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Absolute Asymptotic Stability of Discrete Linear Inclusions

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Abstract. The article is devoted to the study of absolute asymptotic stability of discrete linear inclusions in Banach (both finite and infinite dimensional) space. We establish the relation between absolute asymptotic stability, asymptotic stability, uniform asymptotic stability and uniform exponential stability. It is proved that for asymptotical compact (a sum of compact operator and contraction) discrete linear inclusions the notions of asymptotic stability and uniform exponential stability are equivalent. It is proved that finite-dimensional discrete linear inclusion, defined by matrices $\{A_1, A_2, ..., A_m\}$, is absolutely asymptotically stable if it does not admit nontrivial bounded full trajectories and at least one of the matrices $\{A_1, A_2, ..., A_m\}$ is asymptotically stable. We study this problem in the framework of non-autonomous dynamical systems (cocyles).

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1 Introduction

The aim of this paper is studying of the problem of the absolute asymptotic stability of discrete linear inclusion (see Gurvits [22] and the references therein)

$$x_{t+1} \in F(x_t), \tag{1}$$

where $F(x) = \{A_1x, A_2x, ..., A_mx\}$ for all $x \in E^d$ (E^d is a d-dimensional euclidian space) and A_i $(1 \le i \le m)$ is a $d \times d$-matrix.

The article is devoted to the study of absolute asymptotic stability of discrete linear inclusions in Banach space (both finite and infinite-dimensional case). The problem of asymptotic stability for the discrete linear inclusion arise in a number of different areas of mathematics: control theory – Molchanov [29]; linear algebra – Artzrouni [2], Beyn and Elsner [3], Bru, Elsner and Neumann [7], Daubechies and Lagarias [16], Elsner and Friedland [17], Elsner, Koltracht and Neumann [18], Gurvits [22], Vladimirov, Elsner and Beyn [40]; Markov Chains – Gurvits [19], Gurvits and Zaharin [20, 21]; iteration process – Bru, Elsner and Neumann [7], Opoitsev [30] and see also the bibliography therein.

We establish the relation between absolute asymptotic stability (AAS), asymptotic stability (AS), uniform asymptotic stability (UAS) and uniform exponential stability (UES). It is proved that for asymptotically compact (a sum of compact operator and contraction) discrete linear inclusions these notions of stability are equivalent. We study this problem in the framework of non-autonomous dynamical systems (cocyles). We show that the problem of absolute asymptotic stability for the discrete linear inclusions is related with the compact global attractors of non-autonomous dynamical systems (both ordinary dynamical systems (with uniqueness) and set-valued dynamical systems). We plan to continue the studying of discrete inclusions (both linear and nonlinear) in the framework of non-autonomous dynamical systems. In our future publications we will give the proofs of the followings results:

(i) infinite-dimensional discrete linear inclusion, defined by compact operators $\{A_1, A_2, ..., A_m\}$, is absolutely asymptotically stable if it does not admit non-trivial bounded full trajectories and at least one of the operators $\{A_1, A_2, ..., A_m\}$ is asymptotically stable;

(ii) discrete inclusion, defined by nonlinear (in particularly, affine) contractive mappings $\{A_1, A_2, ..., A_m\}$ admits a compact global chaotic attractor,

amongst others. We consider that this method of studying of discrete inclusions (both linear and nonlinear) is fruitful and it permits to obtain the new and nontrivial results.

This paper is organized as follows.

In Section 2 we give a new approach to the study of discrete linear inclusions (DLI) which is based on non-autonomus dynamical systems (cocycles).

Section 3 is devoted to the study of DLIs in arbitrary Banach spaces. We show that for an infinite-dimensional DLI the notions of asymptotic stability and uniform asymptotic stability are not equivalent (Example 3.1). We prove the equivalence of the uniform asymptotic stability and generalized contraction for DLIs (Theorem 3.5). If a discrete linear inclusion (DLI) is completely continuous (compact), then we prove that absolute asymptotic stability and uniform exponential stability are equivalent. We also give the description of absolute asymptotic stability in term of joint spectral radius.

Section 4 is dedicated to the study of asymptotically compact discrete linear inclusions. We establish the relation between different types of stability for this class of DLIs. The main results of this sections are Theorems 4.16, 4.17 and 4.18.

In Section 5 we study the problem of absolute asymptotic stability for finite-dimensional discrete linear inclusions. We establish some general properties of semi-group non-autonomous linear dynamical systems and we prove that finite-dimensional discrete linear inclusion, defined by matrices $\{A_1, A_2, ..., A_m\}$, is absolute asymptotic stable if it doesn't admit non-trivial bounded full trajectories and at least one of the matrices $\{A_1, A_2, ..., A_m\}$ is asymptotically stable (Theorem 5.24 - the main result of paper).

2 Discrete Linear Inclusions and Cocycles

Let E be a real or complex Banach space with norm $|\cdot|$, $\mathbb{S}$ be a group of real ($\mathbb{R}$) or integer ($\mathbb{Z}$) numbers, $\mathbb{T}$ ($\mathbb{S}_+ := \{s \in \mathbb{S} \ : \ s \geq 0\} \subseteq \mathbb{T}$) be a semi-group of the additive group $\mathbb{S}$. Denote by $[E]$ the space of all bounded operators $A : E \to E$. Consider a set of operators $\mathcal{M} \subseteq [E]$.

Definition 2.1. *A discrete linear (autonomous) inclusion $DLI(\mathcal{M})$ is called (see, for example,[22]) a set of all sequences $\{x(t)\}_{t\in\mathbb{Z}_+}$ of vectors in E such that*

$$x(t+1) = A(t)x(t) \tag{2}$$

for some $A(t) \in \mathcal{M}$, i.e.

$$x(t) = A(t)A(t-1)...A(1)A(0)x(0) \ \text{ all } A(t) \in \mathcal{M},$$

where $A(0) := Id_E$.

Definition 2.2. *The bilateral sequence $\{x(t)\}_{t\in\mathbb{Z}}$ of vectors in E is called a full trajectory of $DLI(\mathcal{M})$ (entire trajectory or trajectory on $\mathbb{Z}$) if $x(t+s+1) = A(t)x(t+s)$ for all $s \in \mathbb{Z}$ and $t \in \mathbb{Z}_+$.*

We may consider this a discrete control problem, where at each moment of time t we may apply a control from the set $\mathcal{M}$, and $DLI(\mathcal{M})$ is the set of possible trajectories of the system. The basic issue for any control system concerns its stability. One of the most important types of stability is so-called absolute asymptotic stability (AAS).

Definition 2.3. *$DLI(\mathcal{M})$ is called absolutely asymptotically stable (AAS) (or convergent) if for any its trajectory $\{x(t)\}$ we have*

$$\lim_{t\to\infty} x(t) = 0.$$

Equivalently, all operator products

$$\lim_{t\to\infty} A(t)A(t-1)...A(1)A(0)x = 0 \quad (\text{all } A(t) \in \mathcal{M}) \tag{3}$$

for very $x \in E$.

Definition 2.4. *The set $\mathcal{M} \subseteq [E]$ of operators is called product bounded (or uniformly stable) if there exists a $M > 0$ such that $\|A(t)A(t-1)...A(1)A(0)\| \leq M$ for all finite sequence $\{A(t)\}_{t\in\mathbb{Z}_+}$ $(A(t) \in \mathcal{M})$.*

Definition 2.5. *$DLI(\mathcal{M})$ is said to be asymptotically stable (AS) if it is product bounded (or uniformly stable) and convergent.*

Let (X,ρ) be a complete metric space with the metric ρ. Denote by $C(X)$ the family of all compact subsets of X. Consider the set-valued function $F : E \to C(E)$ defined by the equality $F(x) := \{Ax \mid A \in \mathcal{M}\}$. Then the discrete linear inclusion $DLI(\mathcal{M})$ is equivalent to the difference inclusion

$$x(t+1) \in F(x(t)). \tag{4}$$

Denote by $\mathcal{F}_{x_0}$ the set of all trajectories of discrete inclusion (4) (or $DLI(\mathcal{M})$) issuing from the point $x_0 \in E$ and $\mathcal{F} := \bigcup\{\mathcal{F}_{x_0} \mid x_0 \in E\}$.

Below we will give a new approach concerning the study of discrete linear inclusions $DLI(\mathcal{M})$ (or difference inclusion (4)). Denote by $C(\mathbb{Z}_+, X)$ the space of all continuous mappings $f : \mathbb{Z}_+ \to X$ equipped with the compact-open topology. This topology can be metrized, for example, by the equality

$$d(f^1, f^2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f^1, f^2)}{1 + d_n(f^1, f^2)}$$

$$(d_n(f^1, f^2) := \max\{|f^1(k) - f^2(k)| \mid 0 \le k \le n\})$$

is defined a complete metric on $C(\mathbb{Z}_+, X)$ which generates compact-open topology. Denote by $(C(\mathbb{Z}_+, X), \mathbb{Z}_+, \sigma)$ a dynamical system of translations (shifts dynamical system or dynamical system of Bebutov [5,12,13,36–38]) on $C(\mathbb{Z}_+, X)$, i.e. $\sigma(k, f) := f_k$ and f_k is a $k \in \mathbb{Z}_+$ shift of f (i.e. $f_k(n) := f(n+k)$ for all $n \in \mathbb{Z}_+$).

Let now $Q \subseteq X$ be a compact. Denote by $C(\mathbb{Z}_+, Q) := \{f \in C(\mathbb{Z}_+, X) \mid f(\mathbb{Z}_+) \subseteq Q\}$. It is easy to see that $C(\mathbb{Z}_+, Q)$ is invariant (with respect to shifts) and closed subset of $(C(\mathbb{Z}_+, X), \mathbb{Z}_+, \sigma)$ and, consequently, on the space $C(\mathbb{Z}_+, Q)$ is defined a dynamical system of shifts $(C(\mathbb{Z}_+, Q), \mathbb{Z}_+, \sigma)$ (induced by the dynamical system of Bebutov $(C(\mathbb{Z}_+, X), \mathbb{Z}_+, \sigma)$). Note that by the theorem of Tikhonoff [26] the space $C(\mathbb{Z}_+, Q)$ is compact.

Let $\mathcal{M}$ be a compact subset of $[E]$ (for example, $\mathcal{M}$ may be a finite set, i.e. $\mathcal{M} = \{A_1, A_2, ..., A_m : A_i \in [E]\ (1 \le i \le m)\}$). Denote by $\Omega := \{f \in C(\mathbb{Z}_+, [E]) \mid f(\mathbb{Z}_+) \subseteq \mathcal{M}\}$. It is clear that Ω is an invariant (with respect to shifts) and closed subset of $C(\mathbb{Z}_+, [E])$ and, hence, on the space Ω is defined a dynamical system of shifts $(\Omega, \mathbb{Z}_+, \sigma)$ (induced by the dynamical system of Bebutov $(C(\mathbb{Z}_+, [E]), \mathbb{Z}_+, \sigma)$). Notice that by the Tikhonoff's theorem the space Ω is compact in $C(\mathbb{Z}_+, [E])$.

We may now rewrite equation (2) in the following way:

$$x(t+1) = \omega(t)x(t),\ \ (\omega \in \Omega) \tag{5}$$

where $\omega \in \Omega$ is the operator-function defined by the equality $\omega(t) := A(t)$ for all $t \in \mathbb{Z}_+$. Denote by $\varphi(t, x_0, \omega)$ a solution of equation (5) issuing from the point $x_0 \in E$ at the initial moment $t = 0$. Note that $\mathcal{F}_{x_0} = \{\varphi(\cdot, x_0, \omega) \mid \omega \in \Omega\}$ and $\mathcal{F} = \{\varphi(\cdot, x_0, \omega) \mid x_0 \in E^d, \omega \in \Omega\}$, i.e. $DLI(\mathcal{M})$ (or inclusion (4)) is equivalent to the family of linear non-autonomous equations (5) $(\omega \in \Omega)$.

From the general properties of linear difference equations it follows that the mapping $\varphi : \mathbb{Z}_+ \times E^d \times \Omega \to E$ satisfies the following conditions:

(i) $\varphi(0, x_0, \omega) = x_0$ for all $(x_0, \omega) \in E \times \Omega$;

(ii) $\varphi(t+\tau, x_0, \omega) = \varphi(t, \varphi(\tau, x_0, \omega), \sigma(\tau, \omega))$ for all $t, \tau \in \mathbb{Z}_+$ and $(x_0, \omega) \in E \times \Omega$;

(iii) the mapping φ is continuous;

(iv) $\varphi(t, \lambda x_1 + \mu x_2, \omega) = \lambda\varphi(t, x_1, \omega) + \mu\varphi(t, x_2, \omega)$ for all $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $x_1, x_2 \in E$ and $\omega \in \Omega$.

Let W, Ω be two complete metric spaces and $(\Omega, \mathbb{Z}_+, \sigma)$ be a discrete semi-group dynamical system on Ω.

Definition 2.6. *Recall [36] that a triplet $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ (or shortly φ) is called a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with the fiber W if φ is a mapping from $\mathbb{Z}_+ \times W \times \Omega$ to W satisfying the following conditions:*

1. $\varphi(0, x, \omega) = x$ *for all* $(x, \omega) \in W \times \Omega$;
2. $\varphi(t+\tau, x, \omega) = \varphi(t, \varphi(\tau, x, \omega), \sigma(\tau, \omega))$ *for all* $t, \tau \in \mathbb{Z}_+$ *and* $(x, \omega) \in W \times \Omega$;
3. *the mapping φ is continuous.*

If W is a real or complex Banach space and

4. $\varphi(t, \lambda x_1 + \mu x_2, \omega) = \lambda\varphi(t, x_1, \omega) + \mu\varphi(t, x_2, \omega)$ *for all* $\lambda, \mu \in \mathbb{R}$ *(or $\mathbb{C}$),* $x_1, x_2 \in W$ *and* $\omega \in \Omega$,

then the cocycle φ is called linear.

Let $X := W \times \Omega$, and define the mapping $\pi : X \times \mathbb{T}_1 \to X$ by the equality: $\pi((u, \omega), t) := (\varphi(t, u, \omega), \sigma(t, \omega))$ (i.e. $\pi = (\varphi, \sigma)$). Then it is easy to check that $(X, \mathbb{T}_1, \pi)$ is a dynamical system on X, which is called a skew-product dynamical system [1,36]; but $h = pr_2 : X \to \Omega$ is a homomorphism of $(X, \mathbb{T}_1, \pi)$ onto $(\Omega, \mathbb{T}_2, \sigma)$.

Definition 2.7. *Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be two dynamical systems and $h : X \to Y$ be a homomorphism from $(X, \mathbb{T}, \pi)$ onto $(Y, \mathbb{T}, \sigma)$. A triplet $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is called a non-autonomous dynamical system.*

Thus, if we have a cocycle $\langle W, \varphi, (\Omega, \mathbb{T}_2, \sigma)\rangle$ over the dynamical system $(\Omega, \mathbb{T}_2, \sigma)$ with the fiber W, then there can be constructed a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h\rangle$ ($X := W \times \Omega$), which we will call a non-autonomous dynamical system generated (associated) by the cocycle $\langle W, \varphi, (\Omega, \mathbb{T}_2, \sigma)\rangle$ over $(\Omega, \mathbb{T}_2, \sigma)$.

From the presented above it follows that every $DLI(\mathcal{M})$ (respectively, inclusion (4)) in a natural way generates a linear cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$, where $\Omega = C(\mathbb{Z}_+, \mathcal{M})$, $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system of shifts on Ω and $\varphi(t, x, \omega)$ is a solution of equation (5) issuing from the point $x \in E$ at the initial moment $n = 0$. Thus, we can study inclusion (4) (respectively, $DLI(\mathcal{M})$) in the framework of the theory of linear cocycles with discrete time.

3 Absolute Asymptotic Stability of Discrete Linear Inclusions in Banach Spaces

In this section we will study $DLI(\mathcal{M})$ in an arbitrary Banach space. Let E be a real or complex Banach space with the norm $|\cdot|$ and $[E]$ be a Banach space of all linear bounded operators acting on the space E and equipped with the operational norm. Below we suppose that $\mathcal{M} := \{A_1, A_2, ..., A_m\}$ and $A_i \in [E]$.

Note that for infinite-dimensional discrete linear inclusions $DLI(\mathcal{M})$ ($\dim(E) < +\infty$) the notion of absolute asymptotic stability (AAS) and the equality

$$\lim_{t\to+\infty} \|A(t)A(t-1)...A(1)A(0)\| = 0 \tag{6}$$

are equivalent. It is easy to see that for infinite-dimensional $DLI(\mathcal{M})$ ($\dim(E) = +\infty$) it is not true. This fact is confirmed by the following example.

Example 3.1. *Let $E := c_0$, $A \in [c_0]$ be the operator defined by the equality*

$$A\xi := \{\xi_{k+1}\}$$

for all $\xi := \{\xi_k\} \in c_0$. It is easy to verify that the operator A possesses the following properties:

(i)

$$A^n\xi \to 0 \tag{7}$$

as $n \to \infty$ for each $\xi \in l_2$, where $A^n := A \circ A^{n-1}$ $(n \geq 1)$ and $A^0 := Id_E$;

(ii)

$$A^n e_{n+1} = e_1, \tag{8}$$

where $e_1 = (1, 0, 0, ...), e_2 = (0, 1, 0, ...), ...$ $(n = 1, 2, ...)$.

Let $\mathcal{M} := \{A\}$, i.e. $m = 1$. In this case $DLI(\mathcal{M})$ is equivalent to the linear autonomous difference equation

$$x(t + 1) = Ax(t).$$

From (7) it follows that $DLI(\mathcal{M})$ (with $\mathcal{M} = \{A\}$) is absolutely asymptotically stable. On the other hand, from equality (8) we have $\|A^n\| \geq 1$ and, consequently, equality (6) does not hold.

Let (X, h, Y) be a locally trivial Banach fiber bundle [4,24].

Definition 3.2. *A non-autonomous dynamical system $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is said to be linear, if the map $\pi^t : X_y \to X_{\sigma(t,y)}$ $(X_y := h^{-1}(y))$ is linear for every $t \in \mathbb{T}$ and $y \in Y$, where $\pi^t := \pi(t, \cdot)$.*

Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a linear cocycle over $(Y, \mathbb{T}, \sigma)$ with the fiber E (or shortly φ). If $X := E \times Y$ and $(X, \mathbb{T}, \pi)$ is a skew-product dynamical system, then the triplet $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$, where $h := pr_2 : X \to Y$, is a linear non-autonomous dynamical system generated by cocycle φ.

Theorem 3.3. *[13] Let E be a Banach space, Ω be a compact metric space and $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ be a linear cocycle over $(\Omega, \mathbb{Z}_+, \sigma)\rangle$. Then the next conditions are equivalent:*

(i) the cocycle φ is uniformly asymptotically stable, i.e.

$$\lim_{n\to+\infty} \sup_{\omega\in\Omega} \|U(n,\omega)\| = 0;$$

(ii) the cocycle φ is uniformly exponentially stable, i.e. there are two positive constants $\mathcal{N}$ and ν such that $\|U(n,\omega)\| \le \mathcal{N}e^{-\nu t}$ for all $t \ge 0$ and $\omega \in \Omega$.

Let $\mathcal{M} \subseteq [E]$ be a nonempty bounded set of operators and denote by $\mathcal{S} = \mathcal{S}(\mathcal{M})$ the semigroup generated by $\mathcal{M}$ augmented with the identity operator $I := Id_E$, so that $\mathcal{S} = \bigcup_{n=0}^{\infty} \mathcal{M}^n$, where $\mathcal{M}^n := \{\prod_{t=1}^{n} A(t) \mid A(t) \in \mathcal{M}\}$.

Definition 3.4. *The number*

$$\rho(\mathcal{M}) := \limsup_{n\to\infty} \|\mathcal{M}^n\|^{\frac{1}{n}} \quad \textit{and} \quad \|\mathcal{M}\| := \sup\{\|A\| \ : \ A \in \mathcal{M}\}$$

is called [16, 22, 32] a joint spectral radius of bounded subset of linear operators $\mathcal{M}$.

Theorem 3.5. *Let $\mathcal{M} \subset [E]$ be a compact subset (in particular, the set $\mathcal{M}$ may consist of finite number of elements, i.e. $\mathcal{M} = \{A_1, A_2, ..., A_m\}$ with $A_i \in [E]$ $(1 \le i \le m)$). Then the following statements are equivalent:*

a) the discrete linear inclusion $DLI(\mathcal{M})$ is uniformly asymptotically stable;

b) $\rho(\mathcal{M}) < 1$.

Proof. Let $\Omega := C(\mathbb{Z}_+, \mathcal{M})$. Then Ω is a compact subset of $C(\mathbb{Z}_+, [E])$ and on Ω there is defined a dynamical system of translations $(\Omega, \mathbb{Z}_+, \sigma)$ induced by the Bebutov's dynamical system $(C(\mathbb{Z}_+, [E]), \mathbb{Z}_+, \sigma)$. Consider the cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)\rangle$ generated by $DLI(\mathcal{M})$, i.e. $\varphi(t, \omega, x) := A(t)A(t-1)...A(1)A(0)x$, where $A(t) \in \mathcal{M}$ and $\omega \in \Omega$ with $\omega(t) := A(t)$ (for all $t \in \mathbb{Z}_+$). According to Theorem 3.3 there exists two positive constants $\mathcal{N}$ and ν such that $|A(t)A(t-1)\dots A(1)A(0)x| \le \mathcal{N}e^{-\nu t}|x|$ for all $A(t) \in \mathcal{M}$ $(t \in \mathbb{Z}_+)$. From the last inequality we obtain $\rho(\mathcal{M}) \le e^{-nu} < 1$.

Let now $\alpha := \rho(\mathcal{M}) < 1$. Then for all $\varepsilon \in (0, 1 - \rho(\mathcal{M}))$ there exists a number $t(\varepsilon) \in \mathbb{N}$ such that

$$\|A(t)A(t-1)...A(1)A(0)\| \le (\alpha + \varepsilon)^t \tag{9}$$

for all $t \ge t(\varepsilon)$. Since $\beta := \alpha + \varepsilon < 1$, then from inequality (9) follows the condition a). The theorem is proved. □

Remark 3.6. *The statement close to Theorem 3.5 it was established before in [22] for infinite-dimensional DLIs and in [22, 30] for finite-dimensional DLIs.*

Lemma 3.7. *[15] Suppose that each operator A of $\mathcal{M}$ is compact, then for any bounded set $B \subset E$ and $t \in \mathbb{N}$ the set $U(t,\Omega)A$ is relatively compact, where $U(t,\omega) := \varphi(t,\cdot,\omega) = A(t)A(t-1)...A(1)A(0)$ ($\omega(t) := A(t)$ for all $t \in \mathbb{Z}_+$).*

Definition 3.8. *A cocycle $\langle E,\varphi,(\Omega,\mathbb{Z}_+,\sigma)$ is called compact (completely continuous) if for each bounded subset $B \subset E$ there exists an integer number $t_0 = t_0(B) \in \mathbb{N}$ such that the set $U(t_0,\Omega)B$ is relatively compact.*

Theorem 3.9. *[13] Let Ω be a compact space and $\langle E,\varphi,(\Omega,\mathbb{Z}_+,\sigma)$ be a compact cocycle. Then the following conditions are equivalent:*

(i) the cocycle φ is convergent, i.e.

$$\lim_{t\to\infty} |\varphi(t,x,\omega)| = 0 \tag{10}$$

for all $(x,\omega) \in E \times \Omega$;

(ii) the cocycle φ is uniformly exponentially stable.

Theorem 3.10. *Let $\mathcal{M} \subseteq [E]$ be a compact subset and suppose that each operator A from $\mathcal{M}$ is compact. Then the next statements are equivalent:*

(i) the discrete linear inclusion $DLI(\mathcal{M})$ is absolutely asymptotically stable;

(ii) $\rho(\mathcal{M}) < 1$.

Proof. Consider the cocycle $\langle E,\varphi,(\Omega,\mathbb{Z}_+,\sigma)$ generated by $DLI(\mathcal{M})$. By Lemma 3.7, under the conditions of the theorem, this cocycle is compact. Now to finish the proof it is sufficient to apply Theorems 3.5 and 3.9. □

Remark 3.11. *In our paper [15] was established the equivalence of absolute asymptotic stability and uniform exponential stability for $DLI(\mathcal{M})$ in Banach spaces if $\mathcal{M} \subseteq [E]$ is compact and every operator A from $\mathcal{M}$ is compact.*

4 Asymptotically Compact Discrete Linear Inclusions

Definition 4.1. *The entire trajectory of the semigroup dynamical system $(X,\mathbb{T},\pi)$ passing through the point $x \in X$ at $t = 0$ is defined as the continuous map $\gamma : \mathbb{S} \to X$ that satisfies the conditions $\gamma(0) = x$ and $\pi^t\gamma(s) = \gamma(s+t)$ for all $t \in \mathbb{T}$ and $s \in \mathbb{S}$, where $\pi^t := \pi(t,\cdot)$.*

Let $\Phi_x(\pi)$ be the set of all entire trajectories of $(X,\mathbb{T},\pi)$ passing through x at $t = 0$ and $\Phi(\pi) = \cup\{\Phi_x(\pi) : x \in X\}$.

Definition 4.2. *A dynamical system $(X,\mathbb{T},\pi)$ is said to be asymptotically compact [23, 28] if for all bounded positively invariant set $M \subset X$ there exists a nonempty compact subset K from X such that $\lim_{t\to+\infty} \rho(\pi(t,M),K) = 0$.*

Definition 4.3. *A measure of non-compactness [23, 34] on a complete metric space X is a function β from the bounded sets of X to the nonnegative real numbers satisfying:*

(i) $\beta(A) = 0$ *for* $A \subset X$ *if and only if* A *is relatively compact;*

(ii) $\beta(A \cup B) = \max[\beta(A), \beta(B)]$;

(iii) $\beta(A + B) \leq \beta(A) + \beta(B)$ *for all* $A, B \subset X$ *if the space* X *is linear.*

Definition 4.4. *The Kuratowsky measure of non-compactness α is defined by*

$$\alpha(A) = \inf\{d : A \textit{ has a finite cover of diameter} < d\}.$$

Definition 4.5. *A dynamical system $(X, \mathbb{T}, \pi)$ (respectively, a cocycle φ) is said to be conditionally β-condensing [23] if there exists $t_0 > 0$ such that $\beta(\pi^{t_0} B) < \beta(B)$ for all bounded sets B in X with $\beta(B) > 0$ (respectively, for any bounded set $B \subseteq E$ the inequality $\alpha(\varphi(t_0, B, Y)) < \alpha(B)$ holds if $\alpha(B) > 0$.).*

Definition 4.6. *A dynamical system $(X, \mathbb{T}, \pi)$ (respectively, a cocycle φ) is said to be β-condensing if it is conditionally β-condensing and the set $\pi^{t_0} B$ is bounded for all bounded sets $B \subseteq X$ (respectively, the set $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded for all bounded set $B \subseteq E$.)*

According to Lemma 2.3.5 in [23, p.15] and Lemma 3.3 in [9] the conditional condensing dynamical system $(X, \mathbb{T}, \pi)$ is asymptotically compact.

Let $X := E \times Y$, $A \subset X$, and $A_y := \{x \in A : pr_2 x = y\}$. Then $A = \cup\{A_y : y \in Y\}$. Let $\tilde{A}_y := pr_1 A_y$ and $\tilde{A} := \cup\{\tilde{A}_y : y \in Y\}$. Note that if the space Y is compact, then a set $A \subset X$ is bounded in X if and only if the set $\tilde{A}$ is bounded in E.

Lemma 4.7. *[11, 13] The equality $\alpha(A) = \alpha(\tilde{A})$ takes place for all bounded sets $A \subset X$, where $\alpha(A)$ and $\alpha(\tilde{A})$ are the Kuratowsky measure of non-compactness for the sets $A \subset X$ and $\tilde{A} \subset E$.*

Definition 4.8. *A cocycle φ is called conditional α-contraction of order $k \in [0, 1)$, if there exists $t_0 > 0$ such that for any bounded set $B \subseteq E$ for which $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded the inequality $\alpha(\varphi(t_0, B, Y)) \leq k\alpha(B)$ holds.*

Definition 4.9. *The cocycle φ is called α-contraction if it is a conditional α-contraction cocycle and the set $\varphi(t_0, B, Y) = \cup\{\varphi(t_0, u, Y) | u \in B, y \in Y\}$ is bounded for all bounded sets $B \subseteq E$.*

Lemma 4.10. *[11, 13] Let Y be compact and the cocycle φ be α-condensing. Then the skew-product dynamical system $(X, \mathbb{T}, \pi)$, generated by the cocycle φ, is α-condensing.*

Denote by $\mathbb{B}(\pi) := \{x \in X \mid \exists \gamma \in \Phi_x(\pi)$, such that $\gamma(\mathbb{S})$ is bounded $\}$ and $\mathbb{B}_x(\pi) := \mathbb{B}(\pi) \bigcap \Phi_x(\pi)$.

Theorem 4.11. *[11, 13] Let $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-autonomous dynamical system, Y be compact and $(X,\mathbb{T},\pi)$ be conditionally α-condensing. Then the following assertions are equivalent:*

(i) (a) the non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is convergent, i.e. $\lim\limits_{t\to+\infty}|\pi(t,x)|=0$ for all $x\in X$;

(b) the dynamical system $(X,\mathbb{T},\pi)$ doesn't admit non-trivial bounded trajectories on $\mathbb{T}$, i.e. $\mathbb{B}(\pi)\subseteq\Theta=\{\theta_y : y\in Y,\theta_y\in X_y,|\theta_y|=0\}$.

(ii) the non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is uniformly exponentially stable, i.e. there are positive constants $\mathcal{N}$ and ν such that $|\pi(t,x)|\le\mathcal{N}e^{-\nu t}|x|$ for all $x\in X$.

Remark 4.12. *If the vector bundle fiber (X,h,Y) is finite dimensional (i.e. every fiber $X_y := h^{-1}(y)$ is finite-dimensional) and Y is a compact metric space, then the non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is conditionally α-condensing. Thus Theorem 4.11 is true for the finite-dimensional linear non-autonomous dynamical system with compact base Y. This fact it was established before by Cheban [8].*

Theorem 4.13. *Let the following conditions be fulfilled:*

(i) the linear non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is convergent;

(ii) Y is compact;

(iii) the dynamical system $(X,\mathbb{T},\pi)$ is conditionally α-condensing;

(iv) there exists a positive number M such that

$$|\pi(t,x)|\le M|x| \tag{11}$$

for all $x\in X$.

Then the non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is uniformly exponentially stable.

Proof. Let $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear convergent non-autonomous dynamical system, Y be compact and there exists a positive number M such that the inequality (11) holds, then by Theorem 2.11.2 [12] (see also [10] or [13]) the trivial section $\Theta := \{\theta_y \mid \theta_y\in X_y := h^{-1}(y),|\theta_y|=0\}$ of the vectorial fiber (X,h,Y) is a maximal compact invariant set of dynamical system $(X,\mathbb{T},\pi)$). Thus this system doesn't admit non-trivial bounded trajectories on $\mathbb{S}$. To finish the proof of Theorem it is sufficient to apply Theorem 4.11. □

Lemma 4.14. *Let $A', A'' \in [E]$, $A := A' + A''$ and the following conditions hold:*

(i) operator A' is contractive, i.e. $\|A'\| < 1$;

(ii) operator A'' is compact.

Then the operator A is α-contraction and $\alpha(A(B)) \le k\alpha(B)$ for all bounded subset $B \subseteq E$, where $k := \|A'\|$.

Proof. Since $A(B) \subseteq A'(B) + A''(B)$, then according to Lemma 2.2 [35] $\alpha(A(B)) \le \alpha(A'(B)) + \alpha(A''(B)) \le \|A'\|\alpha(B) + \alpha(A''(B))$. To finish the proof of Lemma it is sufficient to note that under the conditions of Lemma $\alpha(A''(B)) = 0$. □

Lemma 4.15. *Let $\mathcal{M}$ be a compact subset of $[E]$. Suppose that each operator A of $\mathcal{M}$ may be presented as a sum $A' + A''$, where A' is a contraction and A'' is a compact operator, then $\alpha(U(t, \Omega)B) \le k\alpha(B)$ for any bounded subset $B \subseteq E$ and $n \in \mathbb{N}$, where $U(t, \omega) := \varphi(t, \cdot, \omega) = A(t)A(t-1)...A(1)A(0)$ $(\omega(t) := A \in \mathcal{M}$ for all $t \in \mathbb{Z}_+)$ and $k := \prod_{j=1}^{t} \|A(j)'\| < 1$.*

Proof. Since the set Ω is compact and $U(t, \omega) = \prod_{k=1}^{t} \omega(k)$ $(\omega \in \Omega)$, then for each t the mapping $U(t, \cdot) : \Omega \to [E]$ is continuous. Note that $A(t) = A(t)' + A(t)''$ and, consequently, we have

$$U(t, \omega) := \prod_{j=1}^{t} A(j) = \prod_{j=1}^{t} (A(j)' + A(j)'') = \prod_{j=1}^{t} A(j)' + C,$$

where $C \in [E]$ is some compact operator. By Lemma 4.14 we have

$$\alpha(U(t, \omega)B) \le k_0 \alpha(B)$$

for all bounded subset $B \subseteq E$, where $k_0 := \| \prod_{j=1}^{t} A(j)' \| \le \prod_{j=1}^{t} \|A(j)'\| := k < 1$. The lemma is proved. □

Theorem 4.16. *Let $\mathcal{M}$ be a compact subset of $[E]$. Suppose that each operator A of $\mathcal{M}$ may be presented as a sum $A' + A''$, here $A'$is a contraction and A'' is a compact operator. Then the following assertions are equivalent:*

(i) The discrete linear inclusion $DLI(\mathcal{M})$ is absolute asymptotic stable and $DLI(\mathcal{M})$ doesn't admit non-trivial bounded trajectories on $\mathbb{Z}$;

(ii) The discrete linear inclusion $DLI(\mathcal{M})$ is uniformly exponentially stable.

Proof. Consider the cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)$ generated by $DLI(\mathcal{M})$. By Lemma 4.15, under the conditions of the theorem, this cocycle is α-contraction. Now to finish the proof it is sufficient to apply Theorem 4.11, because every α-contraction cocycle φ is α-condensing. □

Theorem 4.17. *Let $\mathcal{M}$ be a compact subset of $[E]$. Suppose that the following conditions hold:*

(i) each operator A of $\mathcal{M}$ may be presented as a sum $A' + A''$, here $A'$is a contraction and A'' is a compact operator;

(ii) the discrete linear inclusion $DLI(\mathcal{M})$ doesn't admit non-trivial bounded trajectories on $\mathbb{Z}$;

(iii) the set $\mathcal{M} \subseteq [E]$ of operators is product bounded.

Then the discrete linear inclusion $DLI(\mathcal{M})$ is uniformly exponentially stable.

Proof. Consider the cocycle $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma)$ generated by $DLI(\mathcal{M})$ and corresponding skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$, where $X := E \times Y$ and $\pi := (\varphi, \sigma)$. By Lemma 4.15, under the conditions of the theorem, this cocycle is α-contraction and, consequently, the dynamical system $(X, \mathbb{Z}_+, \pi)$ too. It is easy to verify that under the conditions of Theorem this non-autonomous dynamical system is uniformly stable, i.e. $|\pi(t, x)| \le M|x|$ for all $x := (u, y) \in X$ and $t \in \mathbb{Z}_+$ because $|\pi(t, x)| = |U(t, \omega)u| \le M|u| = M|x|$, where $U(t, \omega) := \varphi(t, \cdot, \omega) = A(t)A(t-1) \ldots A(1)A(0)$ ($\omega(j) := A(j) \in \mathcal{M}$ for all $j \in \mathbb{Z}_+$) and $\mathcal{M} \subset [E]$ is product bounded.

Now we will show that non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}_+, \sigma), h\rangle$ is convergent. In our case this means that $\lim_{t\to+\infty} |\pi(t, x)| = 0$ for all $x \in X$. Indeed, the system $(X, \mathbb{Z}_+, \pi)$ is α-contraction and, consequently, it is asymptotically compact. The trajectory $\{\pi(t, x) \mid t \in \mathbb{Z}_+\}$ is bounded and consequently it is relatively compact. Denote by ω_x the ω-limit set of point x. This set is nonempty, compact and invariant. In particular ω_x consists of the full trajectories of $DLI(\mathcal{M})$ bounded on $\mathbb{Z}$. Under the conditions of our Theorem $\omega_x \subseteq \Theta := \{(0, y) \mid y \in Y\}$ and, consequently, $\lim_{t\to+\infty} |\pi(t, x)| = 0$. Now to finish the proof it is sufficient to apply Lemma 4.15 and Theorem 4.13. □

Theorem 4.18. *Let $\mathcal{M}$ be a compact subset of $[E]$and each operator A of $\mathcal{M}$ may be presented as a sum $A' + A''$, here $A'$is a contraction and A'' is a compact operator.*

Then the following affirmations are equivalent:

(i) the discrete linear inclusion $DLI(\mathcal{M})$ is product bounded and absolutely asymptotically stable;

(ii) the set $\mathcal{M}$ is generalized contractive, i.e. there exist positive numbers $\mathcal{N}$ and ν such that $\|A(t)A(t-1) \ldots A(1)\| \le \mathcal{N} e^{-\nu t}$ for all $t \in \mathbb{N}$ and $A(j) \in \mathcal{M}$ ($1 \le j \le t$).

Proof. Consider the cocycle φ generated by $DLI(\mathcal{M})$. By Lemma 4.15 , under the conditions of theorem, this cocycle is α-condensing. Now to finish the proof it is sufficient to refer Theorem 4.13. □

Definition 4.19. *A dynamical system $(X,\mathbb{T},\pi)$ is called locally compact (locally completely continuous) if for any $x \in X$ there are $\delta_x > 0$ and $l_x > 0$ such that $\pi^t B(x,\delta_x)$ $(t \geq l_x)$ is relatively compact.*

Remark 4.20. *Note that the dynamical system $(X,\mathbb{T},\pi)$ is locally compact (completely continuous), if one of the following two conditions holds:*

(i) the phase space X of dynamical system $(X,\mathbb{T},\pi)$ is locally compact;

(ii) there exists a number $t_0 \in \mathbb{T}$ such that the operator π^t is completely continuous, where $\pi^t := \pi(t,\cdot)$.

Theorem 4.21. *[8] Let $(X,\mathbb{T},\pi)$ be locally compact and Y be compact. Then the following conditions are equivalent:*

1. $\lim\limits_{t\to+\infty} |xt| = 0$ *for all $x \in X$;*
2. *all the motions in $(X,\mathbb{T},\pi)$ are relatively compact and $(X,\mathbb{T},\pi)$ does not admit nontrivial compact motions defined on $\mathbb{S}$;*
3. *there are positive numbers $\mathcal{N}$ and ν such that $|xt| \leq \mathcal{N}e^{-\nu t}|x|$ for all $x \in X$ and $t \geq 0$.*

Theorem 4.22. *Let $\mathcal{M}$ be a compact subset of $[E]$. Suppose that each operator A of $\mathcal{M}$ is compact. Then the following assertions are equivalent:*

1. *the discrete linear inclusion $DLI(\mathcal{M})$ is absolute asymptotic stable;*
2. *every solution of $DLI(\mathcal{M})$ is relatively compact and $DLI(\mathcal{M})$ doesn't admit non-trivial bounded trajectories on $\mathbb{Z}$;*
3. *the discrete linear inclusion $DLI(\mathcal{M})$ is uniformly exponentially stable.*

Proof. Consider the cocycle $\langle E,\varphi,(\Omega,\mathbb{Z}_+,\sigma)$ generated by $DLI(\mathcal{M})$. By Lemma 4.15, under the conditions of Theorem, the cocycle φ is α-contraction and, consequently, the skew-product dynamical system $(X,\mathbb{T},\pi)$ ($X := E\times\Omega$, $\pi := (\varphi, \sigma)$), generated by cocycle φ, is completely continuous. Now to finish the proof it is sufficient to apply Theorem 4.21 to non-autonomous dynamical system $\langle (X,\mathbb{T},\pi), (\Omega,\mathbb{Z}_+,\sigma), h\rangle$ $(h := pr_2 : X \to \Omega)$. □

Remark 4.23. *Note that for finite-dimensional Banach space E the equivalence of the statements 1. and 2. it was proved by Kozyakin [27].*

5 Absolute Asymptotic Stability of Finite-Dimensional Discrete Linear Inclusions

Let $(X,\mathbb{T},\pi)$ be a dynamical system.

Definition 5.1. *Let $\mathbb{T}'$ be a subset of group $\mathbb{S}$ and $0\in\mathbb{T}'$. The continuous mapping $\gamma:\mathbb{T}'\to X$ is called a trajectory of the point $x\in X$ on $\mathbb{T}'$ if $\pi^t\gamma(s)=\gamma(t+s)$ for all $t\in\mathbb{T}$ and $s\in\mathbb{T}'$ such that $t+s\in\mathbb{T}'$.*

Lemma 5.2. *[39] Let $\{\mathbb{T}_n\}$ be a family of subsets of $\mathbb{S}$ and the following conditions are fulfilled:*

(i) $\mathbb{T}_n\subseteq\mathbb{T}_{n+1}$ *for all* $n\in\mathbb{N}$;

(ii) γ_n *is the trajectory on* $\mathbb{T}_n$ *of the point* $x_n\in X$;

(iii) the sequence $\{x_n\}\subseteq X$ *converges to* $x\in X$.

Then there exists a trajectory on $\mathbb{T}':=\bigcup\{\mathbb{T}_n\ :n\in\mathbb{N}\}$ of the point $x\in X$ such that $\{\gamma_n\}$ converges to γ uniformly on the compacts from $\mathbb{T}'$, i.e. for every compact $K\subseteq\mathbb{T}'$ and positive number ε there exists a number $n_0=n_0(\varepsilon,K)\in\mathbb{N}$ such that $K\subseteq\mathbb{T}_n$ and $\rho(\gamma_n(s),\gamma(s))<\varepsilon$ for all $n\geq n_0$ and $s\in K$, where ρ is the distance on X.

Remark 5.3. *If $(X,\mathbb{T},\pi)$ is a skew-product dynamical system generated by cocycle φ and $x:=(u,y)\in X:=E\times Y$, then $\gamma\in\Phi_x(\pi)\subseteq\Phi(\pi)$ if and only if there exist a continuous function $\nu:\mathbb{S}\to E$ and $\tilde{\gamma}\in\Phi_{h(x)}(\sigma)$such that $\varphi(t,\nu(s),y)=\nu(t+s)$ and $\gamma(s)=(\nu(s),\tilde{\gamma}(s))$ for all $t\in\mathbb{T}$ and $s\in\mathbb{S}$.*

Definition 5.4. *Let (X,h,Y) be a Banach fiber bundle with norm $|\cdot|$. The non-autonomous dynamical system $\langle(X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ is said to be non-critical [25] (satisfying Favard's condition) if $B(\pi)=\Theta$, where $\Theta:=\{\theta_y\mid\theta_y\in X_y,\ |\theta_y|=0,y\in Y\}$.*

Remark 5.5. *Throughout the rest of this section we assume that the Banach fiber bundle (X,h,Y) is finite-dimensional, Y is compact and invariant (i.e. $\sigma^tY=Y$ for all $t\in\mathbb{T}$, where $\sigma^t:=\sigma(t,\cdot)$) and the non-autonomous linear dynamical system $\langle(X,\ \mathbb{T},\pi),\ (Y,\mathbb{T},\sigma),\ h\rangle$ is non-critical.*

Denote by $\omega_x:=\bigcap\limits_{t\geq0}\overline{\bigcup\{\pi(s,x)\ :\ s\geq t\}}$ and $\alpha_\gamma:=\bigcap\limits_{t\leq0}\overline{\bigcup\{\gamma(s)\ :\ s\leq t\}}$ if $\gamma\in\Phi(\pi)$.

Lemma 5.6. *Let $\langle(X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a linear non-critical non-autonomous dynamical system and $x\in X$. Then the following statements hold:*

(i) if $\sup\{|\pi(t,x)|\ :\ t\in\mathbb{T},t\geq0\}<+\infty$, *then*

$$\lim_{t\to+\infty}|\pi(t,x)|=0; \tag{12}$$

(ii) if $\gamma \in \Phi_x(\pi)$ *and* $\sup\{|\gamma(s)| \ : s \in \mathbb{S}, s \leq 0\} < +\infty$, *then*

$$\lim_{s \to -\infty} |\gamma(s)| = 0. \tag{13}$$

Proof. Let $\pi(\mathbb{T}_+, x)$ ($\mathbb{T}_+ := \{t \in \mathbb{T} \ : t \geq 0\}$) be bounded. Since (X, h, Y) is a locally trivial finite-dimensional fiber bundle and Y is compact, then the set $\pi(\mathbb{T}, x)$ is relatively compact and, consequently, $\omega_x \neq \emptyset$. Suppose that the equality (12) is not true. Then there exists a positive number ε_0 and strictly increasing sequence $t_n \to +\infty$ such that

$$|\pi(t_n, x)| \geq \varepsilon_0. \tag{14}$$

Without loss of generality we may suppose that the sequence $\{\pi(t_n, x)\}$ is convergent. Let $x_0 := \lim_{n \to +\infty} \pi(t_n, x)$. Denote by $\mathbb{T}_n := \{s \in \mathbb{S} \ : \ s \geq -t_n\}$ and $\gamma_n : \mathbb{T}_n \to X$ the continuous mapping defined by equality $\gamma_n(s) := \pi(s + t_n, x)$. It is easy to verify that γ_n is a trajectory of the point $x_n := \pi(t_n, x)$ on $\mathbb{T}_n$. By Lemma 5.2 the sequence $\{\gamma_n\}$ is convergent and its limit $\gamma \in \Phi_{x_0}(\pi)$. Note that $\gamma(s) \in \omega_x$ for all $s \in \mathbb{S}$ and $\gamma(0) = x_0$. According to (14) $x_0 \neq 0$. The obtained contradiction proves the first statement.

The second statement may be proved similarly. □

Denote by $X^s := \{x \in X \ : \ \lim_{t \to +\infty} |\pi(t, x)| = 0\}$ and $X^u := \{x \in X \ : \ \exists \gamma \in \Phi_x(\pi)$ such that $\lim_{t \to -\infty} |\gamma(t)| = 0\}$.

Lemma 5.7. *The following statement hold:*

(i) the set X^s *is vectorial, i.e. every fiber* $X^s_y := X^s \bigcap X_y$ *is a subspace of the linear space* X_y;

(ii) X^s *is a positively invariant subset of dynamical system* $(X, \mathbb{T}, \pi)$, *i.e.* $\pi^t X^s \subseteq X^s$ *for all* $t \in \mathbb{T}$;

(iii) the set X^s *is closed.*

Proof. The first and second statements are evident.

Let $a \in \overline{X^s} \setminus X^s$, then there exists $\tilde{x}_n \in E^s$ such that $\{\tilde{x}_n\} \to a$. Let $l_n := \sup\{|\pi(t, \tilde{x}_n)| \ : \ t \in \mathbb{T}\}$ and $\tau_n \in \mathbb{T}$ such that $l_n = |\pi(\tau_n, \tilde{x}_n)|$. Note that $\{l_n\} \to +\infty$ and $\{\tau_n\} \to +\infty$. We may suppose that the sequence $\{\tau_n\}$ is increasing. Assume that $x_n := l_n^{-1}\pi(\tau_n, \tilde{x}_n)$, then $|x_n| = 1$. Now we define the continuous mapping $\gamma_n : \mathbb{T}_n \to X$ by equality $\gamma_n(\tau) := l_n^{-1}\pi(\tau + \tau_n, \tilde{x}_n)$, where $\mathbb{T}_n := \{t \in \mathbb{S} \ : s \geq -\tau_n\}$. Then

(i) $\gamma_n(0) = x_n$;

(ii) γ_n is the trajectory on $\mathbb{T}_n$ of the point x_n;

(iii)
$$|\gamma_n(t)| \le 1 \tag{15}$$
for all $t \in \mathbb{T}_n$.

Without loss of generality we may suppose that the sequence $\{x_n\}$ is convergent. Let $x := \lim\limits_{n\to+\infty} x_n$, then $|x| = 1$. By Lemma 5.2 the sequence $\{\gamma_n\}$ is convergent uniformly on the compacts from $\mathbb{S}$ and if $\gamma := \lim\limits_{n\to+\infty} \gamma_n$, then $\gamma \in \Phi_x(\pi)$. From the inequality (15) follows that $|\gamma(t)| \le 1$ for all $t \in \mathbb{S}$, i.e. $\gamma \in \mathbb{B}(\pi)$, and $\gamma(0) = x \neq 0$. Under the conditions of the lemma we have $\mathbb{B}(\pi) = \Theta$. The obtained contradiction proves our statement. □

Lemma 5.8. *There exist positives numbers $\mathcal{N}$ and ν such that*
$$|\pi(t,x)| \le \mathcal{N} e^{-\nu t} \tag{16}$$
for all $x \in X^s$ and $t \ge 0$.

Proof. According to Lemma 5.7 the subset X^s is a positively invariant and closed subset of dynamical system $(X, \mathbb{T}, \pi)$ and, consequently, on X^s is induced a dynamical system $(X^s, \mathbb{T}, \pi)$. Now to finish the proof of Lemma it is sufficient to refer to Theorem 3.9. □

Lemma 5.9. *The following statement hold:*

1. *for any $x \in E^u$ the set $\Phi_x(\pi)$ contains a unique trajectory γ with condition $\lim\limits_{t\to-\infty} |\gamma(t)| = 0$;*
2. *the set X^u is vectorial;*
3. *X^u is a positively invariant subset of dynamical system $(X, \mathbb{T}, \pi)$;*
4. *$E^s_y \bigcap E^u_y = \{\theta_y\}$ for all $y \in Y$, where $E^i_y := E^i \bigcap X_y$ $(i = s, u)$;*
5. *the set X^u is closed.*

Proof. Suppose that the first statement of Lemma is not true, then there exist $x_0 \in X^u$ and $\gamma_1, \gamma_2 \in \Phi_{x_0}$ such that $\gamma_1 \neq \gamma_2$ and $\lim\limits_{t\to-\infty} |\gamma_i(t)| = 0$ $(i = 1, 2)$. Let $\gamma := \gamma_1 - \gamma_2$, then by linearity of non-autonomous dynamical system $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ we have $\gamma \in \Phi_{\theta_{y_0}}$, where θ_{y_0} is the zero in the space X_{y_0} and $y_0 := h(x_0)$. In addition we have

(i) $\lim\limits_{t\to-\infty} |\gamma(t)| = 0$;

(ii) $|\gamma(t)| = 0$ for all $t \ge 0$ since $\gamma_1(t) = \gamma_2(t) = \pi(t, x_0)$ for all $t \ge 0$.

Thus we found $\gamma \in \mathbb{B}(\pi) \setminus \Theta$. The obtained contradiction proves our affirmation.

The statements 2–4 are evident.

Let $b \in \overline{X^u} \setminus X^u$, then there exists $\tilde{x}_n \in E^u$ such that $\{\tilde{x}_n\} \to b$. Let $l_n := \sup\{|\pi(t, \tilde{x}_n)| \ : \ t \le 0\}$ and $\tau_n \le 0$ such that $l_n = |\pi(\tau_n, \tilde{x}_n)|$. Note that $\{l_n\} \to +\infty$ and $\{\tau_n\} \to -\infty$. We may suppose that the sequence $\{\tau_n\}$ is decreasing. Assume that $x_n := l_n^{-1}\pi(\tau_n, \tilde{x}_n)$, then $|x_n| = 1$. Now we define the continuous mapping $\gamma_n : \mathbb{S} \to X$ by equality $\gamma_n(\tau) := l_n^{-1}\pi(\tau + \tau_n, \tilde{x}_n)$. Then

(i) $\gamma_n(0) = x_n$;

(ii) γ_n is the full trajectory of the point x_n, i.e. $\gamma_n \in \Phi_{x_n}(\pi)$;

(iii)
$$|\gamma_n(t)| \le 1 \tag{17}$$
for all $t \in \mathbb{T}_n := \{t \in \mathbb{S} \ : \ t \le -\tau_n\}$.

Without loss of generality we may suppose that the sequence $\{x_n\}$ is convergent. Let $x := \lim\limits_{n\to+\infty} x_n$, then $|x| = 1$. By Lemma 5.2 the sequence $\{\gamma_n\}$ is convergent uniformly on the compacts from $\mathbb{S}$ and if $\gamma := \lim\limits_{n\to+\infty} \gamma_n$, then $\gamma \in \Phi_x(\pi)$. From the inequality (17) follows that $|\gamma(t)| \le 1$ for all $t \in \mathbb{S}$, i.e. $\gamma \in \mathbb{B}(\pi)$, and $\gamma(0) = x \ne 0$. The obtained contradiction prove our statement. □

Lemma 5.10. *On the set X^u is defined a group dynamical system $(X^u, \mathbb{S}, \tilde{\pi})$, where the mapping $\tilde{\pi} : \mathbb{S} \times X \to X$ is defined by equality $\tilde{\pi}(t, x) := \gamma_x(t)$ and γ_x is a unique trajectory from $\Phi_x(\pi)$ with condition $\lim\limits_{t\to-\infty} |\gamma_x(t)| = 0$.*

Proof. This statement directly follows from Lemmas 5.2 and 5.9 . □

Corollary 5.11. *The following statements hold:*

(i) there exist positive constants N and ν such that $|\tilde{\pi}(t, x)| \le Ne^{\nu t}|x|$ for all $x \in X^u$ and $t \le 0$ $(t \in \mathbb{S})$;

(ii) $\pi^t X^u_y = X^u_{\sigma(t,y)}$ for all $y \in Y$ and $t \ge 0$.

Proof. First statement of Corollary follows from Lemma 5.8 if we will change t by $-t$ in group dynamical system $(X^u, \mathbb{S}, \pi)$.

The second statement is evident. □

Lemma 5.12. *The following statements hold:*

(i)
$$\limsup_{t\to+\infty} |\pi(t, x)| = \liminf_{t\to+\infty} |\pi(t, x)| \tag{18}$$
for all $x \in X$ and also $\lim\limits_{t\to+\infty} |\pi(t, x)| = +\infty$ for all $x \notin X^s$;

(ii)

$$\limsup_{t\to-\infty}|\gamma(t)| = \liminf_{t\to-\infty}|\gamma(t)| \tag{19}$$

for all $x \in X$ *and* $\gamma \in \Phi_x(\pi)$ *moreover* $\lim\limits_{t\to-\infty}|\gamma(t)| = +\infty$ *for all* $x \notin X^u$.

Proof. Denote by

$$L := \limsup_{t\to+\infty}|\pi(t,x)|;\ \ l := \liminf_{t\to+\infty}|\pi(t,x)|. \tag{20}$$

Then $0 \le l \le L$. It is sufficient to consider the case $L \neq 0$. If $L < +\infty$, then $x \in \mathbb{B}^+(\pi) := \{x \in X \mid \sup\{|\pi(t,x)| \ :\ t \ge 0\} < +\infty\}$ and by Lemma 5.6 $L = l = 0$. Thus, if $L \neq 0$, then $L = +\infty$.

If $L = l = +\infty$, then Lemma is proved. If $l < +\infty$, then there exist sequences $\{\tau_n\}$ and $\{t_n\}$ ($\{t_n\} \subseteq \mathbb{T}$) such that $t_n \le \tau_n \le t_{n+1}$ ($\{t_n\} \to +\infty,\ t_{n+1} - t_n \ge n+1$) and $|\pi(\tau_n, x)| = \nu_n$, where $\nu_n := \max\{|pi(t,x)| \ :\ t \in [t_n, t_{n+1}]\}$. Since $L = +\infty$, then $\{\nu_n\} \to +\infty$. Assume $x_n := \nu_n^{-1}\pi(\tau_n, x)$ and $y_n := \sigma(\tau_n, y)$. Then $|x_n| = 1$

$$|\pi(x_n, t)| = \nu_n^{-1}|\pi(t + \tau_n, x)| \le 1 \tag{21}$$

for all $t \in [t_n, t_{n+1}]$. Let $\mathbb{T}_n := \{t \in \mathbb{S} \ :\ s \ge -\tau_n\}$ and $\gamma_n : \mathbb{T}_n \to X$ be a continuous function defined by equality

$$\gamma_n(t) := \nu_n^{-1}\pi(t + \tau_n, x). \tag{22}$$

Note that

$$|\gamma_n(t)| := \nu_n^{-1}|\pi(t + \tau_n, x)| \le 1 \tag{23}$$

for all $t \in [t_n - \tau_n, t_{n+1} - \tau_n]$. In addition

$$|\gamma_n(t)| \le \nu_n^{-1}(l+1) \tag{24}$$

when $t = t_n - \tau_n$ and $t_{n+1} - \tau_n$. From the inequality (24) it follows that $\{t_n - \tau_n\} \to -\infty$ and $\{t_{n+1} - \tau_n\} \to +\infty$. In fact, without loss of generality we may suppose that the sequence $\{x_n\}$ converges. Denote by $x* := \lim\limits_{n\to+\infty} x_n$. If we suppose, for example, that the sequence $\{s_n\} := \{t_{n+1} - \tau_n\}$ converges to s_0 then according to Lemma 5.2 the sequence $\{\gamma_n\}$ converges and its limit $\gamma \in \Phi_{x*}(\pi)$. From the inequality (24) we have $|\gamma(s_0)| = 0$ and, consequently, $|\gamma(t + s_0)| = |\pi^t\gamma(s_0)| = 0$ for all $t \ge 0$. On the other hand from (23) we have $|\gamma(t)| \le 1$ for all $t \le s_0$, since $t_{n+1} - t_n \ge n+1$ for all $n \in \mathbb{N}$. Thus we found $\gamma \in \mathbb{B}(\pi)$ with $\gamma(0) = x*$. Analogously we will obtain the contradiction if we suppose that the sequence $\{t_n - \tau_n\}$ does not converge to $-\infty$. The obtained contradiction proves required statement.

Thus $\{t_n - \tau_n\} \to -\infty$ and $\{t_{n+1} - \tau_n\} \to +\infty$, then according to inequality (23) we have $|\gamma(t)| \le 1$ for all $t \in \mathbb{S}$ and $\gamma(0) = x*$. This contradiction proves the first affirmation of Lemma.

Now we will prove the second statement. Note that $0 \le l \le L$ and $L \neq 0$. If $L < +\infty$, then $x := \gamma(0) \in X^u_y$ ($y := h(x)$) and, consequently, $0 = l = L$ and the required statement is proved. Let $L = +\infty$. We will show that $l = +\infty$. If we suppose that $l < +\infty$, then there exist sequences $\{\tau_n\}$ and $\{t_n\}$ ($t_n \le 0$, $t_n - t_{n+1} > n+1$) such that

(i) $\tau_n \in [t_{n+1}, t_n]$;

(ii) $|\gamma(t_n)| \le l+1$;

(iii) $|\gamma(\tau_n)| = \nu_n := \max\limits_{t_{n+1} \le t \le t_n} |\gamma(t)|$.

Since $L = +\infty$, then $\{\nu_n\} \to +\infty$. Let $x_n := \nu_n^{-1}\gamma(\tau_n)$, then $|x_n| = 1$. We define $\gamma_n \in \Phi_{x_n}(\pi)$ by equality

$$\gamma_n(t) := \nu_n^{-1}\gamma(t+\tau_n) \ (t \in \mathbb{S}). \tag{25}$$

It is easy to see that

$$|\gamma_n(t)| = \nu_n^{-1}|\gamma(t+\tau_n)| \le 1 \ (\forall t \in [t_{n+1}-\tau_n, t_n - \tau_n]) \tag{26}$$

and

$$|\gamma_n(t)| \le \nu_n^{-1}(l+1) \tag{27}$$

for $t = t_{n+1} - \tau_n$ and $t_n - \tau_n$. We may suppose that the sequence $\{x_n\}$ is convergent. Let $x* := \lim\limits_{n\to+\infty} x_n$, then $|x*| = 1$. Reasoning analogously as in the proof of the first statement we may prove that $\{t_{n+1} - \tau_n\} \to -\infty$ and $\{t_n - \tau_n\} \to +\infty$. By Lemma 5.2 the sequence $\{\gamma_n\}$ converges uniformly on the compacts from $\mathbb{S}$ and its limit $\gamma \in \Phi_{x*}(\pi)$. Therefore from (26) it follows that $|\gamma(t)| \le 1$ for all $t \in \mathbb{S}$. The obtained contradiction completes the proof of Lemma. □

Denote by $k_y^s := dim(X_y^s)$ (respectively, $k_y^u := dim(X_y^u)$) the dimension of the space X_y^s (respectively, X_y^u).

Lemma 5.13. *[6] Let (X, h, y) be a finite-dimensional fiber bundle with compact base Y, $\{y_n\} \to y$ and E_n be a subspace of the fiber X_{y_n} with $dim(E_n) \ge l$. If*

$$L = \limsup E_n := \bigcap_{m=1}^{\infty} \overline{\bigcup \{E_n \ : \ n \ge m\}},$$

then the linear subspace span(L) *generated by L (i.e. the minimal space, containing L) is a subspace of the fiber X_y and* $\dim($span$(L)) \ge l$.

Lemma 5.14. *$k_y^u \le k_p^u$ for all $p \in \omega_y$.*

Proof. Let $p \in \omega_y$, then there exists a sequence $\{t_n\} \to +\infty$ such that $p = \lim\limits_{n\to+\infty} \sigma(t_n, y)$. Let $y_n := \sigma(t_n, y)$, $E_n := X_{y_n}^u$, $U := \limsup E_n$ and $l := k_y^u$. Now to finish the proof of Lemma, according to Lemma 5.13, it sufficient to show that $U \subseteq X_p^u$. Assume that it is not true, i.e. there exists $x \in U \setminus X_p^u$. From the definition of U it follows that $tx \in E$ for all $t \in \mathbb{R}$ and, consequently, we may suppose that $|x| = 1$. Then there exist $x_n \in X^u_{y_n}$ such that $|x_n| = 1$ and $\{x_n\} \to x$. By Lemma 5.12 $\lim_{t\to-\infty} |\tilde{\pi}(t, x)| = +\infty$ and, consequently, for every $L > 0$ there exist $n = n(L) \in \mathbb{N}$ and $t_0 = t_0(L) < 0$ such that $|\tilde{\pi}(t_0, x_n)| \le L$ for all $n \ge n_0$. Since $x \in X^u_{y_n}$, then there exists a unique $\gamma_n \in \Phi_{x_n}(\pi)$ with condition $\lim_{t\to-\infty} |\gamma_n(t)| = 0$, therefore there is $t_1 < t_0$ such that $||\gamma_n(t_1) \le 1$. Let $\nu_n := \max\{|\gamma_n(t)| \ : \ t \in [t_1, 0]\}$, then $\nu_n > L$. We will choose a number $s_n \in [t_1, 0]$ such that $\nu_n = |\gamma_n(s_n)|$. Denote by $z_n := \nu_n^{-1}\gamma_n(s_n)$ and consider the sequence $\{\tilde{\gamma}_n\}$ defined by equality

$$\tilde{\gamma}_n(t) := \nu_n^{-1}\gamma_n(t + s_n) \ (t \in \mathbb{S}).$$

It is clear that $\tilde{\gamma}_n \in \Phi_{z_n}(\pi)$ and $\lim_{t\to-\infty} |\tilde{\gamma}_n(t)| = 0$. Note that

$$|\tilde{\gamma}_n(t)| = \nu_n^{-1}|\gamma_n(t + s_n)| \le 1 \tag{28}$$

for $t \in [t_1 - s_n, -s_n]$. If now $L \to +\infty$, then $s_n \to -\infty$ and $t_1 - s_n \to -\infty$. In fact, from the equality $\nu_n = |\gamma_n(s_n)|$ and inequality $\nu_n > L$ it follows that $s_n \to -\infty$.

Without loss of generality we may suppose that the sequence $\{z_n\}$ is convergent. Let $z := \lim_{n\to+\infty} z_n$, then by Lemma 5.2 the sequence $\{\tilde{\gamma}_n\}$ is convergent too and its limit $\tilde{\gamma} \in \Phi_z(\pi)$. On the other hand by inequality (28) we have $|\tilde{\gamma}(t)| \le 1$ for all $t \in \mathbb{S}$ and $|\tilde{\gamma}(0)| = |z| = 1$. The obtained contradiction completes the proof of Lemma. □

Lemma 5.15. *Let U_y be certain complementary subspace for subspace X^s_y, i.e. $X_y = X^s_y \dot{+} U_y$. Then there exists a positive number δ such that*

$$|\pi(s, x)| \ge \delta|\pi(t, x)| \tag{29}$$

for all $x \in U_y$ and $s \ge t \ge 0$.

Proof. Suppose that it is not true. Then there exist sequences $\{t_k\}$, $\{s_k\}$ and $\{x_k\}$ such that

$$s_k \ge t_k \ge 0, \ x_k \in U_y \ (|x_k| = 1) \tag{30}$$

and

$$|\pi(t_k, x_k)| \ge k|\pi(s_k, x_k)|. \tag{31}$$

Without loss of generality we may assume that the sequence $\{x_k\}$ is convergent and denote by x its limit, then $|x| = 1$.

Let τ_k be chosen so that $0 \le \tau_k \le s_k$ and $|\pi(\tau_k, x_k)| = \max\{|\pi(t, x_k)| \ : \ t \in [0, s_k]\}$. It is easy to see that $\{\tau_k\} \to +\infty$ as $k \to +\infty$. If we suppose that it is not so, then we will have $|\pi(s, x_k)| \le M$ for all $s \in [0, s_k]$ (M is a certain positive constant) and, consequently, $x \in E^s_y$. The obtained contradiction proves our statement.

Denote by $\xi_k := |\pi(\tau_k, x_k)|^{-1}\pi(\tau_k, x_k)$, then $|\xi_k| = 1$ and $\xi_k \in \pi^{\tau_k}U_y$. Let $\mathbb{T}_k := \{t \in \mathbb{S} \ : \ s \ge -\tau_k\}$ and define the mapping $\gamma_k : \mathbb{T}_k \to X$ by equality

$$\gamma_k(t) := |\pi(\tau_k, x_k)|^{-1}\pi(t + \tau_k, x_k), \tag{32}$$

then

$$|\gamma_k(t)| \leq 1 \tag{33}$$

for all $t \in [-\tau_k, s_k - \tau_k]$. Logically there are two possibilities:

a) the sequence $\{s_k - \tau_k\} \to s \geq 0$ (or it contains a convergent subsequence), then

$$|\pi(\tau_k, x_k)| = \max_{0 \leq s \leq s_k} |\pi(s, x_k)| \geq |\pi(t_k, x_k)| \geq k|\pi(s_k, x_k)|$$

and, consequently,

$$\frac{|\pi(s_k, x_k)|}{|\pi(\tau_k, x_k)|} \leq \frac{1}{k}. \tag{34}$$

From (32) and (34) we obtain

$$|\gamma_k(s_k - \tau_k)| = \frac{|\pi(s_k, x_k)|}{|\pi(\tau_k, x_k)|} \leq \frac{1}{k}. \tag{35}$$

We may suppose that the sequence $\{\xi_k\} \to \xi$, then by Lemma 5.2 the sequence $\{\gamma_k\}$ is convergent (uniformly on the compacts from $\mathbb{S}$) too. Denote by $\gamma := \lim\limits_{k \to +\infty} \gamma_k$, then from (35) we have $|\gamma(s)| = 0$ and, consequently, $|\gamma(t)| = 0$ for all $t \geq s$. On the other hand from (33) we obtain $|\gamma(t)| \leq 1$ for all $t \leq 0$ and, consequently, $\gamma \in \mathbb{B}(\pi)$ and $|\gamma(0)| = |x| = 1$.

b) $\{s_k - \tau_k\} \to +\infty$ and from (33) we have $|\gamma(t)| \leq 1$ for all $t \in \mathbb{S}$.

Thus, if we suppose that the statement of Lemma is not true we obtain the contradiction. The lemma is proved. □

Lemma 5.16. *Let $y \in Y$ and $H^+(y) := \overline{\{\sigma(t, y) \ : \ t \in \mathbb{T}\}} = Y$. Then $k_y^u \geq k_y - k_p^s$ for all $p \in \omega_y$, where $k_y := \dim(X_y)$.*

Proof. Let U_y be certain subspace of X_y which is complementary subspace for X_y^s and $p \in \omega_y$. Then there exists a sequence $\{t_n\} \to +\infty$ such that $\{\sigma(t_n, y)\} \to p$. Denote by $U_n := \pi(t_n, U_y)$. By Lemma 5.15 $\dim(U_n) = \dim(U_y)$. Let $U := \limsup U_n$ and $x \in U$ ($|x| = 1$). Then there is $\{x_n\}$ ($x_n \in U_n$, $|x_n| = 1$) such that $x = \lim\limits_{n \to +\infty} x_n$. We will prove that $x \in X_p^u$ (i.e. $U \subseteq E_p^u$). Let $\mathbb{T}_n := \{t \in \mathbb{S} \ : \ s \geq -t_n\}$ and we define the function $\gamma_n : \mathbb{T}_n \to X$ by following equality:

$$\gamma_n(t) := \pi(t, x_n) = \pi(t + t_n, \tilde{x}_n), \tag{36}$$

where $\tilde{x}_n \in U$ and $\pi(t_n, \tilde{x}_n) = x_n$. Since $\gamma_n(0) = x_n \to x$, then by Lemma 5.2 the sequence $\{\gamma_n\}$ is convergent too and its limit $\gamma \in \Phi_x(\pi)$. Note that

$$|x| = |\pi^t \gamma(-t)| = |\pi^t \lim_{n \to +\infty} \pi(-t + t_n, \tilde{x}_n)| = \tag{37}$$

$$\lim_{n \to +\infty} |\pi^t \pi(-t + t_n, \tilde{x}_n)| \geq \delta |\lim_{n \to +\infty} \pi(-t + t_n, \tilde{x}_n)| = \delta |\gamma(-t)|$$

for all $t \geq 0$. From the inequality (37) follows $|\gamma(t)| \leq \delta^{-1}|x|$ for all $t \leq 0$ and by Lemma 5.6 $x \in X_p^u$ ($p = h(x)$). Thus $\dim(X_p^u) \geq k_y - \dim(X_y^s)$. □

Corollary 5.17. *Under the conditions of Lemma 5.16, if additionally the point $y \in Y$ is stable in the sense of Poisson (i.e. $y \in \omega_y$), then $k^s_y + k^u_y = k_y$.*

Proof. In fact, by Lemma 5.16 we have $k^u_p \geq k_y - k^s_y$ for all $p \in \omega_y$. In particular $k^u_y \geq k_y - k^s_y$ because $y \in \omega_y$. On the other hand $k^s_y + k^u_y \leq k_y$ and, consequently, $k^s_y + k^u_y = k_y$. □

Corollary 5.18. *Under the conditions of Corollary 5.17 $k^s_{y\tau} + k^u_{y\tau} = k_y$ for all $\tau \in \mathbb{T}$, where $y\tau := \sigma(\tau, y)$.*

Proof. Note that the point $y\tau$ is stable in the sense of Poisson, $\omega_{y\tau} = \omega$ and $H^+(y\tau) = \omega_{y\tau} = \omega_y = H^+(y) = Y$. According to Corollary 5.17 we have $k^s_{y\tau} + k^u_{y\tau} = k_y$ for all $\tau \in \mathbb{T}$. □

Remark 5.19. *Note that the statements close to Lemmas 5.6 – 5.16 before were established for bilateral (i.e. when $\mathbb{T} = \mathbb{S}$) non-autonomous linear dynamical systems in the works [6, 25, 33].*

Lemma 5.20. *Under the conditions of Lemma 5.16, if additionally the point $y \in \omega_y$, then $k^s_p \geq k^s_y$ for all $p \in \omega_y$.*

Proof. Let $p \in \omega_y$, then there exists $\{t_n\} \to +\infty$ such that $\{yt_n\} \to p$. By Corollary 5.11 $k^u_{yt_n} = k^u y$ and according to Corollary 5.18 we have $k^s_{yt_n} = k_y - k^u_{yt_n} = k_y - k^u_y = k^s_y$ for all $n \in \mathbb{N}$. Let $V := \limsup X^s_{yt_n}$, then by Lemma 5.7 $V \subseteq X^s_p$. Since $\dim(V) = \lim\limits_{n\to+\infty} \dim(X^s_{yt_n}) = k_y - k^u_y = k^s_y$ and $\dim(X^s_p) \geq \dim(V)$, then $k^s_p \geq k^s_y$. □

Corollary 5.21. *Under the conditions of Lemma 5.20 $k^s_p + k^u_p = k_y$ for all $p \in \omega_y$.*

Proof. According to Lemma 5.20 we have

$$k^s_p \geq k^s_y \tag{38}$$

for all $p \in \omega_y$. On the other hand by Lemma 5.16

$$k^s_p \geq k_y - k^s_y \ (p \in \omega_y). \tag{39}$$

From (38) and (39) we obtain $k^s_p + k^u_p \geq k_y$ and, consequently, $k^s_p + k^u_p = k_y$ for all $p \in \omega_y$. □

Theorem 5.22. *Let $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a linear non-autonomous dynamical system and the following conditions be fulfilled:*

(i) the fiber bundle (X, h, Y) is finite-dimensional;

(ii) Y is compact and invariant ($\pi^t Y = Y$ for all $t \in \mathbb{T}$);

(iii) there exists a point $y \in Y$ such that $\omega_y = H^+(y) = Y$;

(iv) there exists at least one asymptotical stable fiber X_{p_0} (i.e. $k^s_{p_0} = k_{p_0}$ or equivalently $k^u_{p_0} = 0$).

Then $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is asymptotically stable, i.e. $X = X^s$.

Proof. According to Corollary 5.21 we have

$$k^s_p + k^u_p = k_y \tag{40}$$

for all $p \in \omega_y$. If $k^u_{p_0} = 0$ for certain $p_0 \in \omega_y$, then by Lemma 5.14 we obtain $k^u_y \le k^u_{p_0} = 0$, i.e. $k^u_y = 0$ and, consequently, $k^s_y = k_y$. From Lemma 5.20 we have $k^s_p \ge k^s_y = k_y$ for all $p \in \omega_y$ and, consequently, $k^s_p = k_y$ for all $p \in Y = \omega_y$. □

Definition 5.23. *Let E be a finite-dimensional ($k := \dim(E)$) Banach space. The linear operator $A \in [E]$ is called asymptotically stable if $|\lambda_j(A)| < 1$ ($j = 1, 2, \dots, k$), where $\sigma(A) := \{\lambda_1(A), \lambda_2(A), \dots, \lambda_k(A)\}$ is a spectrum of A.*

Theorem 5.24. *Let E be a finite-dimensional Banach space, $A_i \in [E]$ ($i = 1, 2, \dots, m$) and $\mathcal{M} := \{A_1, A_2, \dots, A_m\}$. Assume that the following conditions are fulfilled:*

(i) there exists $j \in \{1, 2, \dots, m\}$ such that the operator A_j is asymptotically stable;

(ii) the discrete linear inclusion $DLI(\mathcal{M})$ has not any nontrivial bounded on $\mathbb{Z}$ solutions.

Then the discrete linear inclusion $DLI(\mathcal{M})$ is absolutely asymptotically stable.

Proof. Let $Q := \mathcal{M}$, $Y = \Omega := C(\mathbb{Z}_+, Q)$ and $(Y, \mathbb{Z}_+, \sigma)$ be a semi-group dynamical system of shifts on Y (see Section 2). It is easy to see that $Y = C(\mathbb{Z}_+, Q)$ is topologically isomorphic to $\Sigma_m := \{0, 1, \dots, m-1\}^{\mathbb{Z}_+}$ and $(Y, \mathbb{Z}_+, \sigma)$ is dynamically isomorphic to shift dynamical system on Σ_m (see, for example,[31, 41]) and, consequently, it possesses the following properties:

(i) Y is compact;

(ii) $Y = \overline{Per(\sigma)}$, where $Per(\pi)$ the set of all periodic points of dynamical system $(Y, \mathbb{Z}_+, \sigma)$;

(iii) there exists a Poisson stable point $y \in Y$ such that $Y = H^+(y)$.

Let $\langle E, \varphi, (Y, \mathbb{Z}_+, \sigma)\rangle$ be a cocycle, generated by $DLI(\mathcal{M})$ (i.e. $\varphi(n, u, \omega) := U(n, \omega)u$, where $U(n, \omega) = \prod_{k=1}^{n} \omega(k)$ ($\omega \in \Omega$)), $(X, \mathbb{Z}_+, \pi)$ be a skew-product system associated with cocycle φ (i.e. $X := E \times Y$ and $\pi := (\varphi, \sigma)$) and $\langle (X, \mathbb{Z}_+, \pi),$ $(Y, \mathbb{Z}_+, \sigma),\ h\rangle$ ($h := pr_2 : X \to Y$) be a linear non-autonomous dynamical system, generated by cocycle φ. Denote by $\omega_0 : \mathbb{Z}_+ \to \mathcal{M}$ the mapping defined by equality $\omega_0(i) = A^i_j$ for all $i \in \mathbb{N}$, where $A^i_j := A_j \circ A^{i-1}_j$ ($i = 2, \dots$). Since the operator A_j is asymptotically stable, then the fiber X_{p_0} ($p_0 := \omega_0 \in Y$) is asymptotically stable. Now to finish the proof of Theorem it is sufficient to refer to Theorem 5.22. □

References

[1] Alekseev V.M. *Symbolic Dynamics.* Kiev, Naukova Dumka, 1986.

[2] Artzrouni M. *On the Convergence of Infinite Products of Matrices.* Linear Algebra and its Applications, 1986, **74**, p. 11–21.

[3] Beyn W.-J., Elsner L. *Infinite Products and Paracontracting Matrices.* The Electronic Journal of Linear Algebra, 1997, **2**, p. 1–8.

[4] Bourbaki N. *Variétés Différentielles et Analitiques (Fascicule de résultats).* Herman, Paris, 1971.

[5] Bronsteyn I.U. *Extensions of Minimal Transformation Group.* Noordhoff, 1979.

[6] Bronstein I.U. *Nonautonomous Dynamical Systems.* Chişinău, Stiintsa, 1984.

[7] Bru R., Elsner L., Neumann M. *Convergence of Infinite Products of Matrices and Inner-Outher Iteration Schemes.* Electronic Transactions on Numerical Analysis, 1994, **2**, p. 183–194.

[8] Cheban D.N. *Boundedness, dissipativity and almost periodicity of the solutions of linear and weakly nonlinear systems of differential equations.* Dinamicheskie Sistemy i Kraevye Zadachi: Mat. Issled., vyp. 92. Chişinău, Stiintsa, 1987, p. 143–159.

[9] Cheban D.N. *Global attractors of infinite-dimensional systems, I.* Bulletin of Academy of Sciences of Republic of Moldova, Mathematics, 1994, N 2(15), p. 12–21.

[10] Cheban D.N. *The Asymptotics of Solutions of Infinite Dimensional Homogeneous Dynamical Systems.* Mat. Zametki, 1998, **63**, N 1, p. 115-126 (Translation in Mathematical Notes, 1998, **63**, N 1, p. 115–126).

[11] Cheban D.N. *Uniform exponential stability of linear almost periodic systems in Banach spaces.* Electronic Journal of Differential Equations, 2000, **2000**, N 29, p. 1–18.

[12] Cheban D.N. *Global Attractors of Nonautonomous Dynamical Systems.* State University of Moldova, 2002 (in Russian).

[13] Cheban D.N. *Global Attractors of Non-autonomous Dissipstive Dynamical Systems.* World Scientific, 2004.

[14] Cheban D.N., Mammana C. *Relation between Different Types of Global Attractors of Non-Autonomous Set-Valued Dynamical Systems* (Set-Valued analysis, to appear 2005).

[15] Cheban D.N., Mammana C. *Asymptotic Stability of Autonomous and Non-Autonomous Discrete Linear Inclusions.* Bulletin of Academy of Sciences of Republic of Moldova, Mathematics, 2004, N 3(46), p. 41–52.

[16] Daubechies I., Lagarias J.C. *Sets of Matrices All Infnite Products of Which Converge.* Linear Algebra Appl., 1992, **161**, p. 227–263.

[17] Elsner L., Friedland S. *Norm Conditions for Convergence of Infinite Products.* Linear Algebra Appl., 1997, **250**, p. 133–142.

[18] Elsner L., Koltracht I., Neumann M. *On the Convergence of Asynchronous Paracontractions with Applications to Tomographic Reconstruction from Incomplete Data.* Linear Algebra Appl., 1990, **130**, p. 65–82.

[19] Gurvits L. *Geometrical Approach to Some Problems of Control, Observation and Lumping.* PhD. Thesis, Dept. of Mathematics, Gorky State Univ., USSR, 1985.

[20] Gurvits L.N., Zaharin A.M. *System Theoretic and Graph Theoretic Aspects of Markov Chains Lumping.* 5th International Conference on Mathematical Statistics, Vilnius, 1989.

[21] Gurvits L.N., Zaharin A.M. *Lumpability of Markov Processes and Systems Theory.* MTNS, Amsterdam, 1989.

[22] Gurvits L. *Stability of Discrete Linear Inclusion.* Linear Algebra and its Applications, 1995, **231**, p. 47–85.

[23] Hale J.K. *Asymptotic Behaviour of Dissipative Systems.* Amer. Math. Soc., Providence, RI, 1988.

[24] Husemoller D. *Fibre Bundles.* Springer-Verlag, Berlin–Heidelberg–New York, 1994.

[25] Kato J., Nakajima F. *On Sacker-Sell's theorem for a linear skew product flow.* Thoku Math. J., (2) 28, 1976, N 1, p. 79–88.

[26] Kelley J. *General Topology.* D. Van Norstand Company Inc., Toronto–New York–London, 1957.

[27] Kozyakin V.S. *On the absolute stability of systems with asynchronously operating pulse elements.* Avtomatika i Telemekhanika, 1990, N 10, p. 56–63 (Translation in Automat. Remote Control, 1990, **51**, N 10, part 1, p. 1349–1355.

[28] Ladyzhenskaya O.A. *Attractors for Semigroups and Evolution Equations.* Lizioni Lincei, Cambridge Univ. Press, Cambridge, New-York, 1991.

[29] Molchanov A.P. *Lyapunov function for nonlinear discrete-time control systems.* Avtomatika i Telemekhanika, 1987, **6**, p. 26–35.

[30] Opoitsev V.I. *Stability of nonautonomous systems.* Avtomatika i Telemekhanika, 1986, **10**, p. 56–63.

[31] Robinson C. *Dynamical Systems: Stabilty, Symbolic Dynamics and Chaos.* Studies in Advanced Mathematics, Boca Raton Florida: CRC Press, 1995.

[32] Rota G.C., Strang W.G. *A Note on the Joint Spectral Radius.* Indag. Math., 1969, **22**, p. 379–381.

[33] Sacker R.J., Sell G.R. *Existence of Dichotomies and Invariant Splittings for Linear Differential Systems, I.* Journal of Differential Equations, 1975, **15**, p. 429–458.

[34] Sadovsky B.M. *Condensing and asymptotic compact operators.* Uspekhi Mat. Nauk, 1972, **23**, N 1(163), p. 81–146.

[35] Sell G.R., You Y. *Dynamics of Evolutionary Equations.* Springer-Verlag, New York, 2002.

[36] Sell G.R. *Lectures on Topological Dynamics and Differential Equations.* Van Nostrand Reinhold math. studies, Vol. 2. Van Nostrand–Reinbold, London, 1971.

[37] Shcherbakov B.A. *Topologic Dynamics and Poisson Stability of Solutions of Differential Equations.* Chişinău, Ştiinţa, 1972 (in Russian).

[38] Shcherbakov B.A. *Poisson Stability of Motions of Dynamical Systems and Solutions of Differential Equations*. Chişinău, Ştiinţa, 1985 (in Russian).

[39] Sibirskii K.S., Shube A.S. *Semidynamical Systems.* Chişinău, Ştiinţa, 1987 (in Russian).

[40] Vladimirov A., Elsner L., Beyn W.-J. *Stability and paracontractivity of discrete linear inclusion.* Linear Algebra and its Applications, 2000, **312**, p. 125–134.

[41] De Vries J. *Elements of Topological Dynamics.* Mathematics and Its Applications, Vol. 257. Kluwer Academic Publishers, Dordecht–Boston–London, 1993.

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