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Variety of the center and limit cycles of a cubic system, which is reduced to Lienard form

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Abstract. In the present work for the system $\dot{x} = y(1+Dx+Px^2)$, $\dot{y} = -x+Ax^2+3Bxy+Cy^2+Kx^3+3Lx^2y+Mxy^2+Ny^3$ 25 cases are given when the point $O(0,0)$ is a center. We also consider a system of the form $\dot{x} = yP_0(x)$, $\dot{y} = -x+P_2(x)y^2+P_3(x)y^3$, for which 35 cases of a center are shown. We prove the existence of systems of the form $\dot{x} = y(1+Dx+Px^2)$, $\dot{y} = -x+\lambda y+Ax^2+Cy^2+Kx^3+3Lx^2y+Mxy^2+Ny^3$ with eight limit cycles in the neighborhood of the origin of coordinates.

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1. We will consider the system of differential equations

$$\dot{x} = y(1+Dx+Px^2),\ \dot{y} = -x+Ax^2+3Bxy+Cy^2+Kx^3+3Lx^2y+Mxy^2+Ny^3, \quad (1)$$

where A, B, C, D, K, L, M, N, P are real constants. The origin of coordinates of system (1) is a critical point of the center or focus type. The center-focus problem for (1) in the case of $D = P = 0$ was first investigated by I.S. Kukles in [1]. In [2, 3] for the system (1) for $D = P = 0$ necessary and sufficient center conditions of algebraic nature were given. For $B = D = P = 0$ the solution of the center-focus problem for (1) is in [4–7]. In the case of $N = 0$ the center-focus problem for (1) was solved in [8]. In [9] all the cases of the center for system (1) for $D = P = 0$ were found, although their necessity was not established completely. Using Cherkas method [10; 11, p.70] the center-focus problem for $D = P = 0$ was solved in [12]; on the basis of investigation of focal values the solution of this problem was reduced in [13]. In [13] the existence of cubic systems of nonlinear oscillations with seven limit cycles was also proved. In [14] it was shown that in the case of the existence of invariant straight line the necessary and sufficient center condition is the equality to zero of the first five focal values. The case of reversible system of the type (1) from the class $C\mathbb{R}_3^{10}$ was shown in [15].

Together with the system (1) we consider a system of the form

$$\dot{x} = yP_0(x),\ \dot{y} = -x + P_2(x)y^2 + P_3(x)y^3, \quad (2)$$

where $P_0(x)=1+\sum_{k=1}^{4} c_kx^k$, $P_2(x)=\sum_{k=0}^{3} a_kx^k$, $P_3(x)=\sum_{k=0}^{4} b_kx^k$, a_i, b_j, $c_k \in \mathbb{C}$, $i = \overline{0,3}$, $j = \overline{0,4}$, $k = \overline{1,4}$. System (1) by change $y = (1-Ax-Kx^2)Y/[1+(B+Lx)Y]$ and

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change of time [3] is transformed to the system (2), where

$$
\begin{gathered}
a_0 = A + C, \qquad a_1 = 3B^2 + A(D - C) + 2K + M, \\
a_2 = K(2D - C) + 6BL + A(P - M), \qquad a_3 = 3L^2 + K(2P - M), \\
c_1 = D - A, \qquad c_2 = P - K - AD, \qquad c_3 = -DK - AP, \\
c_4 = -KP, \qquad b_0 = B(A + C) + L + N, \\
b_1 = B[2B^2 + A(D - C) + 2K + M] + L(C + D) - 2AN, \qquad (3) \\
b_2 = B[K(2D - C) + 6BL + A(P - M)] + L(K + P - AC) + N(A^2 - 2K), \\
b_3 = B[6L^2 + K(2P - M)] + L[K(D - C) - AM] + 2AKN, \\
b_4 = L[2L^2 + K(P - M)] + K^2 N.
\end{gathered}
$$

There exists a formal series for system (1)

$$U = x^2 + y^2 + \sum_{i+j=3}^{\infty} q_{i,j} x^i y^j, \qquad (4)$$

for which on account of (1)

$$\dot{U} = \sum_{i=1}^{\infty} f_i (x^2 + y^2)^{i+1},$$

where f_i, $i = 1, 2, \dots$, are the focal values of system (1). If in (4) $q_{0,2i} = 0$, $i = 2, 3, \dots$, then the function U and focal values f_i, $i = 1, 2, \dots$, are defined in a unique way.

Let us form the ideal [16, p. 46] $J = \langle f_1, \dots, f_9, \dots \rangle$, where f_i, $i = 1, 2, \dots$, are the focal values of system (1). Together with the ideal J we will use the ideals $\overline{J}_i = \langle f_1, \dots, f_i \rangle$, $i = 1, 2, \dots$. The first focal value of system (1) has the form: $f_1 = B(A + C) + L + N$, the second focal value f_2 has 38 summands, the third – 192, the 4th – 702, the 5th – 2093, the 6th – 5406, the 7th – 12538, the 8th - 26726, the 9th – 53212. To compute the focal values we use computer package MATHEMATICA 5.0. The program for the computing of the focal values is in the paper [13].

The focal values f_i, $i = 1, 2, \dots$, are the polynomials from the ring $\mathbb{C}[q]$, where $q = (A, B, C, D, K, L, M, N, P)$, that's why $J, \overline{J}_i \subset \mathbb{C}[q]$, $i = 1, 2, \dots$. The variety of ideal J is the set [16, p. 108] $\mathbb{V}(J) = \{q \in \mathbb{C}^9 : \forall f \in J \;\; f(q) = 0\}$, which we name a variety of the center of system (1). For all i, $i = 1, 2, \dots$, $\mathbb{V}(\overline{J}_i) \supset \mathbb{V}(J)$. It is obvious that the critical point $O(0, 0)$ of system (1) is a center if and only if $q \in \mathbb{V}(J)$. Thus a solution of the center-focus problem for system (1) is reduced to finding the variety $\mathbb{V}(J)$.

The next result takes place [8]:

Theorem 1. *The next equality is true:* $\mathbb{V}(N)\bigcap\mathbb{V}(J)=\bigcup\limits_{k=1}^{11}\mathbb{V}(J_k)$, *where*

$J_1=\langle B,\,L,\,N\rangle$, $J_2=\langle A,\,C,\,D,\,L,\,N\rangle$, $J_3=\langle A+C,\,A-D,\,N,\,2K-M,\,K+P,\,L\rangle$, $J_4=\langle A+C,\,N,\,2K(A+2D)-AM,\,M-2P,\,L\rangle$, $J_5=\langle A+2C,\,3A+2D,\,N,\,A^2-2P,\,AB+2L\rangle$, $J_6=\langle 2A+3C,\,N,\,2A^2(A+D)+(7A+6D)K,\,2(A+D)(A+2D)+M,\,(A+D)(A+2D)+P,\,AB+3L\rangle$, $J_7=\langle 4A+5C+D,\,N,\,2(A+C)(A+2C)-K,\,2(A+C)(3A+4C)-M,\,(A+C)(3A+4C)-P,\,B(A+C)+L\rangle$, $J_8=\langle 5A+6C+D,\,N,\,A(A+C)(2A+3C)+(5A+8C)K,(A+C)(2A+3C)+M,\,3(A+C)(2A+3C)-P,\,B(A+C)+L\rangle$, $J_9=\langle 7A+9C+2D,\,N,\,(A+C)(A+3C)^2-(2A+5C)K,\,(A+C)(2A+3C)-2M,\,3(A+C)(2A+3C)-2P,\,B(A+C)+L\rangle$, $J_{10}=\langle N,\,C(A+C)-K,\,C(A+C)(C-D)+(A+2C)M-CP,\,B(A+C)+L\rangle$, $J_{11}=\langle N,\,A(A+C)(2A+C+D)+(5A+4C+2D)K,\,(A+C)(2A+C+D)-M,\,(A+C)(A+C+D)+P,\,B(A+C)+L\rangle$.

For system (2) we can construct the series (4), for which $\dot{U}=\sum\limits_{i=1}^{\infty}g_i(x^2+y^2)^{i+1}$, where g_i, $i=1,2,\ldots$, are the focal values of system (2). The first focal value of system (2) has the form $g_1=b_0$, the second - $g_2=3a_0b_1+b_2$, the third - $g_3=3b_4+b_3(13a_0+2c_1)-3b_1(15a_0^3-2a_0a_1-a_2+5a_0^2c_1+a_0c_2)$, g_4 contains 32 summands, g_5 - 98, g_6 - 241, g_7 - 540, g_8 - 1084, g_9 - 2024, g_{10} - 3581, g_{11} - 6039, g_{12} - 9772, g_{13} - 15325. Let's introduce $h=(a_0,\,a_1,\,a_2,\,a_3,\,b_0,\,b_1,\,b_2,\,b_3,\,b_4,\,c_1,\,c_2,\,c_3,\,c_4)$. The focal values g_i, i=1, 2, ..., are the polynomials from the ring $\mathbb{C}[h]$. Now we form the ideal $I{=}\langle g_1,\,g_2,\ldots\rangle\subset\mathbb{C}[h]$. The variety of the center of system (2) is a set $\mathbb{V}(I){=}\{h{\in}\mathbb{C}^{13}:\forall\, g{\in}I\ \ g(h){=}0\}$. Together with ideal I we will consider the ideals $\overline{I}_j{=}\langle g_1,\ldots,g_j\rangle$, $j=1,2,\ldots$.

The first necessary center condition for system (2) has the form $b_0=0$, then the polynomial $P_3(x)$ can be represented as $P_3(x)=xQ(x)$, where $Q(x)$ is the polynomial of the 3rd degree. Let's denote

$$R_1(x)=Q'(x)P_0(x)+3Q(x)P_2(x)$$

then the second necessary center condition is $R_1(0)=0$. Taking into account this condition we have $R_1(x)\equiv xQ_1(x)$, where $Q_1(x)$ is the polynomial of the 5th degree. The next statement is correct [12]:

Theorem 2. *The origin of system (2) is a center if and only if*

$$b_0=0,\ R_k(0)=0,\ k=1,2,\ldots, \tag{5}$$

where $R_1(x)$ *is expressed by the formula* (4), $R_k(x)\equiv Q'_{k-1}(x)P_0(x)+(2k+1)Q_{k-1}(x)P_2(x)$, $Q_k(x)\equiv R_{k-1}(x)/x$, $k=2,3,\ldots$.

Let for system (2) the first four necessary conditions from (5) be held. Then the next theorem takes place:

Theorem 3 [11, p. 70]. *The origin of system (2) is a center if and only if the system of equations*

$$Q^5(x)R_1^3(y)=R_1^3(x)Q^5(y),\ P_0(x)S(x)R_1^2(y)=R_1^2(x)P_0(y)S(y), \tag{6}$$

where the polynomials $R_1(x)$, $S(x)$ *and coefficients* r_i, s_i, $i=\overline{0,5}$, *have the form:*

$$R_1(x)=[Q'(x)P_0(x)+3Q(x)P_2(x)]/x\equiv\sum_{k=0}^{5}r_kx^k,$$

$$S(x)=[3R_1'(x)Q(x)-5Q'(x)R_1(x)]/x\equiv\sum_{k=0}^{5}s_kx^k,$$

$r_0=2b_3-3b_1(3a_0^2-a_1+a_0c_1)$, $r_1=3b_4+b_3(3a_0+2c_1)-3b_1(3a_0a_1-a_2+a_0c_2)$, $r_2=3b_4(a_0+c_1)+b_3(3a_1+2c_2)-3b_1(3a_0a_2-a_3+a_0c_3)$, $r_3=3b_4(a_1+c_2)+b_3(3a_2+2c_3)-3a_0b_1(3a_3+c_4)$, $r_4=3b_4(a_2+c_3)+b_3(3a_3+2c_4)$, $r_5=3b_4(a_3+c_4)$, $s_0=-2[5(3a_0^2b_1+b_3)r_0-3b_1r_2]$, $s_1=5(7a_0b_3-3b_4)r_0-3b_1(a_0r_2-3r_3)$, $s_2=4[15a_0b_4r_0-b_3r_2-3b_1(a_0r_3-r_4)]$, $s_3=-9b_4r_2-b_3r_3-3b_1(7a_0r_4-5r_5)$, $s_4=-2(3b_4r_3-b_3r_4+15a_0b_1r_5)$, $s_5=-3b_4r_4+5b_3r_5$, *has an analytical in the neighborhood of* $x=0$ *solution* $y=\psi(x)$, $\psi(0)=0$, $\psi'(0)=-1$, *or at least one of the equations of system (6) is an identity.*

2. We will consider the solution of the center-focus problem for system (2) under various assumptions for the coefficients a_i, $i=\overline{0,3}$, b_j, $j=\overline{0,4}$, c_k, $k=\overline{1,4}$.

To formulate a theorem we introduce the ideals $E_k\subset\mathbb{C}[h]$, $k=\overline{1,9}$:

$E_1=\langle b_3,\, b_4\rangle$, $E_2=\langle 9a_0^2b_1-4b_3,\, b_4\rangle$, $E_3=\langle b_4+a_0^3b_1,\, b_3-3a_0^2b_1\rangle$, $E_4=\langle a_2-a_0(3a_0^2-2a_1+a_0c_1+c_2),\, 2(135a_0^3+81a_0^2c_1+17a_0c_1^2+c_1^3)-(4a_0-c_1)c_2-3c_3\rangle$, $E_5=\langle a_3-t(a_2-a_1t+a_0t^2),\, b_3+b_1t(3a_0+t),\, b_4,\, c_3-t(2c_2-3c_1t+4t^2),\, c_4-t^2(c_2-2c_1t+3t^2)\rangle\bigcap\mathbb{C}[h]$, $E_6=\langle a_2-t(a_1-c_2-a_0t+2c_1t-3t^2),\, a_3+t^2(c_2-2c_1t+3t^2),\, b_4-t(b_3+3a_0b_1t+b_1t^2),\, c_3-t(2c_2-3c_1t+4t^2),\, c_4-t^2(c_2-2c_1t+3t^2)\rangle\bigcap\mathbb{C}[h]$, $E_7=\langle a_1,\, a_2,\, a_3,\, c_2,\, c_3,\, c_4\rangle$, $E_8=\langle r_0,\, r_1,\, r_2,\, r_3,\, r_4,\, r_5\rangle$, $E_9=\langle s_0,\, s_1,\, s_2,\, s_3,\, s_4,\, s_5\rangle$.

Let $W=\bigcup_{k=1}^{9}\mathbb{V}(E_k)$, $W\subset\mathbb{C}^{13}$.

Further we will use the next notations: $\alpha=a_1-a_0(a_0+c_1)$, $\beta=a_0(3a_0+c_1)-a_1$, $\gamma=3a_0+c_1$, $\delta=3a_0+2c_1$, $\sigma=5a_0+c_1$, $\mu=3a_0b_1+b_2$, $\nu=9a_0^2b_1-4b_3$, $\tau=a_0(a_0^2+a_0c_1+c_2)-a_2$, $\xi=a_0(3a_0+c_1)+a_1$. We will denote by I_j, $j=\overline{1,13}$, the following ideals:

$I_1=\langle b_0,\, b_1,\, b_2,\, b_3,\, b_4\rangle$, $I_2=\langle a_0,\, a_2,\, a_3,\, b_0,\, b_2,\, b_3,\, b_4\rangle$,

$I_3=\langle a_0,\, a_2,\, c_1,\, c_3,\, b_0,\, b_2,\, b_4\rangle$, $I_4=\langle\beta,\, a_2-a_0(3a_0\gamma+c_2),\, 3a_0(3a_2+c_3)+c_4,\, a_3-a_0(3a_2+c_3),\, b_4,\, b_3,\, \mu,\, b_0\rangle$, $I_5=\langle 2a_1-a_0\delta,\, 4a_2-a_0(3a_0\delta+4c_2),\, 3a_0(3a_2+2c_3)+4c_4,\, 2a_3-a_0(3a_2+2c_3),\, b_4,\, \nu,\, \mu,\, b_0\rangle$, $I_6=\langle 3\alpha(6a_0+c_1)+6a_0(3a_0\delta+c_2)+2c_3,\, 3\beta(3\xi+2c_2)-4c_4,\, 2a_2-\alpha(9a_0+2c_1)-2a_0(a_0c_1+c_2),\, 2a_3+\beta(3\xi+2c_2),\, b_4,\, 2b_3-3\alpha b_1,\, \mu,\, b_0\rangle$, $I_7=\langle 3\xi+2c_2,\, 2a_2-\gamma\beta+2c_3,\, \gamma(a_2+\gamma\beta)+c_4,\, a_3-\gamma(a_2+\gamma\beta),\, b_4+b_1(a_2+\gamma\beta),\, 2b_3-3b_1\beta,\, \mu,\, b_0\rangle$,

$I_8=\langle\alpha^2(2\gamma^2-\alpha)+2\tau(2\alpha\gamma+\tau),\, \delta\alpha^2+2\tau[3a_0(2a_0+c_1)+c_2]-\alpha[5a_2-a_0\gamma(11a_0+6c_1)-c_2(7a_0+2c_1)+2c_3],\, 5\alpha^3-2\alpha^2[13a_0\delta+c_1(a_0+5c_1)-c_2]-2\alpha[a_0(135a_0^3+161a_0^2c_1+56a_0c_1^2+6c_1^3)-a_2(27a_0+8c_1)+c_2(45a_0^2+20a_0c_1+2c_1^2)]-4\tau[a_0(22a_0^2+18a_0c_1+3c_1^2)+\sigma c_2+c_3],\, 3\alpha^2-2\alpha(c_1^2+2a_0\delta-c_2)-2[a_0^2\gamma(a_0+c_1)-a_2(4a_0+c_1)+a_0(\gamma c_2-c_3)-c_4],\, 3\alpha^2-2\alpha(c_1^2+2a_0\delta-c_2)-2[a_3+a_0^2\gamma(a_0+c_1)-a_2(4a_0+c_1)+a_0(\gamma c_2-c_3)],\, 2b_4-b_1[\alpha(9a_0+2c_1)-2(a_0^3-\tau)],\, 2b_3-3b_1\beta,\, \mu,\, b_0\rangle$,

$I_9=\langle a_2-a_0(a_1+c_2-3\alpha),\, a_0[a_0(a_1+c_2-\alpha)+c_3]+c_4,\, a_3-a_0[a_0(a_1+c_2)+$

$c_3]$, $a_0^3b_1+b_4$, $3a_0^2b_1-b_3$, μ, $b_0\rangle$, $I_{10}=\langle\, 2a_0^2(8a_0^2-c_2)+c_4$, $a_0(20a_0^2-3c_2)-c_3$, $a_2+a_0(12a_0^2+2a_1-c_2)$, σ, $a_0(2a_0^2b_1-b_3)-b_4$, μ, $b_0\rangle$, $I_{11}=\langle a_2+a_0(15a_0^2+2a_1-c_2)$, $10a_0(9a_0^2-c_2)-3c_3$, $6a_0+c_1$, $a_0^2(9a_0^2-c_2)+c_4$, $3a_3-a_0^2(18a_0^2+3a_1-c_2)$, $a_0b_3+3b_4$, μ, $b_0\rangle$, $I_{12}=\langle 2a_2+a_0(21a_0^2+4a_1-2c_2)$, $a_0(189a_0^2-34c_2)-12c_3$, $a_0^2(27a_0^2-4c_2)+2c_4$, $3a_3-a_0^2(36a_0^2+3a_1-4c_2)$, $9a_0+2c_1$, $a_0\nu-3b_4$, μ, $b_0\rangle$,

$I_{13}=\langle(13a_0+2c_1)[2(30a_0^2+12a_0c_1+c_1^2)+c_2]+3a_0^3$, $2c_3+a_0(81a_0^2+36a_0c_1+2c_1^2+7c_2)$, $a_3-a_0^2(a_1+2c_2+219a_0^2+87a_0c_1+8c_1^2)$, $a_2+a_0(3\alpha-a_1-c_2)$, $2a_0^2(135a_0^2+54a_0c_1+5c_1^2+c_2)+c_4$, $3b_4-6a_0^2b_1(6a_0+c_1)+b_3(13a_0+2c_1)$, μ, $b_0\rangle$.

Notice that the bases of ideals I_j, $j=\overline{1,13}$, have no more than nine elements. Further we introduce the ideals I_j, $j=\overline{14,35}$, which have the form:

$I_{14}=\langle 5a_0^2+2a_1$, a_2, a_3, b_0, μ, $5a_0^2b_1-4b_3$, b_4, σ, $25a_0^2-4c_2$, c_3, $c_4\rangle$, $I_{15}=\langle 15a_0^2+4a_1$, $75a_0^3-16a_2$, $125a_0^4+64a_3$, b_0, μ, $35a_0^2b_1-16b_3$, b_4, σ, $75a_0^2-8c_2$, $125a_0^3+16c_3$, $625a_0^4-256c_4\rangle$, $I_{16}=\langle 10a_0^2+3a_1$, $25a_0^3-9a_2$, a_3, b_0, μ, $20a_0^2b_1-9b_3$, b_4, σ, $25a_0^2-3c_2$, $125a_0^3+27c_3$, $c_4\rangle$, $I_{17}=\langle a_1$, a_2, a_3, b_0, μ, $10a_0^2b_1+b_3$, b_4, σ, c_2, c_3, $c_4\rangle$,

$I_{18}=\langle a_0(7a_0^2+2a_1)+a_2$, $3a_0^2(2a_0^2+a_1)+a_3$, b_0, μ, b_3, b_4, $4a_0+c_1$, $2a_0^2+c_2$, $12a_0^3-c_3$, $9a_0^4-c_4\rangle$, $I_{19}=\langle 2a_0(2a_0^2+a_1)+a_2$, a_3, b_0, μ, b_3, b_4, $7a_0+c_1$, $16a_0^2-c_2$, $12a_0^3+c_3$, $c_4\rangle$, $I_{20}=\langle 5a_0^2+a_1$, a_2, a_3, b_0, μ, b_3, b_4, $10a_0+c_1$, $25a_0^2-c_2$, c_3, $c_4\rangle$, $I_{21}=\langle 5a_0^2+a_1$, $25a_0^3-4a_2$, a_3, b_0, μ, b_3, b_4, $15a_0+2c_1$, $75a_0^2-4c_2$, $125a_0^3+8c_3$, $c_4\rangle$, $I_{22}=\langle 5a_0^2+a_1$, $25a_0^3-3a_2$, $125a_0^4+27a_3$, b_0, μ, b_3, b_4, $20a_0+3c_1$, $50a_0^2-3c_2$, $500a_0^3+27c_3$, $625a_0^4-81c_4\rangle$,

$I_{23}=\langle 7a_0^2+2a_1$, a_2, a_3, b_0, μ, ν, b_4, $7a_0+c_1$, $49a_0^2-4c_2$, c_3, $c_4\rangle$, $I_{24}=\langle 2a_0(2a_0^2+a_1)+a_2$, a_3, b_0, μ, ν, b_4, $11a_0+2c_1$, $10a_0^2-c_2$, $6a_0^3+c_3$, $c_4\rangle$, $I_{25}=\langle 7a_0^2+2a_1$, $49a_0^3-16a_2$, a_3, b_0, μ, ν, b_4, $21a_0+4c_1$, $147a_0^2-16c_2$, $343a_0^3+64c_3$, $c_4\rangle$, $I_{26}=\langle a_0(13a_0^2+8a_1)+4a_2$, $3a_0^2(2a_0^2+a_1)-4a_3$, b_0, μ, ν, b_4, $4a_0+c_1$, $11a_0^2-2c_2$, $3a_0^3+c_3$, $9a_0^4-16c_4\rangle$, $I_{27}=\langle 7a_0^2+2a_1$, $49a_0^3-12a_2$, $343a_0^4+216a_3$, b_0, μ, ν, b_4, $14a_0+3c_1$, $49a_0^2-6c_2$, $343a_0^3+54c_3$, $2401a_0^4-1296c_4\rangle$,

$I_{28}=\langle 2a_0(2a_0^2+a_1)+a_2$, a_3, b_0, μ, b_3, $4a_0^3b_1-b_4$, γ, c_2, $4a_0^3-c_3$, $c_4\rangle$, $I_{29}=\langle 2a_0(2a_0^2+a_1)+a_2$, a_3, b_0, μ, ν, $a_0^3b_1+2b_4$, $9a_0+2c_1$, $6a_0^2-c_2$, $2a_0^3+c_3$, $c_4\rangle$, $I_{30}=\langle 5a_0^2+2a_1$, $25a_0^3-16a_2$, a_3, b_0, μ, $45a_0^2b_1-16b_3$, $25a_0^3b_1+32b_4$, $15a_0+4c_1$, $75a_0^2-16c_2$, $125a_0^3+64c_3$, $c_4\rangle$,

$I_{31}=\langle 14a_0^2+5a_1$, $49a_0^3-25a_2$, a_3, b_0, μ, $72a_0^2b_1-25b_3$, $112a_0^3b_1+125b_4$, $21a_0+5c_1$, $147a_0^2-25c_2$, $343a_0^3+125c_3$, $c_4\rangle$, $I_{32}=\langle 5a_0^2+3a_1$, a_2, a_3, b_0, μ, $5a_0^2b_1-3b_3$, $25a_0^3b_1-27b_4$, $10a_0+3c_1$, $25a_0^2-9c_2$, c_3, $c_4\rangle$, $I_{33}=\langle 7a_0^2+3a_1$, a_2, a_3, b_0, μ, $5a_0^2b_1-3b_3$, $7a_0^3b_1+27b_4$, $14a_0+3c_1$, $49a_0^2-9c_2$, c_3, $c_4\rangle$, $I_{34}=\langle a_1$, a_2, a_3, b_0, μ, $24a_0^2b_1+b_3$, $28a_0^3+b_4$, $7a_0+c_1$, c_2, c_3, $c_4\rangle$, $I_{35}=\langle a_1$, a_2, a_3, b_0, μ, $15a_0^2b_1+4b_3$, $25a_0^3-2b_4$, $5a_0+2c_1$, c_2, c_3, $c_4\rangle$.

Notice that the bases of ideals $I_{14},...,I_{35}$ contain 10 or 11 elements.

It is significant that $\mathbb{V}(I_k)$, $k=\overline{1,35}$, are irreducible varieties. Let $V=\bigcup\limits_{k=1}^{35}\mathbb{V}(I_k)$.

Theorem 4. *The next equality takes place:* $V=W\bigcap\mathbb{V}(I)$.

The proof of Theorem 4 is given in p. 3.

Theorem 4 gives the solution of the center-focus problem for system (2) in the case of $h \in W$. It is obvious that $V \subset \mathbb{V}(I)$. Question: is it true that $W \supset \mathbb{V}(I)$? If $W \supset \mathbb{V}(I)$ then $V = \mathbb{V}(I)$, i.e. in that case Theorem 4 gives the solution of the center-focus problem for system (2).

We will point out further the solution of the center-focus problem for system (1) under different assumptions for the coefficients A, B, C, D, K, L, M, N, P. Let's construct the ideals $G_i \subset \mathbb{C}[q]$, $i = \overline{1,17}$, in which $a_i\,(i = \overline{0,3})$, $b_j\,(j = \overline{0,4})$, c_k $(k = \overline{1,4})$ are expressed by the formulas (3):

$G_1 = \langle b_3,\, b_4\rangle$, $G_2 = \langle 4b_3 - 9a_0^2 b_1,\, b_4\rangle$, $G_3 = \langle b_4 + a_0^3 b_1,\, b_3 - 3a_0^2 b_1\rangle$, $G_4 = \langle a_2 - a_0(3a_0^2 - 2a_1 + a_0c_1 + c_2),\, 2(135a_0^3 + 81a_0^2c_1 + 17a_0c_1^2 + c_1^3) - (4a_0 - c_1)c_2 - 3c_3\rangle$, $G_5 = \langle a_3 - t(a_2 - a_1t + a_0t^2),\, b_3 + b_1t(3a_0 + t),\, b_4,\, c_3 - t(2c_2 - 3c_1t + 4t^2),\, c_4 - t^2(c_2 - 2c_1t + 3t^2)\rangle \bigcap \mathbb{C}[q]$, $G_6 = \langle a_2 - t(a_1 - c_2 - a_0t + 2c_1t - 3t^2),\, a_3 + t^2(c_2 - 2c_1t + 3t^2),\, b_4 - t(b_3 + 3a_0b_1t + b_1t^2),\, c_3 - t(2c_2 - 3c_1t + 4t^2),\, c_4 - t^2(c_2 - 2c_1t + 3t^2)\rangle \bigcap \mathbb{C}[q]$, $G_7 = \langle a_1,\, a_2,\, a_3,\, c_2,\, c_3,\, c_4\rangle$, $G_8 = \langle r_0,\, r_1,\, r_2,\, r_3,\, r_4,\, r_5\rangle$, $G_9 = \langle s_0,\, s_1,\, s_2,\, s_3,\, s_4,\, s_5\rangle$,

$G_{10} = \langle N\rangle$, $G_{11} = \langle B\rangle$, $G_{12} = \langle K,\, L\rangle$, $G_{13} = \langle A + C,\, 3K + M + P\rangle$, $G_{14} = \langle B(3A+3C+D)+L,\, (2A+3C+D)(3A+3C+D)-K\rangle$, $G_{15} = \langle 2(A+C)(2A+C+D) + 3K+M,\, (A+C)(A+C+D)+P\rangle$, $G_{16} = \langle 3(A+C)(A+3C)-4K,\, 3B(A+C)+2L\rangle$, $G_{17} = \langle Kt - (N - Ct)(N - At - Ct),\, 2N^2 - Nt(3C - D) + t^2[C(C - D) + M - t(t + 3B)],\, 3Lt^4 + N^2(N - 2Ct) + Nt^2(C^2 + M - P + t^2) - t^3[C(M - P) + t^2(A + C)]\rangle \bigcap \mathbb{C}[q]$.

Let $\widetilde{G} = \bigcup\limits_{k=1}^{17} \mathbb{V}(G_i)$ and $T = \bigcup\limits_{k=1}^{25} \mathbb{V}(J_k)$, where the ideals $J_i \subset \mathbb{C}[q]$, $i = \overline{12,25}$, have the form:

$J_{12} = \langle A + C,\, K(A - D) + AM + 3BN,\, K(2K + M) - N^2,\, 3K + M + P,\, L + N\rangle$,

$J_{13} = \langle 3B^2 - (2A + 3C)(4A + 3C + D),\, B(A + C) + N,\, K,\, 2(A + C)(3A + 3C + D) + M,\, 3(A + C)(3A + 3C + D) + P,\, L\rangle$, $J_{14} = \langle 6B^2 - (A + 3C)(A + D),\, B(A + C) - 2N,\, 3(A + C)(A + 3C) - 4K,\, (A + C)(3A + 6C + D) + 2M,\, 3(A + C)(3A + 3C + 2D) + 4P,\, 3B(A + C) + 2L\rangle$, $J_{15} = \langle 3B^2 - (A + D)(2A + 3C + D),\, B(2A + 2C + D) - N,\, (2A + 3C + D)(3A + 3C + D) - K,\, (2A + 3C + D)(3A + 3C + D) + M,\, (3A + 3C + D)(3A + 3C + 2D) + P,\, B(3A + 3C + D) + L\rangle$,

$J_{16} = \langle 3(A + C) + D,\, B(A + 2C) + 2N,\, 3AB^2 + 4K(2A + 3C),\, 6B^2 - A(2A + 3C) + 4K + 3M,\, 3B^2 - 2A(2A + 3C) + 2K + 2P,\, AB + 2L\rangle$, $J_{17} = \langle 3(A + C) + D,\, 4(A + 3C)(2A + 3C)^2 + 9B^2(5A + 7C),\, 3B(A + 2C) - N,\, (A + 3C)(2A + 3C) - K,\, 27B^2 - 4(2A + 3C)(A + 4C) - 4M,\, 27B^2 + 12C(2A + 3C) + 4P,\, B(4A + 7C) + L\rangle$,

$J_{18} = \langle 7A + 9C + 2D,\, 36B^2(11A + 17C) + (A + 3C)(17A + 27C)^2,\, 3B(A + C)(3A + 5C) - 2N(17A + 27C),\, 3(A + C)(A + 3C) - 4K,\, (A - 3C)(171A + 139C) - 4(57A^2 + 27B^2 + 16M),\, 3(A + 3C)(67A + 73C) + 4(16A^2 + 27B^2 - 16P),\, B(A + C) + L + N,\, t(17A + 27C) - 1\rangle \bigcap \mathbb{C}[q]$,

$J_{19} = \langle (A + C)(A + 3C)(2A + C + D) - K(A - C + D) + 3BN,\, (A + C)^2(A + 2C)(2A + C + D)(2A + 2C + D) + K[(A + C)^2 + K] + N[3B(A + C) + N],\, 2(A + C)(2A + C + D) + 3K + M,\, (A + C)(A + C + D) + P,\, B(A + C) + L + N\rangle$,

$J_{20} = \langle 2B^2(A + 2C) - (2A + C + D)(A^2 + 4K) + 4BN,\, A^2(2A + C + D) + 2(2A + C)(B^2 + K) + AM + 2BN,\, B(2B^2 + 2K + M) - (2A + C + D)(BC + N),\, A(4A +$

$C+3D)+2(3B^2+3K+M+P)$, $B(A+C)+L+N$, $t(A(2A+C+D)^2+2B^2(3A+2C+D))-1\rangle \bigcap \mathbb{C}[q]$,

$J_{21}=\langle B^2K-(BC+N)[B(A+C)+N]$, $B(2B^2+2K+M)-(2A+C+D)(BC+N)$, $B[2AB^2-K(A+D)]+(BC+N)[4B^2+A(A+D)+K+P]$, $B(A+C)+L+N$, $tB(BC+N)-1\rangle \bigcap \mathbb{C}[q]$,

$J_{22}=\langle(6A+8C+D)(7A+9C+D)(7A+9C+2D)+2(A+C)(2A+3C)(11A+15C+2D)$, $27B^2-12(2A+3C)(10A+11C)+(7A+21C-23D)(7A+9C+D)$, $3B(4A+5C+D)(5A+7C+D)-(33A+45C+7D)N$, $(4A+6C+D)(5A+6C+D)-K$, $(A+C)(2A+3C)+(2A+3C+D)(7A+9C+D)+M$, $(7A+9C+D)(7A+9C+2D)+P$, $B(A+C)+L+N$, $t(33A+45C+7D)-1\rangle \bigcap \mathbb{C}[q]$,

$J_{23}=\langle(5A+3C+2D)[(A+3C)(11A+18C)+4(2A+3C)(A-3C+D)]-12B^2(4A+3C+2D)+4K(7A+6C+2D)$, $B[(A+3C)(4A+7C)+2(A+2C)(A-3C+D)-4K]-6B^3-2N(2A+D)$, $6B^2(3A+C+2D)-(2A+3C)[(5A+3C+2D)^2+4K]-12BN$, $6B^2-A(A+6C)-C(3C+2D)+2(2K+M)$, $3(A+C)(3A+3C+2D)+4P$, $B(A+C)+L+N$, $t(2A+D)(7A+6C+2D)-1\rangle \bigcap \mathbb{C}[q]$,

$J_{24}=\langle 3B^2(7A+6C+2D)+A[(2A+C+D)(2A+C+2D)-2(A+C)(3A+5C)]-K(A+2D)$, $B[(A+C)(7A+10C)+(A+2C)(2A+C+D)-2K]-3B^3+N(A+3C-D)$, $3B^2(3A+2C+D)-A[(A+C)(5A+8C)+(C-D)(2A+C+D)]+AK-3BN$, $3B^2-2A(A+3C)-C(3C+D)+2K+M$, $3(A+C)(3A+3C+D)+P$, $B(A+C)+L+N$, $t(A+2D)(A+3C+D)-1\rangle \bigcap \mathbb{C}[q]$,

$J_{25}=\langle A[(4A+3C+2D)(8A+9C+3D)-3B^2]+K(13A+12C+6D)$, $B[C(A+6C)+6(2A+C+D)(2A+3C+D)-2K]-3B^3-N(5A+3C+3D)$, $A(4A+3C+2D)(7A+9C+3D)+(A+3C)(3B^2+2K)+9BN$, $3B^2-(2A+3C+D)(4A+3C+2D)+2K+M$, $(3A+3C+D)(3A+3C+2D)+P$, $B(A+C)+L+N$, $t(5A+3C+3D)(13A+12C+6D)-1\rangle \bigcap \mathbb{C}[q]$;

and the ideals J_i, $i=\overline{1,11}$ are from Theorem 1.

Theorem 5. *The next equality takes place:* $T=\mathbb{V}(J)\bigcap \widetilde{G}$.

The proof of Theorem 5 is given in p. 4.

Theorem 6. *There exist systems of the form*

$$\dot{x}=y(1+Dx+Px^2),\ \dot{y}=-x+\lambda y+Ax^2+Cy^2+Kx^3+3Lx^2y+Mxy^2+Ny^3, \quad (7)$$

having eight limit cycles in any infinitely small neighborhood of the origin.

The proof is given in p. 5.

3. Now we will examine the focal values of system (2). From $g_1=0$ we have $b_0=0$, from $g_2=0$ we find $b_2=-3a_0b_1$. Taking into account b_0, b_2 from g_3 we obtain $b_4=b_1(15a_0^3-2a_0a_1-a_2+5a_0^2c_1+a_0c_2)-b_3(13a_0+2c_1)/3$. Using the quantities of b_0, b_2, b_4 we can present g_k, $k=4,5,\ldots$, in the form: $g_k=v_kb_3+w_kb_1$, where $v_k, w_k\in\mathbb{C}[a_0, a_1, a_2, a_3, c_1, c_2, c_3, c_4]$. Construct the ideal $X=\langle v_4, w_4, v_5, w_5, \ldots\rangle+\langle g_1, g_2, g_3\rangle$. It is obvious that $X\supset I$.

Statement 1. *The next formula takes place:* $\mathbb{V}(X)=\mathbb{V}(I_3)\bigcup(\bigcup_{k=10}^{13}\mathbb{V}(I_k))$. *Here the ideals* I_3, I_k, $k=\overline{10,13}$, *are prime.*

Proof. Computing the Groebner basis [16, p. 105] for the ideal $X_7 = \langle v_4,\, w_4, \ldots, v_7,\, w_7\rangle + \langle g_1,\, g_2,\, g_3\rangle$ with the order

$$b_0 > b_2 > b_4 > c_3 > c_4 > a_3 > a_2 > a_1 > c_2 > c_1 > b_3 > b_1 > a_0$$

we get $X_7 = \langle a_0^4(5a_0 + c_1)^7(6a_0 + c_1)^2(9a_0 + 2c_1)^2[783a_0^3 + 432a_0^2c_1 + 74a_0c_1^2 + 4c_1^3 + c_2(13a_0 + 2c_1)],\, h_2, \ldots, h_{78}\rangle$, where $h_i \in \mathbb{C}[h]$, $i = \overline{2, 78}$. Further we find the ideal $\widetilde{X}_7 = \langle a_0(5a_0 + c_1)(6a_0 + c_1)(9a_0 + 2c_1)[783a_0^3 + 432a_0^2c_1 + 74a_0c_1^2 + 4c_1^3 + c_2(13a_0 + 2c_1)],\, \widetilde{h}_2, \ldots, \widetilde{h}_{25}\rangle$; at the same time the radicals of ideals [16, p. 230] X_7 and $\widetilde{X}_7$ are equal, i.e. $\sqrt{X_7} = \sqrt{\widetilde{X}_7}$. Using for $\widetilde{X}_7$ the operations of intersection and division of ideals we find the radical $\sqrt{\widetilde{X}_7} = I_3 \bigcap (\bigcap_{k=10}^{13} I_k)$. In that case $\sqrt{X_7} = \langle (5a_0+c_1)(6a_0+c_1)(9a_0+2c_1)[783a_0^3+432a_0^2c_1+74a_0c_1^2+4c_1^3+c_2(13a_0+2c_1)],\, (5a_0+c_1)[3a_0^2a_1 - 3a_3 + 2(4635a_0^4 + 3681a_0^3c_1 + 1067a_0^2c_1^2 + 133a_0c_1^3 + 6c_1^4) + (149a_0^2 + 61a_0c_1 + 6c_1^2)c_2],\, a_2 - a_0[3a_0(a_0+c_1) - 2a_1 + c_2],\, 29727a_0^5 + 29835a_0^4c_1 + 11766a_0^3c_1^2 + 2278a_0^2c_1^3 + 216a_0c_1^4 + 8c_1^5 + (501a_0^3 + 299a_0^2c_1 + 60a_0c_1^2 + 4c_1^3)c_2 - 3a_0c_4,\, 2(105705a_0^5 + 105705a_0^4c_1 + 41553a_0^3c_1^2 + 8019a_0^2c_1^3 + 758a_0c_1^4 + 28c_1^5) + 2(1755a_0^3 + 1053a_0^2c_1 + 211a_0c_1^2 + 14c_1^3)c_2 + 3c_1c_4,\, 2(135a_0^3 + 81a_0^2c_1 + 17a_0c_1^2 + c_1^3) - (4a_0 - c_1)c_2 - 3c_3,\, 3b_4 - 6a_0^2b_1(6a_0 + c_1) + b_3(13a_0 + 2c_1),\, 3a_0b_1 + b_2,\, b_0\rangle$.

Let us show that $\mathbb{V}(X) = \mathbb{V}(X_7)$. For that it is enough to show that for $h \in \mathbb{V}(I_3) \bigcup (\bigcup_{k=10}^{13} \mathbb{V}(I_k))$ $O(0,0)$ is a center. Let at first $h \in \mathbb{V}(I_{13})$. In that case $P_0(x) = (1 - a_0x)^6\widetilde{P}_0(z)/[1 - (6a_0 + c_1)x]$, $Q(x) = (1 - a_0x)^3Q_0(z)$, $R_1(x) = (1 - a_0x)^5R_0(z)$, where

$$\begin{gathered}\widetilde{P}_0(z) = 1 - [3z(326a_0^3 + 180a_0^2c_1 + 33a_0c_1^2 + 2c_1^3) - 9a_0^3z^2(5a_0 + c_1)^2]/(13a_0 + 2c_1),\\ R_0(z) = [2b_3 - 3b_1(3a_0^2 - a_1 + a_0c_1) + 3z(3a_0^2b_1(615a_0^3 - 13a_0a_1 + 355a_0^2c_1 - 2a_1c_1 + \\ 66a_0c_1^2 + 4c_1^3) - b_3(665a_0^3 - 13a_0a_1 + 375a_0^2c_1 - 2a_1c_1 + 68a_0c_1^2 + 4c_1^3))]/(13a_0 + 2c_1),\\ Q_0(z) = b_1 + z(b_3 - 3a_0^2b_1),\; z = x^2(1 - (13a_0 + 2c_1)x/3]/(1 - a_0x)^3.\end{gathered} \tag{8}$$

The change

$$y = YQ^{-1/3}(x) \tag{9}$$

reduces the system (2) after the excluding of time to the equation:

$$P_0(x)YY' = -x(1 - Y^3)Q^{2/3}(x) + xR_1(x)Y^2/(3Q(x)). \tag{10}$$

Further the change (8) reduces the equation (10) to the form:

$$2\widetilde{P}_0(z)Y\frac{dY}{dz} = Q_0^{2/3}(z)(Y^3 - 1) + R_0(z)Y^2/(3Q_0(z)). \tag{11}$$

So in that case for system (2) there exists an analytical in the neighborhood of $O(0,0)$ integral and the critical point $O(0,0)$ is a center.

Let now $h \in \mathbb{V}(I_{10})$. Then $P_0(x) = (1-a_0x)^5\widetilde{P}_0(z)$, $Q(x) = (1-a_0x)^3Q_0(z)$, $R_1(x) = (1-a_0x)^5R_0(z)$, where

$$\widetilde{P}_0(z) = 1 - z(10a_0^2 - c_2) + a_0^2z^2(9a_0^2 - c_2),\ Q_0(z) = b_1 + z(b_3 - 3a_0^2b_1),$$
$$R_0(z) = 3b_1(2a_0^2 + a_1) + 2b_3 - z[3b_1(12a_0^4 + 4a_0^2a_1 - a_3) + b_3(8a_0^2 - 3a_1 - 2c_2)]- \tag{12}$$
$$z^2(3a_0b_1 - b_3)[3a_3 - a_0^2(42a_0^2 + 3a_1 - 4c_2)],\ z = x^2/(1-a_0x)^2.$$

The change (12) transforms (10) to the form (11), i.e. in that case for system (2) there also exists an analytical in the neighborhood of $O(0,0)$ integral and $O(0,0)$ is a center.

If $h \in \mathbb{V}(I_{11})$ then $P_0(x) = (1-a_0x)^6\widetilde{P}_0(z)$, $Q(x) = (1-a_0x)^3Q_0(z)$, $R_1(x) = (1-a_0x)^5R_0(z)$, where

$$\widetilde{P}_0(z) = 1 - z(15a_0^2 - c_2) + 3a_0^2z^2(12a_0^2 - c_2),\ Q_0(z) = b_1 + z(b_3 - 3a_0^2b_1),$$
$$R_0(z) = 3b_1(3a_0^2 + a_1) + 2b_3 - z[9a_0^2b_1(3a_0^2 + a_1) + b_3(15a_0^2 - 3a_1 - 2c_2)], \tag{13}$$
$$z = x^2(1 - a_0x/3)/(1-a_0x)^3,$$

and using the change (13) equation (10) is reduced to (11).

If $h \in \mathbb{V}(I_{12})$ then $P_0(x) = (1-a_0x)^6\widetilde{P}_0(z)/(2-3a_0x)$, $Q(x) = (1-a_0x)^3Q_0(z)$, $R_1(x) = (1-a_0x)^5R_0(z)$, where

$$\widetilde{P}_0(z) = 2[1 + z(4c_2 - 33a_0^2)/4 + 3a_0^2z^2(15a_0^2 - c_2)/8],\ Q_0(z) = b_1 + z(b_3 - 3a_0^2b_1),$$
$$R_0(z) = [6b_1(3a_0^2 + 2a_1) + 8b_3 + z(9a_0^2b_1(3a_0^2 - 4a_1 - 2c_2) - 4b_3(6a_0^2 - 3a_1 - 2c_2))]/4,$$
$$z = x^2(1 - 4a_0x/3)/(1-a_0x)^3,$$

i.e. in that case (10) also is transformed to (11).

Under $h \in \mathbb{V}(I_3)$ the presence of a center at $O(0,0)$ is obvious. □

Remark. From the proof of Statement 1 it follows that $\langle a_2 - a_0[3a_0(a_0+c_1) - 2a_1 + c_2],\ 2(135a_0^3 + 81a_0^2c_1 + 17a_0c_1^2 + c_1^3) - (4a_0 - c_1)c_2 - 3c_3\rangle \subset X$.

Investigating the first ten focal values with the help of Statement 1, we get

Statement 2. *For the ideal* $\widetilde{I} = I + \langle a_2 - a_0[3a_0(a_0+c_1) - 2a_1 + c_2],\ 2(135a_0^3 + 81a_0^2c_1 + 170a_0c_1^2 + c_1^3) - (4a_0 - c_1)c_2 - 3c_3\rangle$ *the next formula takes place:* $\sqrt{\widetilde{I}} = I_3 \bigcap (\bigcap_{k=10}^{13} I_k) \bigcap (\bigcap_{j=1}^{3} \widehat{I}_j)$, *where* $\widehat{I}_1 = \langle 3a_0^2b_1 - b_3,\ a_2 - a_0[3a_0(a_0+c_1) - 2a_1 + c_2],\ 3(a_0^2a_1 - a_3) + 2a_0(135a_0^3 + 81a_0^2c_1 + 179a_0c_1^2 + c_1^3) - a_0c_2(a_0 - c_1),\ a_0(273a_0^3 + 165a_0^2c_1 + 34a_0c_1^2 + 2c_1^3) - a_0c_2(a_0 - c_1) + 3c_4,\ 2(135a_0^3 + 81a_0^2c_1 + 17a_0c_1^2 + c_1^3) - (4a_0 - c_1)c_2 - 3c_3,\ a_0^3b_1 + b_4,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_2 = \langle b_1,\ b_3,\ a_2 - a_0[3a_0(a_0+c_1) - 2a_1 + c_2],\ 2(135a_0^3 + 81a_0^2c_1 + 17a_0c_1^2 + c_1^3) - (4a_0 - c_1)c_2 - 3c_3,\ b_4,\ b_2,\ b_0\rangle$, $\widehat{I}_3 = \langle a_0, b_3, a_2, a_3, 3c_3 - c_1(2c_1^2 + c_2), b_4, b_2, b_0\rangle$.

Statement 3. *Let* $X = I + \langle b_3,\ b_4\rangle$. *Then the next equality takes place:* $\sqrt{X} = I_1 \bigcap I_2 \bigcap I_4 \bigcap (\bigcap_{k=18}^{22} I_k) \bigcap (\bigcap_{j=1}^{3} \widehat{I}_j)$, *where* $\widehat{I}_1 = \langle a_0,\ a_2,\ c_1,\ c_3,\ b_0,\ b_2,\ b_3,\ b_4\rangle$, $\widehat{I}_2 =$

$\langle 6a_0 + c_1,\ 10a_0(9a_0^2 - c_2) - 3c_3,\ a_2 + a_0(15a_0^2 + 2a_1 - c_2),\ a_0^2(9a_0^2 - c_2) + c_4,\ 3a_3 - a_0^2(18a_0^2 + 3a_1 - c_2),\ b_4,\ b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_3 = \langle 6a_0 + c_1,\ 12a_0^2 - c_2,\ 2a_0^2 + a_1,\ 10a_0^3 + c_3,\ a_0^3 - a_2,\ 3a_0^4 - c_4,\ a_3,\ b_4,\ b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$.

Proof. With the help of operations of division and intersection of ideals one finds the radical of the ideal $X_9 = \langle g_1, \ldots, g_9\rangle + \langle b_3, b_4\rangle$. We have $\sqrt{X_9} = I_1 \bigcap I_2 \bigcap I_4 \bigcap (\bigcap\limits_{k=18}^{22} I_k) \bigcap (\bigcap\limits_{j=1}^{3} \widehat{I}_j)$.

Let us show that for $h \in \mathbb{V}(X_9)$ $O(0,0)$ is a center. Let $h \in \mathbb{V}(I_1) \bigcap \mathbb{V}(I_2) \bigcap \mathbb{V}(I_4)$, then $R_1(h) \equiv 0$, i.e. $h \in \mathbb{V}(I)$. In that case the equation (2) by change (9) after excluding time is transformed to

$$P_0(x)YY' = -x(1 - Y^3)Q^{2/3}(x).$$

Let now $h \in \bigcup\limits_{k=18}^{22} \mathbb{V}(I_k)$. Then $P_0(h)S(h)/R_1^2(h) \equiv const$, so $h \in \mathbb{V}(I)$. Further we have $\mathbb{V}(\widehat{I}_1) \subset \mathbb{V}(I_3)$; $\mathbb{V}(\widehat{I}_2), \mathbb{V}(\widehat{I}_3) \subset \mathbb{V}(I_{11})$, therefore for $h \in \bigcup\limits_{j=1}^{3} \mathbb{V}(\widehat{I}_j)$ the critical point $O(0,0)$ is a center. Thus $\sqrt{X} = \sqrt{X_9}$. □

Statement 4. *Let* $X = I + \langle 9a_0^2b_1 - 4b_3,\ b_4\rangle$. *Then the next formula takes place:* $\sqrt{X} = I_1 \bigcap I_2 \bigcap I_5 \bigcap (\bigcap\limits_{k=23}^{27} I_k) \bigcap (\bigcap\limits_{j=1}^{3} \widehat{I}_j)$, *where* $\widehat{I}_1 = \langle a_0, a_2, c_1, c_3, b_0, b_2, b_3, b_4\rangle$, $\widehat{I}_2 = \langle 9a_0 + 2c_1,\ a_0(189a_0^2 - 34c_2) - 12c_3,\ 2a_2 + a_0(21a_0^2 + 4a_1 - 2c_2),\ a_0^2(27a_0^2 - 4c_2) + 2c_4,\ 3a_3 - a_0^2(36a_0^2 + 3a_1 - 4c_2),\ b_4\ 9a_0^2b_1 - 4b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_3 = \langle 9a_0 + 2c_1,\ 15a_0^2 - 2c_2,\ 25a_0^2 + 8a_1,\ 11a_0^3 + 2c_3,\ 13a_0^3 - 4a_2,\ 3a_0^4 - 2c_4,\ 9a_0^4 + 8a_3,\ b_4,\ 9a_0^2b_1 - 4b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$.

Proof. By means of division and intersection operations we find the radical of ideal $X_9 = \langle g_1, \ldots, g_9\rangle + \langle 9a_0^2b_1 - 4b_3,\ b_4\rangle$. Then we have $\sqrt{X_9} = I_1 \bigcap I_2 \bigcap I_5 \bigcap (\bigcap\limits_{k=23}^{27} I_k) \bigcap (\bigcap\limits_{j=1}^{3} \widehat{I}_j)$. The further is analogous to the proof of Statement 3. □

Statement 5. *Let* $X = I + \langle 3a_0^2b_1 - b_3,\ a_0^3b_1 + b_4\rangle$. *Then the next equality takes place:* $\sqrt{X} = I_1 \bigcap I_2 \bigcap I_9 \bigcap (\bigcap\limits_{j=1}^{5} \widehat{I}_j)$, *where* $\widehat{I}_1 = \langle a_0, a_2, c_1, c_3, b_0, b_2, b_3, b_4\rangle$, $\widehat{I}_2 = \langle 5a_0 + c_1,\ a_0(20a_0^2 - 3c_2) - c_3,\ a_2 + a_0(12a_0^2 + 2a_1 - c_2),\ 2a_0^2(8a_0^2 - c_2) + c_4,\ a_0^3b_1 + b_4,\ 3a_0^2b_1 - b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_3 = \langle 6a_0 + c_1,\ 10a_0(9a_0^2 - c_2) - 3c_3,\ a_2 + a_0(15a_0^2 + 2a_1 - c_2),\ a_0^2(9a_0^2 - c_2) + c_4,\ 3a_3 - a_0^2(18a_0^2 + 3a_1 - c_2),\ a_0^3b_1 + b_4,\ 3a_0^2b_1 - b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_4 = \langle 9a_0 + 2c_1,\ a_0(189a_0^2 - 34c_2) - 12c_3,\ 2a_2 + a_0(21a_0^2 + 4a_1 - 2c_2),\ a_0^2(27a_0^2 - 4c_2) + 2c_4,\ 3a_3 - a_0^2(36a_0^2 + 3a_1 - 4c_2),\ a_0^3b_1 + b_4,\ 3a_0^2b_1 - b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$, $\widehat{I}_5 = \langle 783a_0^3 + 432a_0^2c_1 + 74a_0c_1^2 + 4c_1^3 + c_2(13a_0 + 2c_1),\ a_0[(9a_0 + 2c_1)^2 - 2c_1^2 + 7c_2] + 2c_3,\ a_2 - a_0[3a_0(a_0 + c_1) - 2a_1 + c_2],\ 2a_0^2(135a_0^2 + 54a_0c_1 + 5c_1^2 + c_2) + c_4,\ a_3 - a_0^2[3a_0(73a_0 + 29c_1) + a_1 + 2c_2],\ a_0^3b_1 + b_4,\ 3a_0^2b_1 - b_3,\ 3a_0b_1 + b_2,\ b_0\rangle$.

The proof follows from Statement 1 and Theorem 3.

By direct examination we become sure that the next statement is true.

Statement 6. *The next equalities are right:* $\sqrt{I+E_8} = I_1 \bigcap (\bigcap_{k=4}^{8} I_k) \bigcap \langle a_0,\, a_1,\, a_2,\, a_3,\, b_0,\, b_2,\, b_3,\, b_4\rangle$, $\quad \sqrt{I+E_9} = I_1 \bigcap I_2 \bigcap (\bigcap_{k=4}^{9} I_k)$.

Proof of Theorem 4. Computing the radicals of ideals $E_k \bigcap I$, $k = \overline{5,7}$, we get $\bigcup_{k=5}^{7} \mathbb{V}(E_k \bigcap I) \subset V$. If $h \in \bigcup_{k=14}^{35} \mathbb{V}(I_k)$ then $O(0,0)$ is a center since in that case $P_0(h)S(h)/R_1^2(h) \equiv const$. Further taking into account Statements 1-6 we become sure in the correctness of Theorem 4. □

4. Now we will examine system (1).

Statement 7. *The next formula is true:* $\sqrt{J+\langle B\rangle} = \bigcap_{k=1}^{9} \widetilde{J}_k$, *where the radical ideals* $\widetilde{J}_k$ *have the form:*

$\widetilde{J}_1 = \langle B,\, L,\, N\rangle$, $\widetilde{J}_2 = \langle A+C,\, B,\, (A-D)K+AM,\, K(2K+M)-N^2,\, 3K+M+P,\, L+N\rangle$, $\widetilde{J}_3 = \langle 17A+27C,\, 2A+3D,\, B,\, 100A^4-177147N^2,\, 20A^2+81K,\, 50A^2-243M,\, 10A^2-81P,\, L+N\rangle$, $\widetilde{J}_4 = \langle 2A+D,\, B,\, C(A+3C)^2(2A+3C)+4N^2,\, (A+3C)^2+4K,\, 2C^2+(A+2C)^2-M,\, 3(A-3C)(A+C)-4P,\, L+N\rangle$, $\widetilde{J}_5 = \langle A+3C-D,\, B,\, A(2A+3C)^2(3A+4C)-N^2,\, A(2A+3C)-K,\, (A-2C)(2A+3C)+M,\, 6(A+C)(2A+3C)+P,\, L+N\rangle$, $\widetilde{J}_6 = \langle 2A+C+D,\, B,\, 2K(2A+C)+AM,\, K[A(A+C)-2(2K+M)]+2N^2,\, A(A+C)-3K-M-P,\, L+N\rangle$, $\widetilde{J}_7 = \langle 5A+3C+3D,\, B,\, A(2A+3C)^2(5A+12C)+81N^2,\, A(2A+3C)+3K,\, (2A+3C)(7A+6C)-9M,\, 2(A-3C)(2A+3C)-9P,\, L+N\rangle$, $\widetilde{J}_8 = \langle 73A^2+180AC+117C^2,\, 3(11A+15C)+7D,\, B,\, 100A^3(341A+360C)+415233N^2,\, A(29A+9C)+91K,\, A(157A+237C)-273M,\, 2A(29A+9C)+91P,\, L+N\rangle$, $\widetilde{J}_9 = \langle B,\, (A+C)(A+3C)(2A+C+D)-(A-C+D)K,\, (A+C)^2[(A+2C)(2A+C+D)+K]+K^2+N^2,\, 2(A+C)(2A+C+D)+3K+M,\, (A+C)(A+C+D)+P,\, L+N\rangle$.

Proof. Let's generate the ideal $\widetilde{J} = (J+\langle 2u-v+w-A,\, u-v+w+C,\, 3u-2v+w+D\rangle)\bigcap \mathbb{C}[B,\, K,\, L,\, M,\, N,\, P,\, u,\, v,\, w]$. Using Groebner bases one obtains that the radical of ideal $\widetilde{J}_0 = \widetilde{J}+\langle Nuvw(P-2u^2+2uv-uw)(u+w)(2u-w)(2u-4v+w)(u-2v+w)(v-w)(2v-w)(2v+w)(3v-2w)(4v-3w)(5v-4w)(6v-7w)(6v-5w)(7v-5w)(8v-5w)(8v-3w)(26v-19w)\rangle$ has the form: $\sqrt{\widetilde{J}_0} = \bigcap_{k=1}^{9} \widetilde{\widetilde{J}}_k$, where $\widetilde{\widetilde{J}}_k = (\widetilde{J}_k + \langle 2u-v+w-A,\, u-v+w+C,\, 3u-2v+w+D\rangle)\bigcap \mathbb{C}[B,\, K,\, L,\, M,\, N,\, P,\, u,\, v,\, w]$, $k = \overline{1,9}$. In that way it is proved that $\sqrt{J+\langle B\rangle} \subset \bigcap_{k=1}^{9} \widetilde{J}_k$. Let's show that if $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $O(0,0)$ is a focus.

The first focal value for system (1) in the case $B=0$ has the form: $\widetilde{f}_1 = L+N$. Focal values $\widetilde{f}_i$, $i = \overline{2,8}$, have accordingly 18, 82, 274, 750, 1790, 3854, 7662 summands. Denote by $\widehat{I}$ the ideal $\widehat{I} = \langle \widetilde{f}_1, \ldots, \widetilde{f}_8, \ldots\rangle$. Notice that $\widetilde{f}_i$, $i =$

$1, 2, \dots$ are the polynomials from the ring $\mathbb{C}[A,\ C,\ D,\ K,\ L,\ M,\ N,\ P]$, so $\widehat{I} \subset \mathbb{C}[A,\ C,\ D,\ K,\ L,\ M,\ N,\ P]$.

From the condition $\widetilde{f}_1 = 0$ we get

$$L = -N. \tag{14}$$

Considering the condition (14) for system (7) to be held, exclude from $\widetilde{f}_i$, $i = \overline{2,8}$, the variable L and get $\widetilde{f}_i = \alpha_i N F_i$, $i = \overline{2,8}$, where $\alpha_i \neq 0$, $F_i \in \mathbb{C}[A,\ C,\ D,\ K,\ M,\ N,\ P]$, $i = \overline{2,8}$. As $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$, then $N \neq 0$. Construct the ideal $\widehat{\widehat{I}} = (\langle L+N,\ F_2, \dots, F_8\rangle + \langle 2u - v + w - A,\ u - v + w + C,\ 3u - 2v + w + D\rangle) \bigcap \mathbb{C}[K,\ L,\ M,\ N,\ P,\ u,\ v,\ w]$. The ideal $\widehat{\widehat{I}}$ has the form: $\widehat{\widehat{I}} = \langle L+N,\ \widetilde{F}_2, \dots, \widetilde{F}_8\rangle$, where $\widetilde{F}_i \in \mathbb{C}[K,\ M,\ N,\ P,\ u,\ v,\ w]$. From the condition $\widetilde{F}_2 = 0$ we get:

$$K = [u(2u - 4v + w) - M - P]/3. \tag{15}$$

Taking into account the conditions (14) and (15) we have: $\widetilde{F}_i = \beta_i \widetilde{g}_i$, $i = \overline{3,8}$, where $\beta_i \neq 0$, $\widetilde{g}_i \in \mathbb{C}[M,\ N,\ P,\ u,\ v,\ w]$. Notice that $\widetilde{g}_i = \widetilde{g}_3 \gamma_i + \widetilde{G}_i$, $i = \overline{4,8}$, where $\gamma_i \in \mathbb{C}[M,\ N,\ P,\ u,\ v,\ w]$, $\widetilde{G}_i \in \mathbb{C}[M,\ P,\ u,\ v,\ w]$, $i = \overline{4,8}$. The polynomials $\widetilde{G}_i$, $i = \overline{4,8}$, can be written in the form: $\widetilde{G}_i = \delta_i M^{i-2} w^2 + M^{i-3}\omega_{i,1} + \dots + \omega_{i,i-2}$, where $\delta_i \neq 0$, $\omega_{i,1}$, $\omega_{i,i-2}$, $i = \overline{4,8}$, are the polynomials in $P,\ u,\ v,\ w$.

Since $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $w \neq 0$. And the next relations will be true: $\widetilde{G}_i = \theta_i \widetilde{G}_4 + T_i$, $i = \overline{5,8}$, where $\theta_i \neq 0$, and the variable M has the 1st degree in T_5. Solving the equation $T_5 = 0$ we have:

$$M = T_{5,1}/(w T_{5,2}), \tag{16}$$

where $T_{5,1}$ and $T_{5,2}$ are coprime polynomials in variables $P,\ u,\ v,\ w$. Taking into consideration the condition (16) we get $\widetilde{G}_4 = Y_5/T_{5,2}^2$, $T_i = Y_i/T_{5,2}$, $i = \overline{6,8}$, where the polynomials Y_i, $i = \overline{5,8}$, can be represented in the form: $Y_i = \chi_i uv(P - 2u^2 + 2uv - uw)\widetilde{Y}_i$, at the same time $\chi_i \neq 0$, $i = \overline{5,8}$, $\widetilde{Y}_5, \dots, \widetilde{Y}_8$ are the polynomials from the ring $\mathbb{C}[P,\ u,\ v,\ w]$, containing accordingly 314, 314, 541 and 853 summands.

Since $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $uv(P - 2u^2 + 2uv - uw) \neq 0$. Assume that $T_{5,2} \neq 0$. Then the critical point $O(0,0)$ can be a center if $\widetilde{Y}_i = 0$, $i = \overline{5,8}$. Further we will denote by $R_x(F_1,\ F_2)$ the resultant [16, p. 209] of the polynomials F_1 and F_2 in a variable x. Let's compute two resultants: $\widetilde{R}_1 = R_P(O_5,\ sO_6 + O_7)$ and $\widetilde{R}_2 = R_P(O_5,\ sO_6 + O_8)$, where $O_i = \widetilde{Y}_i|_{w=1}$, $i = \overline{5,8}$. We have $\widetilde{R}_1 = \sum\limits_{i=1}^{6} s^{i-1} S_i$, where S_i, $i = \overline{1,6}$, are the polynomials in variables $u,\ v$ of the form $S_i = \varepsilon_i u^6 (1+u)(-1+2u)(1+2u-4v)(1+u-2v)^3 S_0^2 Z_i$, $i = \overline{1,5}$, $S_6 = \varepsilon_6 u^6 (1+u)(-1+2u)(1+2u-4v)(1+u-2v)^3 S_0^2 \widetilde{S}_0 Z_6$, at that $\varepsilon_i \neq 0$, $i = \overline{1,6}$, S_0 is a polynomial in variables $u,\ v$, $\widetilde{S}_0 = -2736 - 11424u - 15885u^2 - 9625u^3 + 16224v + 46596uv + 42165u^2 v + 5250u^3 v - 33360v^2 - 60792uv^2 - 18180u^2 v^2 + 28736v^3 + 21060uv^3 - 8160v^4$; $Z_1, \dots, Z_6$ are coprime polynomials in $u,\ v$, including accordingly 1792, 1671, 1554, 1404, 1250 and 925 summands.

The resultant $\widetilde{R}_2$ can be represented in the next form: $\widetilde{R}_2 = \sum\limits_{i=1}^{6} s^{i-1} W_i$, where $W_i = \tau_i u^6 (1+u)(-1+2u)(1+2u-4v)(1+u-2v)^3 S_0^2 \xi_i$, $i = \overline{1,5}$, $W_6 = \varepsilon_6 u^6 (1+$

$u)(-1+2u)(1+2u-4v)(1+u-2v)^3 S_0^2 \widetilde{S}_0 \xi_6$, at that $\tau_i \neq 0, i=\overline{1,6}$, and $\xi_1, \ldots, \xi_6$ are coprime polynomials in u, v, including accordingly 2940, 2634, 2344, 1981, 1623 and 925 summands.

As far as $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $u^6(1+u)(-1+2u)(1+2u-4v)(1+u-2v)^3 \neq 0$. Let $S_0 \neq 0$, then the critical point $O(0,0)$ can be a center if the following conditions are held: $Z_i = 0$, $\xi_i = 0$, $i=\overline{1,6}$. Let's compute the next resultants: $\widetilde{r}_i = R_u(Z_6,\ Z_{6-i})$, $i=\overline{1,5}$, and also $\widetilde{r}_0 = R_u(\xi_6,\ \xi_5)$. Here r_i, $i=\overline{1,5}$, are the polynomials in v having accordingly 1986th, 2115th, 2230th, 2341th and 2427th degrees, the coefficients of which are coprime integer numbers of the orders $10^{2128}-10^{3619}$, $10^{2141}-10^{2793}$, $10^{2299}-10^{3003}$, $10^{2654}-10^{4405}$, $10^{2616}-10^{3401}$ accordingly; $\widetilde{r}_0$ is a polynomial in v of 2247th degree, and its coefficients are coprime integer numbers of the order $10^{2252}-10^{2948}$.

The greatest common divisor of the polynomials $\widetilde{r}_1, \ldots, \widetilde{r}_5$ is $(v-1)^3(2v-1)^{39}(2v+1)^3(3v-2)^{39}(4v-3)^{54}(5v-4)^{17}(6v-7)^6(6v-5)^5(7v-5)^5(8v-5)^5(8v-3)^3(26v-19)^3\widehat{P}^3$, where $\widehat{P}$ is a polynomial in v of 77th degree, and its coefficients are coprime integer numbers of the order $10^{78}-10^{103}$.

As $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ and the greatest common divisor of the polynomials $\widehat{P}$ and $\widetilde{r}_0$ is 1 then in that case the origin is a focus.

Let $S_0 = 0$. Notice that $R_P(T_{5,1}, T_{5,2}) = \alpha_0 u^2 v^2 T_0 \widehat{S}_0$, where $\alpha_0 \neq 0$, T_0 is the polynomial in variables u, v, w, $\widehat{S}_0|_{w=1} = S_0$. Denote by $\widetilde{T}_i$, $i=\overline{5,7}$, the following resultants: $\widetilde{T}_i = R_M(T_4, T_i)$, $i=\overline{5,7}$. Here $\widetilde{T}_i$, $i=\overline{5,7}$, are the polynomials in variables P, u, v, w, containing 314, 846 and 1756 summands accordingly. For any $i=6,7$, the equalities are true: $\widetilde{T}_i = \widetilde{S}_i \widehat{S}_0 + \widetilde{\widetilde{T}}_i$, where $\widetilde{\widetilde{T}}_i$, $i=6,7$, are the polynomials in P, u, v, w, including 824 and 1571 summands accordingly. Further using resultants we exclude the variable P: $H_1 = R_P(\widetilde{O}_5, \widetilde{O}_6)$, $H_2 = R_P(\widetilde{O}_5, \widetilde{O}_7)$, where $\widetilde{O}_5 = \widetilde{T}_5|_{w=1}$, $\widetilde{O}_i = \widetilde{\widetilde{T}}_i|_{w=1}$, $i=6,7$. H_1 and H_2 are the polynomials in u, v having 7652 and 12987 summands accordingly. For H_i, $i=1,2$, the next equalities are true: $H_i = \widehat{H}_i S_0 + \widetilde{H}_i$, where $\widetilde{H}_1$ and $\widetilde{H}_2$ are the polynomials in u, v, having accordingly 1474 and 1978 summands. Further we have $\widetilde{Z}_i = R_v(S_0, \widetilde{H}_i)$, $i=1,2$. Here $\widetilde{Z}_1$, $\widetilde{Z}_2$ are the polynomials in one variable u of 1505th and 1988th degrees accordingly, containing 1448 and 1931 summands. The greatest common divisor of $\widetilde{Z}_1$ and $\widetilde{Z}_2$ has the form:

$$u^{58}(1+u)^5(-1+2u)^{18}(1+4u)^9(46-103u-563u^2+60u^3)^6(1540+5011u-35614u^2+$$

$$+51479u^3-24216u^4+1920u^5)^2(-2885120+48860768u-338183580u^2+1252033136u^3-$$

$$-2176161807u^4+3494962821u^5-2544493968u^6+920349000u^7-118729800u^8+$$

$$+4860000u^9)^3(155605184-2227701700u+2040477985u^2+22348142299u^3-$$

$$-64132349961u^4+70372499301u^5-40878190008u^6+14935630500u^7-1932076800u^8+$$

$$+77760000u^9).$$

Since $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $u(1+u)(-1+2u) \neq 0$. Consider the case $1+4u=0$. The next equality is right: $\widetilde{\widetilde{H}}_1 = \widehat{S}_0 \widetilde{X}_0 + \widetilde{V}_1$, where $\widetilde{\widetilde{H}}_1|_{w=1} = \widetilde{H}_1$; $\widetilde{X}_0$, $\widetilde{V}_1 \in \mathbb{C}[u,\ v,\ w]$, the polynomial $\widetilde{V}_1$ has 1474 summands. Let's generate the ideal $\widetilde{U}_1 = \langle 4u + w,\ \widehat{S}_0,\ \widetilde{V}_1\rangle$. The Groebner basis of this ideal is $\widetilde{U}_1 = \langle (8v - 3w)w^{123},\ \widetilde{h}_2, \dots,\ \widetilde{h}_{15}\rangle$, where $\widetilde{h}_i \in \mathbb{C}[u,\ v,\ w]$, $i = \overline{2, 15}$. But according to the condition $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$, so $(8v - 3w)w \neq 0$, i.e. in that case $O(0, 0)$ is a focus.

Denote by $e_0 = 60u^3 - 563u^2w - 103uw^2 + 46w^3$. Then we have $\widehat{S}_0 = e_0\widetilde{X}_1 + \widehat{T}_0$, where $\widetilde{X}_1,\ \widehat{T}_0 \in \mathbb{C}[u,\ v,\ w]$, $\widetilde{\widetilde{H}}_1$ can be written in the form: $\widetilde{\widetilde{H}}_1 = \widehat{T}_0\widetilde{X}_2 + e_0\widetilde{X}_3 + \widetilde{V}_2$, where $\widetilde{X}_2,\ \widetilde{X}_3,\ \widetilde{V}_2 \in \mathbb{C}[u,\ v,\ w]$. Let's generate the ideal $\widetilde{U}_2 = \langle e_0,\ \widehat{T}_0,\ \widetilde{V}_2\rangle$ and compute its Groebner basis. We get $\widetilde{U}_2 = \langle w^{26}u_0,\ \widetilde{h}_2, \dots,\ \widetilde{h}_{52}\rangle$, where $u_0 = 240v^3 - 1486v^2w + 1203vw^2 - 237w^3$, $\widetilde{h}_i \in \mathbb{C}[u,\ v,\ w]$, $i = \overline{2, 52}$. Since $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $w \neq 0$. Computing for the ideal $\widetilde{U}_2 + \langle u_0\rangle$ its Groebner basis we get $\widetilde{U}_2 + \langle u_0\rangle = \langle w^{15}(u - 2v + w),\ \widetilde{h}_2, \dots,\ \widetilde{h}_8\rangle$, where $\widetilde{h}_i \in \mathbb{C}[u,\ v,\ w]$, $i = \overline{2, 8}$. But $w(u - 2v + w) \neq 0$ as far as $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ so $O(0, 0)$ is a focus.

Let now $e_0 = 1920u^5 - 24216u^4w + 51479u^3w^2 - 35u^2w^3 + 5011uw^4 + 1540w^5$. In that case $\widehat{S}_0 = e_0\widetilde{X}_4 + \widehat{T}_1$, where $\widetilde{X}_4,\ \widehat{T}_1 \in \mathbb{C}[u,\ v,\ w]$, then $\widetilde{\widetilde{H}}_1$ can be represented as $\widetilde{\widetilde{H}}_1 = \widehat{T}_1\widetilde{X}_5 + e_0\widetilde{X}_6 + \widetilde{V}_3$, where $\widetilde{X}_5,\ \widetilde{X}_6,\ \widetilde{V}_3 \in \mathbb{C}[u,\ v,\ w]$. The Groebner basis for the ideal $\widetilde{U}_3 = \langle e_0,\ \widehat{T}_1,\ \widetilde{V}_3\rangle$ is $\widetilde{U}_3 = \langle w^{28}u_0,\ \widetilde{h}_2, \dots,\ \widetilde{h}_{94}\rangle$, where $u_0 = 491520v^5 - 3714048v^4w + 6701504v^3w^2 - 4849808v^2w^3 + 1471076vw^4 - 143019w^5$, $\widetilde{h}_i \in \mathbb{C}[u,\ v,\ w]$, $i = \overline{2, 94}$. Further we have $\widetilde{U}_3 + \langle u_0\rangle = \langle w^{18}(2u - 4v + w), \widetilde{h}_2, \dots,\ \widetilde{h}_{22}\rangle$, where $\widetilde{h}_i \in \mathbb{C}[u,\ v,\ w]$, $i = \overline{2, 22}$. As $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $w(2u - 4v + w) \neq 0$. The critical point $O(0, 0)$ is a focus.

Let

$$e_0 = 4860000u^9 - 118729800u^8w + 920349000u^7w^2 - 2544493968u^6w^3 + \\ +3494962821u^5w^4 - 2716161807u^4w^5 + 1252033136u^3w^6 - 338183580u^2w^7 + \\ +48860768uw^8 - 2885120w^9.$$

Then $\widehat{S}_0$ and $\widetilde{\widetilde{H}}_1$ can be represented as $\widehat{S}_0 = e_0\widetilde{X}_7 + \widehat{T}_2$; $\widetilde{\widetilde{H}}_1 = \widehat{T}_2\widetilde{X}_8 + e_0\widetilde{X}_9 + \widetilde{V}_4$, where $\widehat{T}_2,\ \widetilde{X}_7,\ \widetilde{X}_8,\ \widetilde{X}_9,\ \widetilde{V}_4 \in \mathbb{C}[u,\ v,\ w]$. The greatest common divisor of resultants $R_u(e_0,\ \widehat{T}_2)$ and $R_u(e_0,\ \widetilde{V}_4)$ equals $\widetilde{\gamma}u_0$, where $\widetilde{\gamma} \neq 0$, $u_0 = 1244160000v^9 - 20796134400v^8w + 130889433600v^7w^2 - 40702563776v^6w^3 + 725607750864v^5w^4 - 795952371456v^4w^5 + 548902046936v^3w^6 - 232670029920v^2w^7 + 55523773877vw^8 - 5720760000w^9$. Using Groebner basis the ideal $\widetilde{U}_4 = \langle e_0,\ u_0,\ \widehat{T}_2,\ \widetilde{V}_4\rangle$ can be represented in the next form: $\widetilde{U}_4 = \langle w^{24}(u - 2v + w), \widetilde{h}_2, \dots,\ \widetilde{h}_{72}\rangle$, where $\widetilde{h}_i \in \mathbb{C}[u,\ v,\ w]$, $i = \overline{2, 72}$. Since $\widetilde{q} \notin \mathbb{V}(\widetilde{J}_0)$ then $w(u - 2v + w) \neq 0$, so in that case $O(0, 0)$ is a focus.

Consider now the last possible case of the center for $S_0 = 0$. Let's denote $e_0 = 77760000u^9 - 1932076800u^8w + 14935630500u^7w^2 - 40878190008u^6w^3 + 70372499301u^5w^4 - 64132349961u^4w^5 + 22348142299u^3w^6 + 2040477985u^2w^7 - 2227701700uw^8 + 155605184w^9$. Then $\widehat{S}_0$ and $\widetilde{\widetilde{H}}_1$ are represented in the form: $\widehat{S}_0 = e_0\widetilde{X}_{10} + \widehat{T}_3$; $\widetilde{\widetilde{H}}_1 = \widehat{T}_3\widetilde{X}_{11} + e_0\widetilde{X}_{12} + \widetilde{V}_5$, where $\widehat{T}_3,\ \widetilde{V}_5,\ \widetilde{X}_{10},\ \widetilde{X}_{11},\ \widetilde{X}_{12}, \in \mathbb{C}[u,\ v,\ w]$. Finding the greatest common divisor of the resultants $R_u(e_0,\ \widehat{T}_3)$ and $R_u(e_0,\ \widetilde{V}_5)$ we have $\widetilde{\mu}u_0$, where $\widetilde{\mu} \neq 0$,

$$u_0 = 79626240000v^9 - 1168382361600v^8w + 5981127091200v^7w^2 -$$

$$-137592210378246v^6w^3+18275585731200v^5w^4-14958694725792v^4w^5+$$
$$+7438311225796v^3w^6-2117102790235v^2w^7+303888967255vw^8-15968920230w^9.$$

The Groebner basis of ideal $\widetilde{U}_4=\langle e_0,\, u_0,\, \widehat{T}_3,\, \widetilde{V}_5\rangle$ has the form: $\widetilde{U}_4=\langle w^{24}(2u-4v+w), \widetilde{h}_2, \dots, \widetilde{h}_{72}\rangle$, where $\widetilde{h}_i\in\mathbb{C}[u,\, v,\, w]$, $i=\overline{2,72}$. As far as $\widetilde{q}\notin\mathbb{V}(\widetilde{J}_0)$ then $w(2u-4v+w)\neq 0$; hence, the critical point $O(0,0)$ is a focus. □

Further through $\widetilde{I}_k$, $k=\overline{1,35}$, we will denote the ideals obtained from I_k, $k=\overline{1,35}$, if the coefficients $a_i\,(i=\overline{0,3})$, $b_j\,(j=\overline{0,4})$, $c_m\,(m=\overline{1,4})$ are expressed by the formulas (). Notice that $\widetilde{I}_k\subset\mathbb{C}[q]$, $k=\overline{1,35}$. Using Groebner basis we become sure in the truth of the next statement.

Statement 8. *The next equalities take place:*

$$\sqrt{\widetilde{I}_2}=J_4\bigcap J_{12}\bigcap\langle A,\, C,\, N,\, K,\, L\rangle\bigcap\langle A+C,\, N,\, K,\, M-P,\, L\rangle,$$
$$\sqrt{\widetilde{I}_9}=J_{10}\bigcap J_{11}\bigcap J_{19},$$
$$\sqrt{\widetilde{I}_3}=J_2\bigcap J_3\bigcap\langle A,\, C,\, D,\, B,\, K(2K+M)-N^2,\, 3K+M+P,\, L+N\rangle,$$

and at the same time the next inclusions are true:

$$\mathbb{V}(\langle A,\, C,\, N,\, K,\, L\rangle)\subset\mathbb{V}(J_{10}),\ \mathbb{V}(\langle A+C,\, N,\, K,\, M-P,\, L\rangle)\subset\mathbb{V}(J_{10}),$$
$$\mathbb{V}(\langle A,\, C,\, D,\, B,\, K(2K+M)-N^2,\, 3K+M+P,\, L+N\rangle)\subset\mathbb{V}(J_{12}).$$

Statement 9. *Denote by $\widehat{J}_1$ and $\widehat{J}_2$ the following ideals: $\widehat{J}_1=\langle A,\, C,\, D,\, N,\, K,\, L\rangle$, $\widehat{J}_2=\langle A,\, C,\, D,\, B,\, K^2+N^2,\, 3K+M,\, P,\, L+N\rangle$. Then the radical of ideal $\widetilde{I}_{10}$ can be written in the next form:*

$$\sqrt{\widetilde{I}_{10}}=J_5\bigcap J_7\bigcap\widehat{J}_1\bigcap\langle 2A+3C,\, 2A+3D,\, N,\, 2A^2+9K,\, M-2P,\, AB+3L\rangle\bigcap\langle 4A+5C+D,\, N,\, C(A+C)-K,\, M,\, 2(A+C)(2A+3C)-P,\, B(A+C)+L\rangle\bigcap\langle A+2C,\, 3A+2D,\, B,\, K(A^2+4K)+4N^2,\, 3K+M,\, A^2-2P,\, L+N\rangle\bigcap\langle 2A+3C,\, 2A+3D,\, AB-3N,\, 2A^2+9K,\, 2(A-3B)(A+3B)-9M,\, A^2+18B^2-9P,\, 2AB+3L\rangle\bigcap\langle 4A+5C+D,\, 2B^2+(A+C)^2,\, B(A+2C)+N,\, C(A+C)-K,\, 3(A+C)^2-M,\, 2(A+C)(2A+3C)-P,\, BC-L\rangle,$$

and at the same time the next inclusions are correct:

$$\mathbb{V}(\langle 2A+3C,\, 2A+3D,\, N,\, 2A^2+9K,\, M-2P,\, AB+3L\rangle)\bigcup\mathbb{V}(\langle 4A+5C+D,\, N,\, C(A+C)-K,\, M,\, 2(A+C)(2A+3C)-P,\, B(A+C)+L\rangle)\subset\mathbb{V}(J_{10}),$$
$$\mathbb{V}(\langle A+2C,\, 3A+2D,\, B,\, K(A^2+4K)+4N^2,\, 3K+M,\, A^2-2P,\, L+N\rangle)\subset\mathbb{V}(J_{19}),$$
$$\mathbb{V}(\langle 2A+3C,\, 2A+3D,\, AB-3N,\, 2A^2+9K,\, 2(A-3B)(A+3B)-9M,\, A^2+18B^2-9P,\, 2AB+3L\rangle)\bigcup\mathbb{V}(\langle 4A+5C+D,\, 2B^2+(A+C)^2,\, B(A+2C)+N,\, C(A+C)-K,\, 3(A+C)^2-M,\, 2(A+C)(2A+3C)-P,\, BC-L\rangle)\subset\mathbb{V}(J_{21}),$$

and $\mathbb{V}(\widehat{J}_1)\subset\mathbb{V}(J_{10})$, $\mathbb{V}(\widehat{J}_2)\subset\mathbb{V}(J_{12})$.

Statement 10. *Let $\widehat{J}_1$ and $\widehat{J}_2$ be the ideals from Statement 9, then the radicals of ideals $\widetilde{I}_{11}$, $\widetilde{I}_{12}$ and $\widetilde{I}_{13}$ can be written in the form:*

$$\sqrt{\widetilde{I}_{11}}=J_8\bigcap\widehat{J}_1\bigcap\widehat{J}_2\bigcap\langle A+2C,\, 2A+D,\, N,\, A^2+4K,\, 3A^2-4P,\, AB+2L\rangle\bigcap\langle 3A+4C,\, A+2D,\, N,\, 3A^2+16K,\, 3A^2-16M,\, A^2-16P,\, AB+4L\rangle\bigcap\langle A,\, 6C+D,\, B^2+$$

$3C^2$, $BC+N$, K, $6C^2-M$, $9C^2-P$, $L\rangle$, $\sqrt{\widetilde{I}_{12}} = J_9 \bigcap J_{18} \bigcap \widehat{J}_1 \bigcap \widehat{J}_2 \bigcap \langle A+2C$, $5A+4D$, N, A^2+4K, $3A^2-8P$, $AB+2L\rangle \bigcap \langle 3A+5C$, $4A+5D$, N, $6A^2+25K$, $6A^2-25M$, $4A^2-25P$, $2AB+5L\rangle$,

$\sqrt{\widetilde{I}_{13}} = J_6 \bigcap J_{17} \bigcap J_{22} \bigcap \widehat{J}_1 \bigcap \widehat{J}_2 \bigcap \langle A$, C, N, K, $2D^2+P$, $L\rangle \bigcap \langle A+2C$, $3A+2D$, N, A^2+4K, A^2-2P, $AB+2L\rangle \bigcap \langle A$, $6C+D$, N, K, $3C^2+M$, $9C^2-P$, $BC+L\rangle \bigcap \langle A+C$, $A-2D$, N, K, M, P, $L\rangle \bigcap \langle A+C$, $A-D$, N, K, M, P, $L\rangle \bigcap \langle A+C$, $2A-D$, N, K, M, P, $L\rangle \bigcap \langle A+3C$, $2A+D$, N, K, A^2-3M, A^2-P, $2AB+3L\rangle$,

and at the same time the following inclusions are held:

$\mathbb{V}(\langle A+2C$, $2A+D$, N, A^2+4K, $3A^2-4P$, $AB+2L\rangle) \bigcup \mathbb{V}(\langle 3A+4C$, $A+2D$, N, $3A^2+16K$, $3A^2-16M$, A^2-16P, $AB+4L\rangle) \bigcup \mathbb{V}(\langle A+2C$, $5A+4D$, N, A^2+4K, $3A^2-8P$, $AB+2L\rangle) \bigcup \mathbb{V}(\langle 3A+5C$, $4A+5D$, N, $6A^2+25K$, $6A^2-25M$, $4A^2-25P$, $2AB+5L\rangle) \bigcup \mathbb{V}(\langle A$, C, N, K, $2D^2+P$, $L\rangle) \bigcup \mathbb{V}(\langle A+2C$, $3A+2D$, N, A^2+4K, A^2-2P, $AB+2L\rangle) \bigcup \mathbb{V}(\langle A+C$, $A-2D$, N, K, M, P, $L\rangle) \bigcup \mathbb{V}(\langle A+C$, $A-D$, N, K, M, P, $L\rangle) \bigcup \mathbb{V}(\langle A+C$, $2A-D$, N, K, M, P, $L\rangle) \subset \mathbb{V}(J_{10})$, $\mathbb{V}(\langle A$, $6C+D$, N, K, $3C^2+M$, $9C^2-P$, $BC+L\rangle) \subset \mathbb{V}(J_8)$, $\mathbb{V}(\langle A+3C$, $2A+D$, N, K, A^2-3M, A^2-P, $2AB+3L\rangle) \subset \mathbb{V}(J_9)$, $\mathbb{V}(\langle A$, $6C+D$, B^2+3C^2, $BC+N$, K, $6C^2-M$, $9C^2-P$, $L\rangle) \subset \mathbb{V}(J_{13})$.

To formulate the next statements we denote by $\widehat{J}_1, ..., \widehat{J}_6$ the ideals of the form:

$\widehat{J}_1 = \langle A$, C, N, K, $3B^2+M$, $L\rangle$, $\widehat{J}_2 = \langle A+C$, N, K, $3B^2+A(A+D)+M$, $3B^2+A(A+D)+P$, $L\rangle$,

$\widehat{J}_3 = \langle A$, B, N, K, $C(3C+D)-M$, $C(3C+D)+P$, $L\rangle$, $\widehat{J}_4 = \langle A$, $3C+D$, B, N, K, $3M+2P$, $L\rangle$,

$\widehat{J}_5 = \langle A+2C$, N, A^2+4K, $A(A+2D)+4(3B^2+M)$, $A(A+2D)+4P$, $AB+2L\rangle$,

$\widehat{J}_6 = \langle A+3C$, B, N, K, $A(A+D)+3M$, $A(A+D)+P$, $L\rangle$.

For the ideals $\widehat{J}_1, ..., \widehat{J}_6$ the inclusions are held: $\mathbb{V}(\widehat{J}_1) \bigcup \mathbb{V}(\widehat{J}_2) \bigcup \mathbb{V}(\widehat{J}_5) \subset \mathbb{V}(J_{10})$, $\mathbb{V}(\widehat{J}_3) \bigcup \mathbb{V}(\widehat{J}_4) \bigcup \mathbb{V}(\widehat{J}_6) \subset \mathbb{V}(J_1)$.

Statement 11. *For the radicals of ideals $\widetilde{I}_1$, $\widetilde{I}_4$, $\widetilde{I}_5$, $\widetilde{I}_7$ the next equalities are true:*

$\sqrt{\widetilde{I}_1} = J_1 \bigcap J_{20} \bigcap J_{21}$, $\sqrt{\widetilde{I}_4} = J_{13} \bigcap J_{24} \bigcap \widehat{J}_1 \bigcap \widehat{J}_2 \bigcap \langle 5A+6C$, B, N, A^2+4K, $5A(A+2D)+12M$, $A(A+2D)+4P$, $L\rangle$,

$\sqrt{\widetilde{I}_5} = J_{14} \bigcap J_{23} \bigcap \widehat{J}_1 \bigcap \widehat{J}_2 \bigcap \langle A$, B, N, K, $C(3C+2D)-2M$, $3C(3C+2D)+4P$, $L\rangle$,

$\sqrt{\widetilde{I}_7} = J_{15} \bigcap (\bigcap\limits_{k=3}^{6} \widehat{J}_k) \bigcap \langle A$, $3C+D$, $BC+N$, K, $2B^2+M$, $3B^2+2P$, $L\rangle \bigcap \langle A$, $B^2-C(3C+D), DC+N, K$, $2C(3C+D)+M$, $3C(3C+D)+P$, $L\rangle \bigcap \langle 3(A+C)+D$, B, N, K, $A(2A+3C)-3M$, $A(2A+3C)-P$, $L\rangle \bigcap \langle 5A+6C+2D$, B, N, A^2+4K, $A(2A+3C)-2M$, $A(2A+3C)-2P$, $L\rangle$,

and at the same time the inclusions take place:

$\mathbb{V}(\langle 5A+6C,\ B,\ N,\ A^2+4K,\ 5A(A+2D)+12M,\ A(A+2D)+4P,\ L\rangle)\bigcup \mathbb{V}(\langle A,\ B,\ N,\ K,\ C(3C+2D)-2M,\ 3C(3C+2D)+4P,\ L\rangle)\bigcup\mathbb{V}(\langle 3(A+C)+D,\ B,\ N,\ K,\ A(2A+3C)-3M,\ A(2A+3C)-P,\ L\rangle)\bigcup\mathbb{V}(\langle 5A+6C+2D,\ B,\ N,\ A^2+4K,\ A(2A+3C)-2M,\ A(2A+3C)-2P,\ L\rangle)\subset\mathbb{V}(J_1)$, and also $\mathbb{V}(\langle A,\ 3C+D,\ BC+N,\ K,\ 2B^2+M,\ 3B^2+2P,\ L\rangle)\bigcup\mathbb{V}(\langle A,\ B^2-C(3C+D),\ DC+N,\ K,\ 2C(3C+D)+M,\ 3C(3C+D)+P,\ L\rangle)\subset\mathbb{V}(J_{21})$.

Statement 12. *The radicals of ideals $\widetilde{I}_6$ and $\widetilde{I}_8$ can be represented in the form:*

$\sqrt{\widetilde{I}_6}=J_{13}\bigcap J_{14}\bigcap J_{16}\bigcap \widehat{J}_1\bigcap \widehat{J}_3\bigcap \widehat{J}_4\bigcap\langle 2A+3C,\ B,\ N,\ A^2+4K,\ A(A+2D)+3M,\ A(A+2D)+4P,\ L\rangle\bigcap\langle 3(A+C)+D,\ B,\ N,\ A^2+4K,\ A(5A+6C)-3M,\ A(5A+6C)-4P,\ L\rangle$,

$\sqrt{\widetilde{I}_8}=J_{13}\bigcap J_{15}\bigcap J_{25}\bigcap \widehat{J}_1\bigcap \widehat{J}_3\bigcap \widehat{J}_5\bigcap \widehat{J}_6\bigcap\langle 3B^2-C(A+D),\ N,\ C(A+C)-K,\ C(A+C)-M,\ (A+C)(A+C+D)+P,\ B(A+C)+L\rangle\bigcap\langle A,\ B,\ N,\ K,\ C(3C+2D)-2M,\ 3C(3C+2D)+4P,\ L\rangle\bigcap\langle A,\ 2B^2-C(3C+2D),\ BC+N,\ K,\ C(3C+2D)+M,\ 3C(3C+2D)+4P,\ L\rangle\bigcap\langle 5A+3C+2D,\ B,\ N,\ K,\ A(A+C)-2M,\ 3A(A+C)-2P,\ L\rangle\bigcap\langle 7A+6C+4D,\ B,\ N,\ A^2+4K,\ A(5A+6C)-8M,\ A(5A+6C)-8P,\ L\rangle$,

and the inclusions are true:

$\mathbb{V}(\langle A,\ 2B^2-C(3C+2D),\ BC+N,\ K,\ C(3C+2D)+M,\ 3C(3C+2D)+4P,\ L\rangle)\subset\mathbb{V}(J_{21})$, $\mathbb{V}(\langle 2A+3C,\ B,\ N,\ A^2+4K,\ A(A+2D)+3M,\ A(A+2D)+4P,\ L\rangle)\bigcup\mathbb{V}(\langle 3(A+C)+D,\ B,\ N,\ A^2+4K,\ A(5A+6C)-3M,\ A(5A+6C)-4P,\ L\rangle)\bigcup\mathbb{V}(\langle A,\ B,\ N,\ K,\ C(3C+2D)-2M,\ 3C(3C+2D)+4P,\ L\rangle)\bigcup\mathbb{V}(\langle 5A+3C+2D,\ B,\ N,\ K,\ A(A+C)-2M,\ 3A(A+C)-2P,\ L\rangle)\bigcup\mathbb{V}(\langle 7A+6C+4D,\ B,\ N,\ A^2+4K,\ A(5A+6C)-8M,\ A(5A+6C)-8P,\ L\rangle)\subset\mathbb{V}(J_1)$, $\mathbb{V}(\langle 3B^2-C(A+D),\ N,\ C(A+C)-K,\ C(A+C)-M,\ (A+C)(A+C+D)+P,\ B(A+C)+L\rangle)\subset\mathbb{V}(J_{10})$.

Proof. To find the radicals $\sqrt{\widetilde{I}_6}$ and $\sqrt{\widetilde{I}_8}$ we will consider the ideals $\widetilde{\widetilde{I}}_6=\widetilde{I}_6+\langle 3(3a_0^2-2a_1+2a_0c_1)+\widetilde{u}^2\rangle$ and $\widetilde{\widetilde{I}}_8=\widetilde{I}_8+\langle 2(a_0^2-a_1+a_0c_1)+\widetilde{u}^2\rangle$. Using Groebner bases we find the radicals $\sqrt{\widetilde{\widetilde{I}}_6}$ and $\sqrt{\widetilde{\widetilde{I}}_8}$ and get $\sqrt{\widetilde{I}_6}=\sqrt{\widetilde{\widetilde{I}}_6}\bigcap\mathbb{C}[q]$, $\sqrt{\widetilde{I}_8}=\sqrt{\widetilde{\widetilde{I}}_8}\bigcap\mathbb{C}[q]$. □

Statement 13. *The radicals of ideals $J+G_{12}$, $J+G_{13}$, $J+G_{14}$ have the form:*

$\sqrt{J+G_{12}}=J_{13}\bigcap\langle B(A+C)+N,\ K,\ 2B^2+M+A(2A+C+D),\ 2B^2+P+A(A+D),\ L\rangle\bigcap\langle A+C,\ N,\ K,\ M-P,\ L\rangle\bigcap\langle B,\ N,\ K,\ L\rangle\bigcap\langle A,\ BC+N,\ K,\ 2B^2+M,\ L\rangle\bigcap\langle A,\ C,\ N,\ K,\ L\rangle$,

$\sqrt{J+G_{13}}=J_{12}\bigcap\langle A+C,\ B,\ N,\ 3K+M+P,\ L\rangle\bigcap\langle A+C,\ 2A+3D,\ B,\ 7A^4-81N^2,\ A^2-3K,\ A^2+9M,\ 8A^2+9P,\ L+N\rangle\bigcap\langle A,\ C,\ 3B^2-D^2,\ BD-N,\ D^2-K,\ D^2+M,\ 2D^2+P,\ BD+L\rangle\bigcap\langle A,\ C,\ D,\ N,\ 3K+M+P,\ L\rangle$,

$\sqrt{J+G_{14}}=J_{15}\bigcap\langle B(2A+2C+D)-N,\ (2A+3C+D)(3A+3C+D)-K,\ 2B^2+(2A+3C+D)(4A+5C+D)+M,\ (2A+3C)(2A+3C+D)^2+2B^2(5A+6C+2D)+(2A+3C+D)P,\ B(3A+3C+D)+L\rangle\bigcap\langle 2A+2C+D,\ N,\ C(A+C)-K,\ C(A+C)(2A+3C)+(A+2C)M-CP,\ B(A+C)+L\rangle\bigcap\langle B,\ N,\ (2A+3C+D)(3A+3C+D)-K,\ L\rangle$, *and the next inclusions are true:*

$\mathbb{V}(\langle B(A+C)+N,\ K,\ 2B^2+M+A(2A+C+D),\ 2B^2+P+A(A+D),\ L\rangle)\bigcup\bigcup\mathbb{V}(\langle A,\ BC+N,\ K,\ 2B^2+M,\ L\rangle)\bigcup\mathbb{V}(\langle B(2A+2C+D)-N,\ (2A+3C+D)(3A+3C+D)-K,\ 2B^2+(2A+3C+D)(4A+5C+D)+M,\ (2A+3C)(2A+3C+D)^2+2B^2(5A+6C+2D)+(2A+3C+D)P,\ B(3A+3C+D)+L\rangle)\subset\mathbb{V}(J_{21})$, $\mathbb{V}(\langle A+C,\ N,\ K,\ M-P,\ L\rangle)\bigcup\mathbb{V}(\langle A,\ C,\ N,\ K,\ L\rangle)\bigcup\mathbb{V}(\langle 2A+2C+D,\ N,\ C(A+C)-K,\ C(A+C)(2A+3C)+(A+2C)M-CP,\ B(A+C)+L\rangle)$, $\mathbb{V}(\langle B,\ N,\ K,\ L\rangle)\bigcup\mathbb{V}(\langle A+C,\ B,\ N,\ 3K+M+P,\ L\rangle)\bigcup\mathbb{V}(\langle B,\ N,\ (2A+3C+D)(3A+3C+D)-K,\ L\rangle)\subset\mathbb{V}(J_1)$, $\mathbb{V}(\langle A+C,\ 2A+3D,\ B,\ 7A^4-81N^2,\ A^2-3K,\ A^2+9M,\ 8A^2+9P,\ L+N\rangle)\subset\mathbb{V}(J_{25})$, $\mathbb{V}(\langle A,\ C,\ 3B^2-D^2,\ BD-N,\ D^2-K,\ D^2+M,\ 2D^2+P,\ BD+L\rangle)\subset\mathbb{V}(J_{15})$, $\mathbb{V}(\langle A,\ C,\ D,\ N,\ 3K+M+P,\ L\rangle)\subset\mathbb{V}(J_2)$.

Statement 14. *The radicals of ideals $J+G_{15}$, $J+G_{16}$, $J+G_{17}$ can be written in the next form:*

$\sqrt{J+G_{15}}=J_{19}\bigcap\langle A+2C,\ 3A+2D,\ N,\ 3K+M,\ A^2-2P,\ AB+2L\rangle\bigcap\langle A+2C,\ 5A+4D,\ A^2-48B^2,\ AB-4N,\ 3A^2+16K,\ 5A^2-16M,\ 3A^2-8P,\ 3AB+4L\rangle\bigcap\langle A+2C,\ 2A+D,\ A^2-12B^2,\ AB+2N,\ K,\ A^2-2M,\ 3A^2-4P,\ L\rangle\bigcap\langle B,\ N,\ 2(A+C)(2A+C+D)+3K+M,\ (A+C)(A+C+D)+P,\ L\rangle\bigcap\langle 5A+7C,\ 8A+7D,\ B,\ A^4-343N^2,\ A^2+7K,\ 17A^2-49M,\ 12A^2-49P,\ L+N\rangle\bigcap\langle A+5C,\ 2A+D,\ B,\ 7A^4-625N^2,\ A^2+25K,\ 11A^2-25M,\ 24A^2-25P,\ L+N\rangle$,

$\sqrt{J+G_{16}}=J_{14}\bigcap\langle B(A+C)-2N,\ 3(A+C)(A+3C)-4K,\ 4B^2+(A+3C)(A+2C-D)+2M,\ 16B^2(2A+3C)+(A+3C)^2(A+3C-2D)+4(A+3C)P,\ 3B(A+C)+2L\rangle\bigcap\langle A+C,\ N,\ K,\ M-P,\ L\rangle\bigcap\langle B,\ N,\ 3(A+C)(A+3C)-4K,\ L\rangle\bigcap\langle A,\ C,\ N,\ K,\ L\rangle$,

$\sqrt{J+G_{17}}=J_{10}\bigcap J_{12}\bigcap J_{17}\bigcap J_{18}\bigcap J_{19}\bigcap J_{21}\bigcap J_{22}\bigcap\langle A+3C,\ 2A+D,N,K,A^2-3M,\ A^2-P,\ 2AB+3L\rangle$,

and the following inclusions take place:

$\mathbb{V}(\langle A+2C,\ 3A+2D,\ N,\ 3K+M,\ A^2-2P,\ AB+2L\rangle)\subset\mathbb{V}(J_5)$, $\mathbb{V}(\langle A+2C,\ 5A+4D,\ A^2-48B^2,\ AB-4N,\ 3A^2+16K,\ 5A^2-16M,\ 3A^2-8P,\ 3AB+4L\rangle)\subset\mathbb{V}(J_{14})$, $\mathbb{V}(\langle A+2C,\ 2A+D,\ A^2-12B^2,\ AB+2N,\ K,\ A^2-2M,\ 3A^2-4P,\ L\rangle)\subset\mathbb{V}(J_{13})$, $\mathbb{V}(\langle 5A+7C,\ 8A+7D,\ B,\ A^4-343N^2,\ A^2+7K,\ 17A^2-49M,\ 12A^2-49P,\ L+N\rangle)\subset\mathbb{V}(J_{24})$, $\mathbb{V}(\langle A+5C,\ 2A+D,\ B,\ 7A^4-625N^2,\ A^2+25K,\ 11A^2-25M,\ 24A^2-25P,\ L+N\rangle)\subset\mathbb{V}(J_{23})$, $\mathbb{V}(\langle B(A+C)-2N,\ 3(A+C)(A+3C)-4K,\ 4B^2+(A+3C)(A+2C-D)+2M,\ 16B^2(2A+3C)+(A+3C)^2(A+3C-2D)+4(A+3C)P,\ 3B(A+C)+2L\rangle)\subset\mathbb{V}(J_{21})$, $\mathbb{V}(\langle A+C,\ N,\ K,\ M-P,\ L\rangle)\bigcup\mathbb{V}(\langle A,\ C,\ N,\ K,\ L\rangle)\bigcup\mathbb{V}(\langle A+3C,\ 2A+D,\ N,\ K,\ A^2-3M,\ A^2-P,\ 2AB+3L\rangle)\subset\mathbb{V}(J_{10})$.

The proof of Theorem 5. The proof follows directly from Theorem 3 and Statements 8–14.

5. The polynomial $\widehat{P}$ has 23 real roots. Let's introduce a vector $p(u,v)$. The system of equations $Z_i=0$, $i=\overline{1,6}$, has 45 real solutions $p=p_k$, $k=\overline{1,45}$. Here

$p_1 = (-2.98291\ldots, 0.61354\ldots)$, $p_2 = (0.28767\ldots, 0.61354\ldots)$,
$p_3 = (2.92233\ldots, 0.61354\ldots)$, $p_4 = (-2.24140\ldots, 1.80824\ldots)$,
$p_5 = (0.70470\ldots, 1, 80824\ldots)$, $p_6 = (4.15318\ldots, 1.80824\ldots)$,
$p_7 = (-0.84828\ldots, 0.59506\ldots)$, $p_8 = (0.11443\ldots, 0.64371\ldots)$,
$p_9 = (0.38246\ldots, 0.70488\ldots)$, $p_{10} = (0.13270\ldots, 0.71955\ldots)$,
$p_{11} = (0.50158\ldots, 0.86047\ldots)$, $p_{12} = (1.23899\ldots, 1.37056\ldots)$,
$p_{13} = (1.83858\ldots, 1.79718\ldots)$.

All values $p = p_k$, $k = \overline{1,45}$, were computed to within 300 digits after the decimal point. Notice that this system has no other real solutions. We substitute p_k in the system $O_5 = 0$, $O_6 = 0$ and find $P = P_k$, $k = \overline{1,45}$. Further replacing w by 1 and computed values $p = p_k$, $P = P_k$, $k = \overline{1,45}$, in $T_{5,1}$ and $T_{5,2}$, one gets accordingly $M = M_k$, $k = \overline{1,45}$. From $\widetilde{g}_3 = 0$ after substitution $w = 1$, $p = p_k$, $P = P_k$, $M = M_k$, $k = \overline{1,45}$, we obtain the next equalities: $\widetilde{\alpha}_k N^2 + \widetilde{\beta}_k = 0$ ($\widetilde{\alpha}_k, \widetilde{\beta}_k \in \mathbb{R}$, $k = \overline{1,45}$), which for $k = \overline{1,13}$ have real roots $N_{2k-1} = -N_{2k}$, and for $k = \overline{14,45}$ - complex roots. Thus we obtain 26 real solutions of system $\widetilde{g}_i = 0$, $i = \overline{3,8}$, of the type $(M_j, N_j, P_j, u_j, v_j)$, $j = \overline{1,26}$. Let $n = (A, D, K, L, M, N, P)$. As a result we get 26 real solutions $n = n_j$ of system $\widetilde{f}_i = 0$, $i = \overline{1,8}$. As a case in point we give one of this solutions: $n_1 = (-2.1488396095\ldots C, 3.5339790637\ldots C, -0.6697945306\ldots C^2, -0.6271954771\ldots C^2, 2.4276250887\ldots C^2, 0.627195477147\ldots C^2, 2.86484242\ldots C^2)$, where $C \neq 0$.

As $\widehat{P}$ and $\widetilde{r}_0$ are coprime polynomials in v then $\widetilde{f}_8|_{n=n_j} \neq 0$, $j = \overline{1,26}$. Direct computations give $\widetilde{f}_8|_{n=n_j} = \widetilde{f}_{8,j}$, $j = \overline{1,26}$, where $\widetilde{f}_{8,1} = 3.3665\ldots \cdot 10^{14} C^{16}$. For $n = n_j$, $j = \overline{1,13}$, $\widetilde{f}_i = 0$, $i = \overline{1,7}$, but $\widetilde{f}_8 \neq 0$. So we get

Statement 15. *When $n = n_j$, $j = \overline{1,13}$, the critical point $O(0,0)$ of system (7), where $\lambda = 0$ is a focus of 8th order.*

Statement 16. *For any $\widetilde{\varepsilon}, \widetilde{\delta} > 0$, $j = \overline{1,26}$, there exist $n \in V_{\widetilde{\delta}}(n_j)$ and $\lambda \in V_{\widetilde{\delta}}(0)$, where $V_{\widetilde{\delta}}(n_j)$ is $\widetilde{\delta}$-neighborhood n_j, $V_{\widetilde{\delta}}(0)$ - $\widetilde{\delta}$-neighborhood of zero, at which system (7) has in $\widetilde{\varepsilon}$-neighborhood of the point $O(0,0)$ 8 limit cycles.*

Proof. Let $e = (\lambda, n) = (\lambda, A, D, K, L, M, N, P)$. Denote $e_k = (0, n_k)$, $k = \overline{1,26}$. For system (7) there exists the only polynomial $\widetilde{W} = (1 + \lambda^2/2)x^2 - \lambda xy + y^2 + \sum\limits_{i+j=3}^{18} P_{i,j} x^i y^j$, where $P_{0,2k} = 0$, $k = \overline{2,9}$, for which on account of system (7) $\dot{\widetilde{W}} = \sum\limits_{i=0}^{8} \widehat{f}_i(e)(x^2 + y^2)^{i+1} + m_{19}(x,y) + m_{20}(x,y)$, where $\widehat{f}_0(e) = \lambda$, $\widehat{f}_i(e)|_{\lambda=0} = \widetilde{f}_i$, $i = \overline{1,8}$, m_i, $i = 19, 20$, are homogeneous polynomials of i^{th} degree. Let's generate $\widehat{f}(e) = (\widehat{f}_0(e), \widehat{f}_1(e), \ldots, \widehat{f}_7(e))$. Then $det\ \partial \widehat{f}(e_k)/\partial e = \rho_k$, where $\rho_k \neq 0$, $k = \overline{1,26}$. The further is analogous to the proof of Theorem 3 from [13].

Proof of Theorem 6. The proof directly follows from Statements 15–16.

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