BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(46), 2004, Pages 25–40
ISSN 1024–7696

$GL(2,R)$-orbits of the polynomial sistems of differential equations

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Abstract. In this work we study the orbits of the polynomial systems $\dot{x} = P(x_1, x_2)$, $\dot{x} = Q(x_1, x_2)$ by the action of the group of linear transformations $GL(2, R)$. It is shown that there are not polynomial systems with the dimension of GL-orbits equal to one and there exist GL-orbits of the dimension zero only for linear systems. On the basis of the dimension of GL-orbits the classification of polynomial systems with a singular point $O(0, 0)$ with real and distinct eigenvalues is obtained. It is proved that on GL-orbits of the dimension less than four these systems are Darboux integrable.

Mathematics subject classification: 34C05, 58F14.
Keywords and phrases: Polynomial differential system, $GL(2, R)$-orbit, resonance, integrability.

1 Center-affine transformations

Consider the polynomial system

$$\dot{x}_1 = \sum_{k=0}^{n} P_k(x_1, x_2), \quad \dot{x}_2 = \sum_{k=0}^{n} Q_k(x_1, x_2), \tag{1}$$

where P_k, Q_k are homogeneous polynomial of degree k:

$$P_k = \sum_{i+j=k} a_{ij} x_1^i x_2^j, \;\; Q_k = \sum_{i+j=k} b_{ij} x_1^i x_2^j. \tag{2}$$

Denote by E the space of coefficients

$$a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30}, ..., a_{0n}; b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, b_{30}, ..., b_{0n})$$

of system (1) and by GL(2,R) the group of center-affine transformations of the phase space Ox, $x = (x_1, x_2)$. Applying in (1) the transformation $X = qx$, where $X = (X_1, X_2)$, $q \in GL(2, R)$, i.e.

$$q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \; \alpha, \beta, \gamma, \delta \in R, \; \Delta = det(q) \neq 0, \; q^{-1} = \frac{1}{\Delta} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}, \tag{3}$$

we obtain the system

$$\dot{X}_1 = \sum_{k=0}^{n} P_k^*(X_1, X_2), \quad \dot{X}_2 = \sum_{k=0}^{n} Q_k^*(X_1, X_2), \tag{4}$$

where

$$P_k^* = \alpha \cdot P_k(q^{-1}x) + \beta \cdot Q_k(q^{-1}x) = \sum_{i+j=k} a_{ij}^* X_1^i X_2^j,$$
$$Q_k^* = \gamma \cdot P_k(q^{-1}x) + \delta \cdot Q_k(q^{-1}x) = \sum_{i+j=k} b_{ij}^* X_1^i X_2^j. \quad (5)$$

Remark 1. *It is easy to see from (5) that every transformation $q \in GL(2,R)$ acts separately on the homogeneities of the same order from (1).*

The coefficients a^* of system (4) can be expressed linearly by the coefficients of system (1): $a^* = L_{(q)}(a)$, $detL_{(q)} \neq 0$. The set $L = \{L_{(q)} | q \in GL(2,R)\}$ forms a 4-parameter group with the operation of composition. L is called the representation of the group $GL(2,R)$ of center-affine transformations of the phase space Ox in the space of coefficients E of system (1).

Let $a \in E$. A set $L(a) = \{L_{(q)}(a) | q \in GL(2,R)\}$ is called *the GL-orbit* of the point a or of the differential system (1) corresponding to this point.

2 Monoparametric transformations

Consider the function $g : R \times E \to E$ such that for every $\tau \in R$ the transformation $g^\tau : E \to E$, where $g^\tau(a) = g(\tau, a)$, $a \in E$, is a diffeomorphism. We say that $(E, \{g^\tau\})$ is a differentiable flow if:

1) $g^0 = id$;
2) $g^{\tau+s} = g^\tau g^s \quad \forall \tau, s \in R$;
3) $(g^\tau)^{-1} = g^{-\tau} \quad \forall \tau \in R$;
4) $g : R \times E \to E$ is a differentiable function.

By [1], [6] the 4-parameter transformation q (see(3)) can be represented as a product of four monoparametric transformations:

$$q^{\alpha_1^*} = \begin{pmatrix} \alpha_1^* & 0 \\ 0 & 1 \end{pmatrix}, q^{\alpha_2} = \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix}, q^{\alpha_3} = \begin{pmatrix} 1 & 0 \\ \alpha_3 & 1 \end{pmatrix}, q^{\alpha_4^*} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_4^* \end{pmatrix},$$

where $\alpha_1^*, \alpha_4^* \in R \backslash \{0\}$; $\alpha_2, \alpha_3 \in R$. Denote

$$q^{\alpha_1} = \begin{pmatrix} e^{\alpha_1} & 0 \\ 0 & 1 \end{pmatrix}, q^{\alpha_4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\alpha_4} \end{pmatrix}, \alpha_1, \alpha_4 \in R;$$

$$L_l = L_{(q^{\alpha_l})},\ l = \overline{1,4};\ L_1^* = L_{\left(q^{\alpha_1^*}\right)},\ L_4^* = L_{\left(q^{\alpha_4^*}\right)}.$$

To every group of monoparametric transformations q^{α_l}, $l = \overline{1,4}$; $q^{\alpha_1^*}$, $q^{\alpha_4^*}$ of the phase space Ox corresponds a system of the form (4) with a_{ij}^*, b_{ij}^*, respectively.

It is easy to verify that $(E, \{L_{(q^{\alpha_l})}\}), l = \overline{1,4}$, are differential flows. They define in E the following systems of linear equations

$$\frac{da}{d\alpha_l} = \Big(\frac{dL_l(a)}{d\alpha_l}\Big)\Big|_{\alpha_l = 0},\ l = \overline{1,4}, \quad (6)$$

or in coordinates

$$q^{\alpha_1}:\begin{cases} \frac{da_{ij}}{d\alpha_l}=\left(\frac{da^*_{ij}}{d\alpha_l}\right)\Big|_{\alpha_l=0}\equiv A^l_{ij}(a),\\ \frac{db_{ij}}{d\alpha_l}=\left(\frac{db^*_{ij}}{d\alpha_l}\right)\Big|_{\alpha_l=0}\equiv B^l_{ij}(a), \qquad l=\overline{1,4}.\\ i+j=\overline{0,n}; \end{cases} \tag{7}$$

In the cases $l=1$ and $l=4$ the matrix of coefficients of the system (7) is diagonal. Indeed, in these cases we have

$$\begin{aligned} A^1_{ij}(a)&=(1-i)a_{ij}, \quad B^1_{ij}(a)=-ib_{ij},\\ A^4_{ij}(a)&=-ja_{ij}, \qquad B^4_{ij}(a)=(1-j)b_{ij}. \end{aligned} \tag{8}$$

Note that $(E,\{L_{(q^{\alpha^*_1})}\})$ and $(E,\{L_{(q^{\alpha^*_4})}\})$ are not flows.

Consider the systems

$$q^{\alpha^*_1}:\frac{da}{d\alpha^*_l}=\left(\frac{dL^*_l(a)}{d\alpha^*_l}\right)\Big|_{\alpha^*_l=1}, \quad l=1,4. \tag{9}$$

Remark 2. *The system* $((9),l=1)$ $(((9),l=4))$ *coincides with the system* $((6),l=1)$ $(((6),l=4))$.

The vector fields

$$V_l=\sum_{i+j=0}^{n}A^l_{ij}(a)\frac{\partial}{\partial a_{ij}}+B^l_{ij}(a)\frac{\partial}{\partial b_{ij}}, \quad l=\overline{1,4},$$

generate a Lie algebra. By [5], [7], [6] the dimension of orbit $O(a)$ is equal with the dimension of this algebra, i.e. with the rank of a matrix M composed from the coordinates of vectors V_l, $l=\overline{1,4}$.

3 The orbits of dimension zero

Consider the homogeneous system

$$\dot{x}_1=P_k(x_1,x_2), \quad \dot{x}_2=Q_k(x_1,x_2), \tag{10}$$

where $0\le k\le n$ and P_k,Q_k are given in (2). For (10) we have the vector fields

$$W_l=\sum_{i+j=k}A^l_{ij}(a)\frac{\partial}{\partial a_{ij}}+B^l_{ij}(a)\frac{\partial}{\partial b_{ij}},\ l=\overline{1,4}. \tag{11}$$

Denote by M_k the matrix of dimension $4\times(2k+2)$ composed from the coordinates of vectors (11). For example,

$$M_0=\begin{pmatrix} a_{00} & 0\\ b_{00} & 0\\ 0 & a_{00}\\ 0 & b_{00}\end{pmatrix}, M_1=\begin{pmatrix} 0 & a_{01} & -b_{10} & 0\\ b_{10} & b_{01}-a_{10} & 0 & -b_{10}\\ -a_{01} & 0 & a_{10}-b_{01} & a_{01}\\ 0 & -a_{01} & b_{10} & 0\end{pmatrix}. \tag{12}$$

We have $M = (M_0, M_1, \ldots, M_n)$ and therefore

$$rankM \geq rankM_k,\ k = \overline{0,n}. \tag{13}$$

Hence, the dimension of orbits of system (10) does not exceed the dimension of orbits of the corresponding system (1).

In the work [6], in each of the cases $k = 0, 1, 2, 3$ the systems (10) are classified in dependence of the dimension of orbits $O(a)$. So, it is shown that if $k = 0$, 2 or 3, then $dimO(a) = 0$ if and only if $P_k \equiv 0$, $Q_k \equiv 0$ and in the case $k = 1$ the dimension of $O(a)$ orbit is equal to zero if and only if the following conditions are satisfied

$$a_{10} - b_{01} = a_{01} = b_{10} = 0. \tag{14}$$

Lemma 1. *In the case $k \neq 1$ the dimension of $O(a)$ orbit of the system (10) is equal to zero if and only if $P_k \equiv 0$, $Q_k \equiv 0$.*

Proof. Assume $k \neq 1$. The orbit $O(a)$ of system (10) has the dimension zero if and only if a is at the same time a singular point for systems (7), $l = \overline{1,4}$, i.e. $A^l_{ij}(a) = B^l_{ij}(a) = 0,\ \forall i + j = k,\ l = \overline{1,4}$. From here, $j = k - i$ and (8) we have that

$$(1 - i)a_{i,k-i} = ib_{i,k-i} = 0,\ i = \overline{0,k}; \tag{15}$$

$$(k - i)a_{i,k-i} = (k - i - 1)b_{i,k-i} = 0, i = \overline{0,k}. \tag{16}$$

From (15) and $k \neq 1$ it follows that $a_{i,k-i} = 0,\ \forall i \neq 1$ and $b_{i,k-i} = 0,\ \forall i \neq 0$, but from (16) we also obtain that $a_{1,k-1} = b_{0k} = 0$. Therefore, $P_k \equiv 0$, $Q_k \equiv 0$. □

According to (13), Lemma 1 and (14) we have

Theorem 1. *The polynomial system (1) has the dimension of GL-orbit equal to zero if and only if it is of the form $\dot{x}_1 = bx_1,\ \dot{x}_2 = bx_2,\ b = const$.*

4 The absence of orbits of the dimension one

We consider system (10). In [6], it is shown that in the cases $k = 0, 1, 2, 3$, the orbits of system (10) have the dimensions not equal to one. We bring here our proof of this fact establishing simultaneously that every two-dimensional polynomial system possesses this property. By Theorem 1, we shall assume that $P_k \not\equiv 0$ or $Q_k \not\equiv 0$ and if $k = 1$, then $a_{10} \neq b_{01}$ or $|a_{01}| + |b_{10}| \neq 0$. From these conditions immediately follows that $rankM_0 = 2$ and $rankM_1 \geq 2$ (see(12)).

Next we consider $k \geq 2$ and $P_k \not\equiv 0$. Let, for example, $a_{\nu,k-\nu} \neq 0$, where ν is equal to one of the numbers $0, 1, 2, \ldots, k$. We will show that the matrix M_k has at least one non–zero minor of the second order. Let us assume the contrary, i.e. all the second order minors of M_k are equal to zero. For the beginning, we will examine the following minors constructed from the coordinates of vectors W_1 and W_4 (see(11),(8)):

$$\begin{aligned}\Delta^1_{\nu,i} &= \begin{vmatrix} (1-\nu)a_{\nu,k-\nu} & (1-i)a_{i,k-i} \\ (\nu-k)a_{\nu,k-\nu} & (i-k)a_{i,k-i}\end{vmatrix} = \\ &= (k-1)(\nu-i)a_{\nu,k-\nu}a_{i,k-i},\ i\neq\nu; \\ \Delta^2_{\nu,i} &= \begin{vmatrix} (1-\nu)a_{\nu,k-\nu} & -ib_{i,k-i} \\ (\nu-k)a_{\nu,k-\nu} & (1-k+i)b_{i,k-i}\end{vmatrix} = \\ &= (k-1)(\nu-i-1)a_{\nu,k-\nu}b_{i,k-i},\ i=\overline{0,k}.\end{aligned} \tag{17}$$

From $\Delta^1_{\nu,i}=0$ it follows that $a_{i,k-i}=0,\ \ \forall i\neq\nu$ and from $\Delta^2_{\nu,i}=0$ we have that $b_{i,k-i}=0,\ \ \forall i$ if $\nu=0$, and that $b_{i,k-i}=0,\ \ \forall i\neq\nu-1$ if $\nu\geq 1$. Hence, the system (10) can have one of the forms

$$\dot{x}_1=a_{0,k}x_2^k,\quad \dot{x}_2=0,\quad a_{0,k}\neq 0; \tag{18}$$

$$\dot{x}_1=a_{\nu,k-\nu}x_1^{\nu}x_2^{k-\nu},\quad \dot{x}_2=b_{\nu-1,k-\nu+1}x_1^{\nu-1}x_2^{k-\nu+1},\quad a_{\nu,k-\nu}\neq 0. \tag{19}$$

For (18) we have $W_1=a_{0,k}\frac{\partial}{\partial a_{0,k}}$ and determine W_3. To this end we apply in (18) the transformation of coordinates q^{α_3}: $X_1=x_1,\ X_2=\alpha_3x_1+x_2$:

$$\dot{X}_1=\dot{x}_1=a_{0,k}x_2^k=a_{0,k}(X_2-\alpha_3X_1)^k=a_{0,k}X_2^k-k\alpha_3a_{0,k}X_1X_2^{k-1}+o(\alpha_3),$$

$$\dot{X}_2=\alpha_3\dot{x}_1+\dot{x}_2=\alpha_3a_{0,k}x_2^k=\alpha_3a_{0,k}(X_2-\alpha_3X_1)^k=\alpha_3a_{0,k}X_2^k+o(\alpha_3).$$

Hence, $W_3=-ka_{0,k}\frac{\partial}{\partial a_{1,k-1}}+a_{0,k}\frac{\partial}{\partial b_{0,k}}$ and the minor $\begin{vmatrix} a_{0,k} & 0 \\ 0 & a_{0,k}\end{vmatrix}\neq 0$. The last inequality contradicts the assumption that all the second order minors of the matrix M_k are null.

We consider in (19) $\nu=1$. We have $W_4=(1-k)\Big(a_{1,k-1}\frac{\partial}{\partial a_{1,k-1}}+b_{0,k}\frac{\partial}{\partial b_{0,k}}\Big)$. Let us calculate W_3:

$$\dot{X}_1=\dot{x}_1=a_{1,k-1}x_1x_2^{k-1}=a_{1,k-1}X_1(X_2-\alpha_3X_1)^{k-1}=a_{1,k-1}X_1X_2^{k-1}+$$
$$+(1-k)\alpha_3a_{1,k-1}X_1^2X_2^{k-2}+o(\alpha_3),$$
$$\dot{X}_2=\alpha_3\dot{x}_1+\dot{x}_2=\alpha_3a_{1,k-1}x_1x_2^{k-1}+b_{0,k}x_2^k=\alpha_3a_{1,k-1}X_1(X_2-\alpha_3X_1)^{k-1}+$$
$$+b_{0,k}(X_2-\alpha_3X_1)^k=b_{0,k}X_2^k+\alpha_3(a_{1,k-1}-kb_{0,k})X_1X_2^{k-1}+o(\alpha_3).$$

Hence, $W_3=(1-k)a_{1,k-1}\frac{\partial}{\partial a_{2,k-2}}+(a_{1,k-1}-kb_{0,k})\frac{\partial}{\partial b_{1,k-1}}$ and

$\begin{vmatrix} (1-k)a_{1,k-1} & 0 \\ 0 & (1-k)a_{1,k-1}\end{vmatrix}\neq 0$. We obtain contradiction.

Let us investigate now the case when in (19) $\nu\geq 2$. We have

$$W_1=(1-\nu)a_{\nu,k-\nu}\frac{\partial}{\partial a_{\nu,k-\nu}}+(\nu-k-1)b_{\nu-1,k-\nu+1}\frac{\partial}{\partial b_{\nu-1,k-\nu+1}}. \tag{20}$$

Taking in (19) the transformation q^{α_2}: $X_1 = x_1 + \alpha_2 x_2$, $X_2 = x_2$ we obtain:

$$\dot{X}_1 = \dot{x}_1 + \alpha_2 \dot{x}_2 = a_{\nu,k-\nu} x_1^{\nu} x_2^{k-\nu} + \alpha_2 b_{\nu-1,k-\nu+1} x_1^{\nu-1} x_2^{k-\nu+1} =$$

$$= (X_1 - \alpha_2 X_2)^{\nu-1} X_2^{k-\nu} [a_{\nu,k-\nu} X_1 + \alpha_2 (b_{\nu-1,k-\nu+1} - a_{\nu,k-\nu}) X_2] =$$

$$= a_{\nu,k-\nu} X_1^{\nu} X_2^{k-\nu} + \alpha_2 (b_{\nu-1,k-\nu+1} - \nu a_{\nu,k-\nu}) X_1^{\nu-1} X_2^{k-\nu+1} + o(\alpha_2),$$

$$\dot{X}_2 = \dot{x}_2 = b_{\nu-1,k-\nu+1} x_1^{\nu-1} x_2^{k-\nu+1} = b_{\nu-1,k-\nu+1} (X_1 - \alpha_2 X_2)^{\nu-1} X_2^{k-\nu+1} =$$

$$= b_{\nu-1,k-\nu+1} X_1^{\nu-1} X_2^{k-\nu+1} + \alpha_2 (1-\nu) b_{\nu-1,k-\nu+1} X_1^{\nu-2} X_2^{k-\nu+2} + o(\alpha_2).$$

From here it follows that

$$W_2 = (b_{\nu-1,k-\nu+1} - \nu a_{\nu,k-\nu}) \frac{\partial}{\partial a_{\nu-1,k-\nu+1}} + (1-\nu) b_{\nu-1,k-\nu+1} \frac{\partial}{\partial b_{\nu-2,k-\nu+2}}.$$

Taking into account that $\nu \geq 2$ and that $a_{\nu,k-\nu} \neq 0$, the following two minors consisting of the coordinates of the vectors (20) and W_2:

$$\begin{vmatrix} (1-\nu) a_{\nu,k-\nu} & 0 \\ 0 & (1-\nu) b_{\nu-1,k-\nu+1} \end{vmatrix}, \quad \begin{vmatrix} (1-\nu) a_{\nu,k-\nu} & 0 \\ 0 & b_{\nu-1,k-\nu+1} - \nu a_{\nu,k-\nu} \end{vmatrix}$$

can not be equal to zero simultaneously.

Hence, we proved that when $P_k \not\equiv 0$ the dimension of every orbit of the system (10) can not be equal to one. The case $Q_k \not\equiv 0$ can be reduced to the case $P_k \not\equiv 0$ if we change in (10) the variables x_1 and x_2.

From Theorem 1, the inequality (13) and from what has been said above in this section, the following conclusion may be drawn

Theorem 2. *The dimension of GL-orbit of every polynomial system (1) is not equal to one.*

It is easy to check that the matrix M_1 from (12) can have the rank at most two. This fact, Theorems 1 and 2 lead to

Theorem 3. *The dimension of the GL-orbit of the linear system* $\dot{x}_1 = a_{10}x_1 + a_{01}x_2$, $\dot{x}_2 = b_{10}x_1 + b_{01}x_2$ *is equal to zero if and only if* $a_{10} - b_{01} = a_{01} = b_{10} = 0$ *and is two in other cases.*

Let us consider the system

$$\dot{x}_1 = a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2, \quad \dot{x}_2 = b_{20}x_1^2 + b_{11}x_1x_2 + b_{02}x_2^2.$$

Its matrix consists of the coordinates of vectors X_l, $\overline{1,4}$, and is of the form

$$M_2 = \begin{pmatrix} -a_{20} & 0 & a_{02} & -2b_{20} & -b_{11} & 0 \\ b_{20} & b_{11} - 2a_{20} & b_{02} - a_{11} & 0 & -2b_{20} & -b_{11} \\ -a_{11} & -2a_{02} & 0 & a_{20} - b_{11} & a_{11} - 2b_{02} & a_{02} \\ 0 & -a_{11} & -2a_{02} & b_{20} & 0 & -b_{02} \end{pmatrix}. \tag{21}$$

It is easy to see that for the system $\dot{x}_1 = 0$, $\dot{x}_2 = x_1x_2$ the rank of the matrix M_2 is equal to three, and for the system $\dot{x}_1 = x_2^2$, $\dot{x}_2 = x_1^2 + x_1x_2$ we have that $rankM_2 = 4$.

From here, Theorems 1, 2, 3 and the inequality (13), follows

Lemma 2. *If the right–hand sides of system (1) have at least one nonlinear term, then the dimension of the GL-orbit is equal to two, three or four.*

Next, this work is dedicated to the classification of systems (1) with a singular point $(0,0)$ with real and distinct eigenvalues λ_1 and λ_2, i.e.

$$\lambda_1, \lambda_2 \in R, \quad \lambda_1 \neq \lambda_2, \tag{22}$$

in dependence of the dimension of GL-orbits.

In this case $P_0 \equiv 0$, $Q_0 \equiv 0$ and according to [2] by transformation of coordinates $q \in GL(2,R)$, the system (1) can be brought to the form

$$\dot{x}_1 = \lambda_1 x_1 + \sum_{k=2}^{n} P_k(x_1,x_2), \quad \dot{x}_2 = \lambda_2 x_2 + \sum_{k=2}^{n} Q_k(x_1,x_2). \tag{23}$$

In (23) the notations (2) of the homogeneities P_k, Q_k, $k = \overline{2,n}$, were preserved. From (12) we have that for (23): $rankM_1 = 2$. From here and (13) it follows that the dimension of every GL-orbits of system (23) with conditions (22) can be equal to two, three or four.

5 The GL-orbits of system (23) of the dimension two

We consider the system

$$\dot{x}_1 = \lambda_1 x_1 + P_k(x_1,x_2), \quad \dot{x}_2 = \lambda_2 x_2 + Q_k(x_1,x_2), \tag{24}$$

where λ_1, λ_2 verify (22) and $2 \leq k \leq n$. In (24) the polynomials P_k, Q_k coincide with the polynomials P_k and Q_k, respectively, from (23). Evidently holds

Remark 3. *The dimension of every GL-orbit of system (23) is not smaller than the corresponding dimension of GL-orbit of system (24).*

From (12) and (8) we have that for (24) the matrix $M = (M_1, M_k)$ consisting of coordinates of vectors V_l, $l = \overline{1,4}$, after some elementary transformations takes the form

$$M \sim \begin{pmatrix} 0 & 0 & 0 & 0 & (1-k)a_{k,0} & (2-k)a_{k-1,1} & \dots & -b_{1,k-1} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{k-1,1} & \dots & (2-k)b_{1,k-1} & (1-k)b_{0,k} \end{pmatrix}. \tag{25}$$

Consider the minors of the third order of the matrix (25):

$$\begin{vmatrix} 0 & 0 & (1-i)a_{ij} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 0 & -ib_{ij} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -ja_{ij} \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-j)b_{ij} \end{vmatrix},$$

$i+j=k$, we observe that they are simultaneously equal to zero if and only if $a_{ij}=b_{ij}=0,\ \forall i+j=k$. From here, Remark 3 and Theorem 3, follows

Lemma 3. *The dimension of the GL-orbit of system (23) with conditions (22) is equal to two if and only if* $P_k \equiv 0,\ \ Q_k \equiv 0,\ \ \forall k \geq 2.$

Next, taking into account this lemma and Remark 1, we obtain

Theorem 4. *Let the origin $O(0,0)$ be a singular point of (1) with real and distinct eigenvalues. Then the GL-orbit of system (1) has the dimension equal to two if and only if* $P_k \equiv 0,\ \ Q_k \equiv 0,\ \ \forall k \geq 2.$

6 The GL-orbits of system (23) of the dimension three

In this section we shall distinguish those systems of the form (23), (22) which have the dimension of the GL-orbit equal to three. Reasoning as above, we shall consider system (24). From (25) we have that $rankM = 2 + rank\tilde{M}_k$, where

$$\tilde{M}_k = \begin{pmatrix} (1-k)a_{k,0} & (2-k)a_{k-1,1} & \dots & -b_{1,k-1} & 0 \\ 0 & -a_{k-1,1} & \dots & (2-k)b_{1,k-1} & (1-k)b_{0k} \end{pmatrix}. \qquad (26)$$

The minors of the second order from (17) of the matrix $\tilde{M}_k$ are $\Delta^1_{\nu,i}$, $\Delta^2_{\nu,i}$ and

$$\Delta^3_{\nu,i} = \begin{vmatrix} -\nu b_{\nu,k-\nu} & -ib_{i,k-i} \\ (1+\nu-k)b_{\nu,k-\nu} & (1+i-k)b_{i,k-i} \end{vmatrix} = (k-1)(\nu-i)b_{\nu,k-\nu}b_{i,k-i}, i \neq \nu$$

(see (8)). If $a_{0,k} \neq 0$ ($b_{0,k} \neq 0$), then from $\Delta^1_{0,i}=0$, $i=\overline{1,k}$ ($\Delta^3_{k,i}=0$, $i=\overline{0,k-1}$) it follows that $a_{i,k-i}=0$ ($b_{i,k-i}=0$), and from $\Delta^2_{0,i}=0$ ($\Delta^2_{i,k}=0$), $i=\overline{0,k}$, we have that ($b_{i,k-i}=0$) ($a_{i,k-i}=0$). In these cases the system (24) looks as

$$S_n(k:1): \ \ \dot{x}_1 = \lambda_1 x_1 + a_{0,k}x_2^k, \quad \dot{x}_2 = \lambda_2 x_2, \quad a_{0,k} \neq 0 \quad (k \geq 2); \qquad (27)$$

$$S_n(1:k): \ \ \dot{x}_1 = \lambda_1 x_1, \quad \dot{x}_2 = \lambda_2 x_2 + b_{k,0}x_1^k, \quad b_{k,0} \neq 0 \quad (k \geq 2). \qquad (28)$$

We suppose now that $a_{\nu,k-\nu} \neq 0$ ($b_{\nu-1,k-\nu+1} \neq 0$) for a certain $\nu \in \{1,2,...,k\}$. From $\Delta^1_{\nu,i}=0$ ($\Delta^3_{\nu,i}=0$), $i \neq \nu$, and $\Delta^2_{\nu,i}=0$ ($\Delta^2_{i,\nu}=0$), $i \neq \nu-1$, it results that $a_{i,k-i}=0,\ \forall i \neq \nu$, and $b_{i,k-i}=0,\ \forall i \neq \nu-1$. These cases lead us to the systems

$$\begin{cases} \dot{x}_1 = x_1\Big(\lambda_1 + a_{\nu,k-\nu}x_1^{\nu-1}x_2^{k-\nu}\Big), \\ \dot{x}_2 = x_2\Big(\lambda_2 + b_{\nu-1,k-\nu+1}x_1^{\nu-1}x_2^{k-\nu}\Big), \qquad \nu=\overline{1,k}. \\ |a_{\nu,k-\nu}| + |b_{\nu-1,k-\nu+1}| \neq 0; \end{cases} \qquad (29)$$

Hence, is proved

Lemma 4. *The GL-orbit of system (24) has the dimension equal to three if and only if it has one of the forms (27)–(29).*

In passing, we will examine the system (23). As usual, by M we will denote the matrix consisting of coordinates of the vectors V_l, $j=\overline{1,4}$, corresponding to system (23), and by $\tilde{M}$ the matrix $(\tilde{M}_2, \tilde{M}_3, \ldots, \tilde{M}_n)$, where $\tilde{M}_k$, $k=\overline{2,n}$, are given in (26). Evidently,

$$rankM = 2 + rank\tilde{M} \geq 2 + rank\tilde{M}_k,\ k=\overline{2,n}. \tag{30}$$

If $rankM = 3$, then from (30) it follows that there exist $k : 2 \leq k \leq n$ such that $rank\tilde{M}_k = 1$. Hence

$$|P_k(x_1,x_2)| + |Q_k(x_1,x_2)| \not\equiv 0. \tag{31}$$

In the case if $P_j \equiv 0,\ \ Q_j \equiv 0,\ \ \forall j \neq k,\ \ 2 \leq j \leq n$, apply Lemma 4. Suppose that together with homogeneities of order k, the right–hand sides of system (23) contain also and homogeneities of other order, for example, of order l, where $l \neq k$, $2 \leq l \leq n$. Hence

$$|P_l(x_1,x_2)| + |Q_l(x_1,x_2)| \not\equiv 0. \tag{32}$$

The condition $rank\tilde{M}_k = rank\tilde{M}_l = 1$ implies that both P_k, Q_k and P_l, Q_l have the form like the right-hand sides of one of systems (27)–(29). In the case P_l, Q_l in (27)–(29) we substitute l for k.

Let $P_k = a_{0,k}x_2^k$, $a_{0,k} \neq 0$ and $Q_k \equiv 0$. The following minors of the matrix $\tilde{M}$:

$$\begin{vmatrix} a_{0,k} & (1-\mu)a_{\mu,l-\mu} \\ -ka_{0,k} & (\mu-l)a_{\mu,l-\mu} \end{vmatrix} = [1-l+(1-\mu)(k-1)]a_{0,k}a_{\mu,l-\mu},$$

$$\begin{vmatrix} a_{0,k} & -\mu b_{\mu,l-\mu} \\ -ka_{0,k} & (1+\mu-l)b_{\mu,l-\mu} \end{vmatrix} = [1-l+\mu(k-1)]a_{0,k}b_{\mu,l-\mu},$$

$0 \leq \mu \leq l$, are simultaneously equal to zero if and only if $a_{\mu,l-\mu} = b_{\mu,l-\mu} = 0$, $\mu = \overline{0,l}$, that is when $P_l \equiv 0,\ Q_l \equiv 0$, contradicting to (32).

Similarly, through examination of the minors

$$\begin{vmatrix} -kb_{k,0} & (1-\mu)a_{\mu,l-\mu} \\ b_{k,0} & (\mu-l)a_{\mu,l-\mu} \end{vmatrix},\quad \begin{vmatrix} -kb_{k,0} & -\mu b_{\mu,l-\mu} \\ b_{k,0} & (1+\mu-l)b_{\mu,l-\mu} \end{vmatrix},$$

it is shown that the case $P_k \equiv 0,\ Q_k = b_{k,0}x_1^k,\ b_{k,0} \neq 0$ is not realized in the condition (32).

Taking into account Lemmas 3, 4 and the conditions (31), (32), it remains to investigate the case when

$$P_k = a_{\nu,k-\nu}x_1^{\nu}x_2^{k-\nu},\ Q_k = b_{\nu-1,k-\nu+1}x_1^{\nu-1}x_2^{k-\nu+1},\ P_l = a_{\mu,l-\mu}x_1^{\mu}x_2^{l-\mu},$$

$$Q_l = b_{\mu-1,l-\mu+1}x_1^{\mu-1}x_2^{l-\mu+1},\ 1 \leq \nu \leq k,\ 1 \leq \mu \leq l.$$

We consider the minors:

$$\Omega^1_{\nu,\mu} = \begin{vmatrix} (1-\nu)a_{\nu,k-\nu} & (1-\mu)a_{\mu,l-\mu} \\ (\nu-k)a_{\nu,k-\nu} & (\mu-l)a_{\mu,l-\mu} \end{vmatrix} = \omega_{\nu,\mu}a_{\nu,k-\nu}a_{\mu,l-\mu},$$

$$\Omega^2_{\nu,\mu} = \begin{vmatrix} (1-\nu)a_{\nu,k-\nu} & (1-\mu)b_{\mu-1,l-\mu+1} \\ (\nu-k)a_{\nu,k-\nu} & (\mu-l)b_{\mu-1,l-\mu+1} \end{vmatrix} = \omega_{\nu,\mu}a_{\nu,k-\nu}b_{\mu-1,l-\mu+1},$$

$$\Omega^3_{\nu,\mu} = \begin{vmatrix} (1-\nu)b_{\nu-1,k-\nu+1} & (1-\mu)b_{\mu-1,l-\mu+1} \\ (\nu-k)b_{\nu-1,k-\nu+1} & (\mu-l)b_{\mu-1,l-\mu+1} \end{vmatrix} = \omega_{\nu,\mu}b_{\nu-1,k-\nu+1}b_{\mu-1,l-\mu+1},$$

where $\omega_{\nu,\mu} = (\nu-1)(l-1)-(\mu-1)(k-1)$, $1 \le \nu \le k$, and $1 \le \mu \le l$. Evidently, $\omega_{1,1} = \omega_{k,l} = 0$.

If $\nu = 1$ ($\nu = k$), then from (31) and (32) it follows that the equalities $\Omega^1_{1,\mu} = \Omega^2_{1,\mu} = \Omega^3_{1,\mu} = 0$ hold if and only if $\mu = 1$ ($\mu = l$). Hence, the dimension of the GL-orbit of each of the systems

$$S_n(\lambda_1:0): \begin{cases} \dot{x}_1 = x_1\Big(\lambda_1 + \sum_{j=1}^{n-1} a_{1,j}x_2^j\Big), \\ \dot{x}_2 = x_2\Big(\lambda_2 + \sum_{j=1}^{n-1} b_{0,j+1}x_2^j\Big), \\ \sum_{j=1}^{n-1} |a_{1,j}| + |b_{0,j+1}| \ne 0; \end{cases} \tag{33}$$

$$S_n(0:\lambda_2): \begin{cases} \dot{x}_1 = x_1\Big(\lambda_1 + \sum_{j=1}^{n-1} a_{j+1,0}x_1^j\Big), \\ \dot{x}_2 = x_2\Big(\lambda_2 + \sum_{j=1}^{n-1} b_{j,1}x_1^j\Big), \\ \sum_{j=1}^{n-1} |a_{j+1,0}| + |b_{j,1}| \ne 0, \end{cases} \tag{34}$$

is equal to three.

Next, suppose that $2 \le \nu \le k-1$, $2 \le \mu \le l-1$. From (31), (32) and $\Omega^j_{\nu,\mu} = 0$, $j = \overline{1,3}$, it follows that $\omega_{\nu,\mu} = 0$, Therefore, we have that $\frac{l-1}{\mu-1} = \frac{k-1}{\nu-1} > 1$. Hence, there exist integer positive numbers p, q, i, j such that

$$(p,q) = 1,\ k = (p+q)i+1,\ \nu = qi+1,\ l = (p+q)j+1,\ \mu = qj+1.$$

Hence, for any natural reciprocal prim numbers p and q, the system

$$S_n(p:-q): \begin{cases} \dot{x}_1 = x_1\left[\lambda_1 + \sum\limits_{i=1}^{n^*} a_{qi+1,pi}\Big(x_1^q x_2^p\Big)^i\right], \\ \dot{x}_2 = x_2\left[\lambda_2 + \sum\limits_{i=1}^{n^*} b_{qi,pi+1}\Big(x_1^q x_2^p\Big)^i\right], \\ \sum\limits_{i=1}^{n^*} |a_{qi+1,pi}| + |b_{qi,pi+1}| \ne 0, \quad (p,q) = 1, \end{cases} \tag{35}$$

where $n^* = \left[\frac{n-1}{p+q}\right]$, has the dimension of the GL-orbit equal to three.

Hence, is proved

Theorem 5. *The dimension of the GL-orbit of system (23) with the conditions (22) is equal to three if and only if it has one of the following forms (27), (28), (33), (34) or (35).*

Corollary 1. *The cubic system* ($n = 3$) *of the form (22), (23) has the dimension of the GL-orbit equal to three if and only if it has one of the forms* $S_3(2:1)$, $S_3(3:1)$, $S_3(1:2)$, $S_3(1:3)$, $S_3(\lambda_1:0)$, $S_3(0:\lambda_2)$, $S_3(1:-1)$, *that is*

$$\dot{x}_1 = \lambda_1 x_1 + a_{02}x_2^2, \quad \dot{x}_2 = \lambda_2 x_2, \quad a_{02} \neq 0; \tag{36}$$

$$\dot{x}_1 = \lambda_1 x_1 + a_{03}x_2^3, \quad \dot{x}_2 = \lambda_2 x_2, \quad a_{03} \neq 0; \tag{37}$$

$$\dot{x}_1 = \lambda_1 x_1, \quad \dot{x}_2 = \lambda_2 x_2 + b_{20}x_1^2, \quad b_{20} \neq 0; \tag{38}$$

$$\dot{x}_1 = \lambda_1 x_1, \quad \dot{x}_2 = \lambda_2 x_2 + b_{30}x_1^3, \quad b_{30} \neq 0; \tag{39}$$

$$\begin{cases} \dot{x}_1 = x_1\Big(\lambda_1 + a_{11}x_2 + a_{12}x_2^2\Big), \\ \dot{x}_2 = x_2\Big(\lambda_2 + b_{02}x_2 + b_{03}x_2^2\Big), \ |a_{11}| + |a_{12}| + |b_{02}| + |b_{03}| \neq 0; \end{cases} \tag{40}$$

$$\begin{cases} \dot{x}_1 = x_1\Big(\lambda_1 + a_{20}x_1 + a_{30}x_1^2\Big), \\ \dot{x}_2 = x_2\Big(\lambda_2 + b_{11}x_1 + b_{21}x_1^2\Big), \ |a_{20}| + |a_{30}| + |b_{11}| + |b_{21}| \neq 0; \end{cases} \tag{41}$$

$$\dot{x}_1 = x_1\Big(\lambda_1 + a_{21}x_1x_2\Big), \dot{x}_2 = x_2\Big(\lambda_2 + b_{12}x_1x_2\Big), |a_{21}| + |b_{12}| \neq 0. \tag{42}$$

The assertion of Corollary 1 can be obtained and by direct method, that is if we equate to zero all the minors of the order four of the matrix $M = (M_1, M_2, M_3)$ with condition that at least one of the minors of the order three is not equal to zero. Here, M_1 coincides with the matrix M_1 from (12) if in the last matrix we put $a_{01} = b_{10} = 0$, $a_{10} = \lambda_1$, $b_{01} = \lambda_2$; the matrix M_2 is given in (21) and

$$M_3 = \left(\begin{array}{cccc} -2a_{30} & -a_{21} & 0 & a_{03} \\ b_{30} & b_{21} - 3a_{30} & b_{12} - 2a_{21} & b_{03} - a_{12} \\ -a_{21} & -2a_{12} & -3a_{03} & 0 \\ 0 & -a_{21} & -2a_{12} & -3a_{03} \end{array} \right.$$

$$\left. \begin{array}{cccc} -3b_{30} & -2b_{21} & -b_{12} & 0 \\ 0 & -3b_{30} & -2b_{21} & -b_{12} \\ a_{30} - b_{21} & a_{21} - 2b_{12} & a_{12} - 3b_{03} & a_{03} \\ b_{30} & 0 & -b_{12} & -2b_{03} \end{array} \right).$$

7 The resonance

By $\varphi(x_1,x_2)$ and $\psi(x_1,x_2)$ we shall denote, respectively, the nonlinearities from the right–hand side of each equation of system (23), i.e.

$$\varphi(x_1,x_2)=\sum_{k=2}^{n} P_k(x_1,x_2), \quad \psi(x_1,x_2)=\sum_{k=2}^{n} Q_k(x_1,x_2), \tag{43}$$

where the polynomials P_k and Q_k, $k=\overline{2,n}$, are shown in (2).

Let λ_1 and λ_2 be two real and distinct numbers. If there exist integer nonnegative numbers $m_1,m_2;\ m_1+m_2\geq 2$ $(n_1,n_2;\ n_1+n_2\geq 2)$ such that

$$\lambda_1 = m_1\lambda_1 + m_2\lambda_2 \tag{44}$$

or

$$\lambda_2 = n_1\lambda_1 + n_2\lambda_2, \tag{45}$$

then the couple of numbers (λ_1,λ_2) is called *resonant*.

Taking into account (44) ((45)), we say that $a_{m_1,m_2}x_1^{m_1}x_2^{m_2}$ $(b_{n_1,n_2}x_1^{n_1}x_2^{n_2})$ is *a resonant term* of the polynomial $\varphi(x_1,x_2)$ $(\psi(x_1,x_2))$ corresponding to the resonant couple (λ_1,λ_2).

A *couple* of polynomials (φ,ψ) is call *resonant* if they contain only resonant terms corresponding to the same resonant couple of the numbers (λ_1,λ_2), considering $\psi\equiv 0$ $(\varphi\equiv 0)$ if λ_1 and λ_2 verify (44) ((45)) and do not verify (45) ((44)) for any integer numbers $n_1,n_2\geq 0$, $n_1+n_2\geq 2$ $(m_1,m_2\geq 0, m_1+m_2\geq 2)$.

In passing, in this section, we will describe a couple of resonant polynomials. Suppose that (λ_1,λ_2) is a resonant couple. We will distinguish the following four possible cases: 1) $\lambda_1\cdot\lambda_2>0,\ \lambda_1\neq\lambda_2$; 2) $\lambda_1\neq 0,\ \lambda_2=0$; 3) $\lambda_1=0,\ \lambda_2\neq 0$ and 4) $\lambda_1\cdot\lambda_2<0$.

1) $\lambda_1\cdot\lambda_2>0$, $\lambda_1\neq\lambda_2$. In this case the equalities (44) and (45) do not hold simultaneously. If we consider the equality (44), then it looks as:

$$\lambda_1 = 0\cdot\lambda_1 + k\cdot\lambda_2, \tag{46}$$

where k is one of the numbers $2,3,\ldots$. To the couple (λ_1,λ_2) which verifies (46) the resonant couple of polynomials

$$\varphi(x_1,x_2)=a_{0,k}x_2^k, \quad \psi(x_1,x_2)\equiv 0$$

corresponds.

Similarly, if we have the equality (45), then it looks as: $\lambda_2=k\cdot\lambda_1+0\cdot\lambda_2$ and leads to the resonant couple of polynomials

$$\varphi(x_1,x_2)\equiv 0, \quad \psi(x_1,x_2)=b_{k,0}x_1^k.$$

2) $\lambda_1 \neq 0$, $\lambda_2 = 0$. In these condition the relation (44) holds for $m_1 = 1$ and any $m_2 \in \{1,2,3,\dots\}$ and the relation (45) holds for $n_1 = 0$ and $n_2 \in \{2,3,\dots\}$. To the resonant couple (λ_1, λ_2) the couple of resonant polynomials

$$\varphi(x_1,x_2) = x_1 \sum_{j=1}^{n-1} a_{1,j} x_2^j, \quad \psi(x_1,x_2) = x_2 \sum_{j=1}^{n-1} b_{0,j+1} x_2^j$$

corresponds.

3) $\lambda_1 = 0$, $\lambda_2 \neq 0$. The equality (44) holds for $m_1 \in \{2,3,\dots\}$ and $m_2 = 0$, and (45) for $n_1 \in \{1,2,3,\dots\}$ and $n_2 = 1$. Hence, we come to the resonant couple of polynomials

$$\varphi(x_1,x_2) = x_1 \sum_{j=1}^{n-1} a_{j+1,0} x_1^j, \quad \psi(x_1,x_2) = x_2 \sum_{j=1}^{n-1} b_{j,1} x_1^j.$$

4) $\lambda_1 \cdot \lambda_2 < 0$. Every of the relations (44) and (45) can hold only in the case when λ_1/λ_2 is a rational number. Let $\lambda_1 : \lambda_2 = p : (-q)$, where p and q are integer positive reciprocal prime numbers, i.e. $(p,q) = 1$. Denote by n^* the integer part of the number $(n-1)/(p+q)$. In this case, the equality (44) holds for $m_1 = qi + 1$, $m_2 = pi$, and (45) for $n_1 = qi$, $n_2 = pi + 1$, $i = \overline{1, n^*}$. The resonant couple of polynomials (φ, ψ) corresponding to (λ_1, λ_2) is

$$\varphi(x_1,x_2) = x_1 \sum_{i=1}^{n^*} a_{qi+1,pi} \left(x_1^q x_2^p\right)^i, \quad \psi(x_1,x_2) = x_2 \sum_{i=1}^{n^*} b_{qi,pi+1} \left(x_1^q x_2^p\right)^i.$$

From what have been said above and Theorem 5, follows

Theorem 6. *The dimension of GL-orbit of system (23) with conditions (22) is equal to three if and only if the polynomials φ and ψ from (43) are not simultaneously equal to zero and the pair (φ, ψ) is resonant.*

Taking into account Theorems 1, 2, 4 and 6, we obtain the following characteristic of systems (23) with the dimension of orbit equal to four:

Theorem 7. *The dimension of GL-orbit of system (23) with the conditions (22) is equal to four if and only if $|\varphi(x_1,x_2)| + |\psi(x_1,x_2)| \not\equiv 0$ and the pair of polynomials (φ, ψ) is not resonant.*

8 The integrability on the GL-orbits of the dimension three of system (23)

We consider the polynomial system

$$\dot{x}_1 = P(x_1,x_2), \quad \dot{x}_2 = Q(x_1,x_2). \tag{47}$$

Let $n = max\{degP, degQ\}$ and $D = P\partial/\partial x_1 + Q\partial/\partial x_2$. A curve $f(x_1,x_2) = 0$, $f \in C[x_1,x_2]$, (an expression $f = exp[h(x_1,x_2)/g(x_1,x_2)]$, where $h, g \in C[x_1,x_2]$), is called an *algebraic invariant curve* (an *exponential invariant curve*) for (47) if there exists a polynomial $K \in C[x_1, x_2]$ of the order at most $n-1$ such that the following identity $D(f) \equiv f \cdot K$ holds. The polynomial $K(x_1, x_2)$ is called *the cofactor* of the invariant curve f. By [4], if $f = exp(h/g)$ is an exponential invariant curve for a system (47), then $g(x_1, x_2) = 0$ is an algebraic invariant curve for the same system.

Let $f_1, \ldots, f_s$ be a collection of algebraic invariant curves and exponential invariant curves of system (47) and, respectively, $K_1, \ldots, K_s$ their cofactors. If there exist such numbers $\beta_1, \beta_2, \ldots, \beta_s \in C$ that $F \equiv f_1^{\beta_1} f_2^{\beta_2} \ldots f_s^{\beta_s} = const$ ($\mu = f_1^{\beta_1} f_2^{\beta_2} \ldots f_s^{\beta_s}$) is a first integral (an integrating factor) for (47), that is $D(F) \equiv 0$ ($D(\mu) + \mu(P'_{x_1} + Q'_{x_2}) \equiv 0$), then we say that the system of differential equations (47) is *Darboux integrable in the generalized sense.* If among $f_1, \ldots, f_s$ there are not an exponential invariant curve, then we shall speak on Darboux integrability of (47).

It easy to show that F (μ) is a first integral (an integrating factor) of the Darboux type for (47) if and only if the following identity

$$\sum_{i=1}^{s} \beta_i K_i(x_1, x_2) \equiv 0 \quad \left(\sum_{i=1}^{s} \beta_i K_i(x_1, x_2) \equiv -(P'_{x_1} + Q'_{x_2}) \right)$$

is verified.

Next, we will examine on integrability the systems of the form (23), (22) which have the dimension of GL-orbit equal to three, i.e. systems (27), (28), (33)–(35). Because the system (28) ((34)) can be reduced to the system (27) ((33)) by a substitution $x_1 \to x_2$, $x_2 \to x_1$, we shall consider only the problem of integrability of systems (27), (33) and (35).

By [3], the systems of normal form are integrable in quadratures. The aim of this section is to show that the given systems are Darboux integrable in the generalized sense.

The system (27). a) Let $\lambda_1 \neq k\lambda_2$. It is easy to check that the curves $f_1 = x_2$ and $f_2 = (\lambda_1 - k\lambda_2)x_1 + a_{0,k}x_2^k$ are algebraic invariant curves for (27) and have the cofactors $K_1(x_1, x_2) = \lambda_2$ and $K_2(x_1, x_2) = \lambda_1$, respectively. Evidently, the identity $\beta_1 \cdot K_1 + \beta_2 \cdot K_2 \equiv 0$ holds for $\beta_1 = \lambda_1$, $\beta_2 = -\lambda_2$ and therefore $F = f_1^{\lambda_1} f_2^{-\lambda_2}$ is a first integral of system (27).

b) $\lambda_1 = k\lambda_2$. In this case besides the invariant curve $f_1 = x_2$ with $K_1 = \lambda_2$, we have also an exponential invariant curve $f_2 = exp(x_1/x_2^k)$ with $K_2 = a_{0,k}$. The first integral is $F = f_1^{a_{0,k}} f_2^{-\lambda_2}$.

The system (33). Let

$$\tilde{\varphi} = \lambda_1 + \sum_{j=1}^{n-1} a_{1,j} x_2^j, \quad \tilde{\psi} = x_2 \left(\lambda_2 + \sum_{j=1}^{n-1} b_{0,j+1} x_2^j \right).$$

If $\tilde{\varphi} \equiv 0$ ($\tilde{\psi} \equiv 0$), then $F = x_1$ ($F = x_2$) is a first integral of (33) and if $\tilde{\psi} \not\equiv 0$, this integral looks

$$F = x_1 exp[-\int (\tilde{\varphi}/\tilde{\psi}) dx_2].$$

Let $\tilde{\varphi} \not\equiv 0$, $\tilde{\psi} \not\equiv 0$, $r = deg\tilde{\psi}$, $s = max\{0, deg\tilde{\psi} - deg\tilde{\varphi} + 1\}$, $\tilde{\psi} = b_{0,r}(x_2 - b_1)^{r_1} \dots (x_2 - b_m)^{r_m}$, where $b_1 = 0$, $b_j \in C \setminus \{0\}$, $j = \overline{2,m}$, $r_1 + \dots + r_m = r$. For system (33) $f_0 \equiv x_1 = 0$, $f_i \equiv x_2 - b_i = 0$, $i = \overline{1,m}$, are invariant lines, and

$$f_{m+1} = exp\frac{1}{x_2 - b_1}, \dots, f_{m+r_1-1} = exp\frac{1}{(x_2 - b_1)^{r_1-1}}, \dots,$$

$$f_r = exp\frac{1}{(x_2 - b_m)^{r_m-1}},\ f_{r+1} = exp(x_2), \dots, f_{r+s} = exp(x_2^s)$$

are exponential invariant curves. Because

$$\int \frac{\tilde{\varphi}}{\tilde{\psi}} dx_2 = -\Bigg[\beta_1 ln\,|x_2 - b_1| + \dots + \beta_m ln\,|x_2 - b_m| + \frac{\beta_{m+1}}{x_2 - b_1} + \dots$$

$$+ \frac{\beta_r}{(x_2 - b_m)^{r_m-1}} + \beta_{r+1}x_2 + \dots + \beta_{r+s}x_2^s\Bigg],$$

the integral F of (33) can be written in the Darboux form: $F = \prod\limits_{i=0}^{r+s} f_i^{\beta_i}$.

In the investigated case it is more easy to find an integrating factor which looks $\mu = 1/(x_1\tilde{\psi})$.

The system (35). Because p and q are reciprocal prime numbers, for them such integer positive numbers u and v can be found that $pu - qv = 1$. The transformation $z_1 = x_1^u x_2^v$, $z_2 = x_1^q x_2^p$ [3] reduces (35) to a system similar with (33):

$$\dot{z}_1 = z_1\left[u\lambda_1 + v\lambda_2 + \sum_{i=1}^{n^*}(ua_{qi+1,pi} + vb_{qi,pi+1})z_2^i\right],$$

$$\dot{z}_2 = z_2\left[q\lambda_1 + p\lambda_2 + \sum_{i=1}^{n^*}(qa_{qi+1,pi} + pb_{qi,pi+1})z_2^i\right].$$

Thus, we shall integrate directly system (35). If

$$\lambda_1 : \lambda_2 = a_{qi+1,pi} : b_{qi,pi+1} = -p : q, \quad i = \overline{1,n^*}, \tag{48}$$

then the right–hand sides of (35) have a common factor $\lambda_1 + \sum\limits_{i=1}^{n^*} a_{qi+1,pi}(x_1^q x_2^p)^i$. After their cancelation by this factor, we obtain the system $\dot{x}_1 = x_1$, $\dot{x}_2 = \frac{\lambda_2}{\lambda_1}x_2$ which has a general integral $x_1^{\lambda_2}x_2^{-\lambda_1} = const$. In the case when (48) is not satisfied we have an integrating factor

$$\mu = \left[x_1x_2\Big(q\lambda_1 + p\lambda_2 + \sum_{i=1}^{n^*}(qa_{qi+1,pi} + pb_{qi,pi+1})(x_1^q x_2^p)^i\Big)\right]^{-1}.$$

From what has been said above, follows

Theorem 8. *On GL-orbits of dimension three the system (23) with the conditions (22) has a generalized Darboux first integral (a Darboux integrating factor).*

In the case of cubic systems (36) and (37) we have the first integrals

$$x_2^{\lambda_1}\left[(\lambda_1 - j\lambda_2)x_1 + a_{0,j}x_2^j\right]^{-\lambda_2} \quad if \quad \lambda_1 \neq j\lambda_2,$$

and

$$x_2^{a_{0,j}} exp(-\lambda_2 x_1/x_2^j) \quad if \quad \lambda_1 = j\lambda_2, \quad j = \overline{2,3}.$$

The system (40) has a first integral $x_2 = c$ if $\lambda_2 = b_{02} = b_{03} = 0$ and an integrating factor $\mu = \left[x_1x_2(\lambda_2 + b_{02}x_2 + b_{03}x_2^2)\right]^{-1}$ if $|\lambda_2| + |b_{02}| + |b_{03}| \neq 0$. At the same time, the system (42) has a first integral $x_1^{\lambda_2}x_2^{-\lambda_1} = const$ if $\lambda_1 + \lambda_2 = a_{21} + b_{12} = 0$ and an integrating factor $\mu = [x_1x_2(\lambda_1 + \lambda_2 + (a_{21} + b_{12})x_1x_2)]^{-1}$ in other cases. The cubic systems (38), (39) and (41) can be reduced to the systems investigated above by substitution $x_1 \to x_2$, $x_2 \to x_1$.

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Received November 2, 2004

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