 4(1 + \mu)xy + \mu y^2, \quad \frac{dy}{ds} = x + 2y + 4xy + (2 + 3\mu)y^2,$$

with $(-71 + 17\sqrt{17})/32 < \mu < 0$ possesses the invariant algebraic curve

$$x^2 + x^3 + x^2y + 2\mu xy^2 + 2\mu xy^3 + \mu y^4 = 0.$$

The conditions of compatibility of equations for asymptotic lines of the variety defined by the corresponding Pfaff equation lead to a following conditions on the functions

$$(3\mu^2 + 4\mu + 2)(y(s))^2 + (9\mu + 6)x(s)y(s) + 6(x(s))^2 = 0,$$

$$(y(s))^2\mu + (-2\mu - 1)x(s)y(s) - 3(x(s))^2 = 0.$$

The values of parameter μ determined by the conditions

$$(3\mu + 10)(\mu + 4)(\mu - 2) = 0, \quad \mu = 0$$

are special.

Next example is the system with at least a four limit cycles.

$$\frac{dx}{ds} = kx - y - 10x^2 + bxy + y^2, \quad \frac{dy}{ds} = x + x^2 + fxy,$$

The studying of asymptotic lines for this system give a following conditions on parameters

$$10k^2 + 11fk^2 + f^2k^2 - 7k + 2fk - 9b + k^3 - 80 - 81f + b^2k + 2bk^2 + 9bk + 9fbk = 0$$

and functions

$$x^3 + 10y(x)x^2 - (y(x))^3 + fx^2y(x) - bx(y(x))^2 = 0,$$

$$f(y(x))^2 - fy(x)kx + (y(x))^2 + y(x)bx + y(x)x - kx^2 - 10x^2 = 0.$$

Remark that these equations equivalent the equations of direct lines.

6 Cubic systems

By analogy can be investigated the properties of the asymptotic lines of the surfaces connected with the system

$$\frac{dx}{ds} = p(x, y), \quad \frac{dy}{ds} = q(x, y), \quad (30)$$

where $p(x, y)$ and $q(x, y)$ are polynomials of degree 3.

Let us consider the examples.

The system

$$\frac{dx}{ds} = y, \quad \frac{dy}{ds} = -x - x^2y + mu^2y \quad (31)$$

connects with the Van der Pol equation.

After extension on the projective plane we get an integrable Pfaff equation

$$(-x^2z^2 - x^3y + x\mu^2yz^2 - y^2z^2) dz + (z^3x + zx^2y - z^3\mu^2y) dx + yz^3 dy = 0.$$

The equations for the asymptotic lines of corresponding surface give the conditions

$$x^2 + (y(x))^2 - x\mu^2y(x) = 0$$

and

$$-x^2 + 2(y(x))^2 + 4x\mu^2y(x) = 0.$$

From the first condition we find

$$y(x) = \left(1/2\mu^2 + (-)1/2\sqrt{\mu^4 - 4}\right) x,$$

and the second gives us

$$y(x) = \left(-\mu^2 + (-)1/2\sqrt{4\mu^4 + 2}\right) x.$$

For the system

$$\frac{dx}{ds} = -y + ax(x^2 + y^2 - 1), \quad \frac{dy}{ds} = x + by(x^2 + y^2 - 1) \quad (32)$$

the point $(0, 0)$ is node at the condition $ab > -1$ and $(a - b)^2 \geq 4$ and it has the limit cycle around this point.

After extension we get an integrable Pfaff equation

$$\begin{aligned} & (x^2z^2 + byx^3 + xby^3 - xbyz^2 + y^2z^2 - yax^3 - axy^3 + yaxz^2) dz + \\ & + (-z^3y + zax^3 + zaxy^2 - z^3ax) dy + (-z^3x - zbyx^2 - zby^3 + z^3by) dx = 0 \end{aligned}$$

with a developing surfaces as general integral.

The conditions on parameters for existence of asymptotic lines are

$$ab + 1 = 0, \quad -a + b = 0.$$

For the corresponding functions one get the equations of direct lines

$$xby(x) - x^2 - y(x)ax - (y(x))^2 = 0,$$

or

$$y(x) = xby(x) - x^2 - y(x)ax - (y(x))^2,$$

and

$$\begin{aligned} & (30ab + 8a^3b - 25a^2 - 9b^2 - 12a^4)x^6 + \\ & + (36b^3 + 76ba^2 - 108ab^2 - 4a^3 - 32a^3b^2 + 32ba^4)yx^5 + \\ & + (69b^2 + 53a^2 + 36a^4 - 64a^3b - 134ab + 24ab^3 - 8a^2b^2)y^2x^4 + \\ & + (-24b^3 - 8ba^2 + 64a^3b^2 + 8ab^2 + 24a^3 - 64a^2b^3)y^3x^3 + \\ & + (-8a^2b^2 - 134ab + 24a^3b + 36b^4 - 64ab^3 + 69a^2 + 53b^2)y^4x^2 + \\ & + (-32b^4a - 36a^3 - 76ab^2 + 32a^2b^3 + 4b^3 + 108ba^2)y^5x + \\ & + (-25b^2 - 9a^2 + 30ab + 8ab^3 - 12b^4)y^6 = 0 \end{aligned}$$

with a very complicate relations between the parameters a, b .

7 Quatric systems

The system

$$\frac{dx}{ds} = -Ay + yx^2 - x^4, \quad \frac{dy}{ds} = ax - x^3 \quad (33)$$

has a center and the limit cycle at the conditions $a = 2A + A^2$, $0 < A < 5.10^{-5}$.

After extension we get an integrable Pfaff equation

$$\begin{aligned} & (z^2x^3 - z^4ax) \frac{d}{ds}x(s) + (-x^4z - z^4Ay + z^2x^2y) \frac{d}{ds}y(s) + \\ & + (ax^2z^3 - x^4z + Ay^2z^3 - zx^2y^2 + yx^4) \frac{d}{ds}z(s) = 0 \end{aligned}$$

with a developing surfaces as general integral.

For the functions one get the equations of direct lines

$$x^{10} + (3a - 9A)z^2x^8 + (24A^2 - 3Aa)z^4x^6 + (A^2 - 36A^3 - 2Aa + a^2)z^6x^4 +$$

$$(6A^3 + 36A^3a - 12A^2a + 6a^2A)z^8x^2 + (9a^2A^2 - 18A^3a + 9A^4)z^{10} = 0$$

and the family of conics

$$(z(s))^2 a - (x(s))^2 = 0$$

$$(x(s))^2 - A(z(s))^2 = 0.$$

8 Geodesics of the first kind

A geodesics of the first kind on non holonomic variety are defined by the system of equations (5)-(7) .

Let us consider the solutions of this system of equations for the variety V^2 defined by the vector field

$$\vec{F} = [ayz, bxz, cxy],$$

where a, b, c are parameters.

In this case the condition $\vec{N} = 0$ holds and variety is holonomic.

The system of equations for geodesics is

$$\begin{aligned} \frac{d^2}{ds^2}x(s) + \frac{ayzx(c+b) \left(\frac{d}{ds}y(s)\right) \frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \frac{ay^2z(a+c) \left(\frac{d}{ds}x(s)\right) \frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \\ + \frac{ayz^2(a+b) \left(\frac{d}{ds}x(s)\right) \frac{d}{ds}y(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} = 0, \end{aligned}$$

$$\begin{aligned} \frac{d^2}{ds^2}y(s) + \frac{bx^2z(c+b) \left(\frac{d}{ds}y(s)\right) \frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \frac{bxzy(a+c) \left(\frac{d}{ds}x(s)\right) \frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \\ + \frac{bxz^2(a+b) \left(\frac{d}{ds}x(s)\right) \frac{d}{ds}y(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} = 0, \end{aligned}$$

$$\begin{aligned} \frac{d^2}{ds^2}z(s) + \frac{cx^2y(c+b) \left(\frac{d}{ds}y(s)\right) \frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \frac{cxy^2(a+c) \left(\frac{d}{ds}x(s)\right) \frac{d}{ds}z(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} + \\ + \frac{cxyz(a+b) \left(\frac{d}{ds}x(s)\right) \frac{d}{ds}y(s)}{a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2} = 0. \end{aligned}$$

This system of equations has an integral

$$ay(s)z(s) \frac{d}{ds}x(s) + bx(s)z(s) \frac{d}{ds}y(s) + cx(s)y(s) \frac{d}{ds}z(s) = 0$$

and can be integrated.

In fact after substitution of the expression $\frac{d}{ds}z(s)$ into the above equations one get the algebraic relations with respect the variable $z(s)$ which are compatible at the conditions

$$(ab + b^2) (z(y))^2 + 2bz(y)cy \frac{d}{dy}z(y) + (ay^2c + c^2y^2) \left(\frac{d}{dy}z(y)\right)^2 = 0$$

and lead to the solution

$$z(y) = y^K - C1$$

where

$$K = -\frac{cb}{c^2 + ac} - \frac{\sqrt{-cab(a+c+b)}}{c^2 + ac}.$$

Finally we present the expression for the Chern-Simons invariant of affine connection defined by the equations of geodesics (5-7) in case of the Lorenz system of equations.

It has been obtained with the help of a six dimensional Riemann extension of corresponding space and has the form

$$CS(\Gamma) = \int \frac{L}{2M^2} dx dy dz,$$

where

$$\begin{aligned} L = & (2b + 2 - 2\sigma)x^2y^2 + (3\sigma^2 + 4\sigma r - 4rb - 2b\sigma - 4r)zx^2 + (2b + 2 - 2\sigma)z^2x^2 + \\ & + (-3r\sigma^2 + 4\sigma^2b - 5\sigma^3 + 2b\sigma r - 2\sigma r^2 + 4\sigma^2 + 2br^2 + 2r^2)x^2 + \\ & + (-2rb - 4r + 9\sigma^3 + 2r\sigma^2)yx + (-2\sigma + 4 - 2\sigma^2 - 2\sigma r + 2b\sigma - 4b^2)zyx - \\ & - ((b\sigma r + 2\sigma^2 + \sigma^2b - \sigma z^2 - 2\sigma r - \sigma r^2)y - \sigma y^3)x + (-2\sigma b^2 + 2b^3 - 2b^2)z^2 + \\ & + (-4\sigma^3 - \sigma^2b + 2 - 2b - \sigma^2 + \sigma r + (-\sigma + b\sigma)z - b\sigma r - 2\sigma)y^2 + \\ & + (-2\sigma b^2 + 2rb^2 + 2b^2r\sigma - 2b^2\sigma^2)z, \end{aligned}$$

and

$$\begin{aligned} M = & (x^2 + b^2)z^2 + ((-2b + 2)xy - 2rx^2)z + \\ & + (\sigma^2 + 1 + x^2)y^2 + (-2\sigma^2 - 2r)xy + (\sigma^2 + r^2)x^2. \end{aligned}$$

References

- [1] SINTSOV D.M. *Raboty po negolonomnoi geometrii*. Kiev, 1972.
- [2] SLUCHAEV V.V. *Geometriya vektornykh polei*. Tomsk, Izd. Tomskogo universiteta, 1982.
- [3] AMINOV YU.A. *Geometriya vektornogo polya*. Moskva, Nauka, 1990.
- [4] DRYUMA V. *The Riemann Extensions in theory of differential equations and their applications*. Matematicheskaya fizika, analiz, geometriya, 2003, **10**, N 3, p. 1-19.
- [5] DRYUMA V. *The Riemann and Einstein-Weyl geometries in the theory of ODE's, their applications and all that*. New Trends in Integrability and Partial Solvability, p. 115-156 (eds. A.B. Shabat et al.). Kluwer Academic Publishers (ArXiv: nlin: SI/0303023, 11 March, 2003, p. 1-37).
- [6] DRYUMA V. *Applications of Riemannian and Einstein-Weyl Geometry in the theory of second order ordinary differential equations*. Theoretical and Mathematical Physics, 2001, **128**, N 1, p. 845-855.

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