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The optimal flow in dynamic networks with nonlinear cost functions on edges

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Abstract. In this paper we study the dynamic version of the nonlinear minimum-cost flow problem on networks. We consider the problem on dynamic networks with nonlinear cost functions on edges that depend on time and flow. Moreover, we assume that the demand function and capacities of edges also depend on time. To solve the problem we propose an algorithm, which is based on reducing the dynamic problem to the classical minimum-cost problem on a time-expanded network. We also study some generalization of the proposed problem.

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1 Introduction

In this paper we study the dynamic version of the nonlinear minimum-cost flow problem on networks, in which flows from supply nodes should be sent, in minimum cost, to demand nodes such that the flows on used links do not exceed their capacities. This problem generalizes the well-known classical minimum-cost flow problem on static networks [1] and extends some dynamic models from [2–5].

Classical static network flow models have been well known as valuable tools for many applications. However, they fail to capture the property of many real-life problems. The static flow can not properly consider the evolution of the system in time. The time is an essential component, either because the flows of some commodity take time to pass from one location to another, or because the structure of network changes over time. To tackle this problem, we use dynamic network flow models instead of the static ones.

The minimum cost flow problem is the problem of sending flows in a network from supply nodes to demand nodes with minimum total cost such that link capacities are not exceeded. This problem has been studied extensively in the context of static networks. In this paper, we study the minimum cost flow problem in dynamic networks.

We consider the problem on dynamic networks with nonlinear cost functions on edges that depend on time and on flow. Moreover, we assume that the demand function and capacities of edges also depend on time. We propose an algorithm for solving the problem, which extends the algorithms from [2, 3] and is based on reducing the dynamic problem to the classical minimum-cost problem on a time-expanded network.

2 Problem formulation

A dynamic network $N = (V, E, u, \tau, d, \varphi)$ consists of directed graph $G = (V, E)$ with the set of vertices V and the set of edges E, capacity function u: $E \times \mathbb{T} \to R$, transit time function τ_e: $E \to R_+$, demand function d: $V \times \mathbb{T} \to R$ and cost function φ: $E \times R_+ \times \mathbb{T} \to R_+$, where $\mathbb{T} = \{0, 1, 2, \ldots, T\}$. The demand function $d_v(t)$ satisfies the following conditions:

a) there exists $v \in V$ with $d_v(0) < 0$;

b) if $d_v(t) < 0$ for a node $v \in V$ then $d_v(t) = 0$, $t = 1, 2, \ldots, T$;

c) $\sum_{t\in\mathbb{T}} \sum_{v\in V} d_v(t) = 0$.

Nodes $v \in V$ with $\sum_{t\in\mathbb{T}} d_v(t) < 0$ are called sources, nodes $v \in V$ with $\sum_{t\in\mathbb{T}} d_v(t) > 0$ are called sinks and nodes $v \in V$ with $\sum_{t\in\mathbb{T}} d_v(t) = 0$ are called intermediate.

A feasible dynamic flow on N is a function x: $E \times \mathbb{T} \to R_+$ that satisfies the following conditions:

$$\sum_{\substack{e\in E^+(v)\\ t-\tau_e\geq 0}} x_e(t - \tau_e) - \sum_{e\in E^-(v)} x_e(t) = d_v(t),\ \forall\, t \in \mathbb{T},\ \forall\, v \in V; \tag{1}$$

$$0 \leq x_e(t) \leq u_e(t), \quad \forall\, t \in \mathbb{T},\ \forall\, e \in E; \tag{2}$$

$$x_e(t) = 0,\ \forall\, e \in E,\ t = \overline{T - \tau_e + 1, T}; \tag{3}$$

where $E^+(v) = \{(u, v) \,|\, (u, v) \in E\}$, $\quad E^-(v) = \{(v, u) \,|\, (v, u) \in E\}$.

Here the function x defines the value $x_e(t)$ of flow entering edge e at time t. It is easy to observe that the flow does not enter edge e at time t if it will have to leave the edge after time T; this is ensured by condition (3).

To model transit costs, which may change over time, we define the cost function $\varphi_e(x_e(t), t)$ with the meaning that flow of value $\xi = x_e(t)$ entering edge e at time t will incur a transit cost of $\varphi_e(\xi, t)$. We consider the discrete time model, in which all times are integral and bounded by horizon T. The time horizon (finite or infinite) is the time until which the flow can travel in the network and defines the makespan $\mathbb{T} = \{0, 1, \ldots, T\}$ of time moments we consider.

The integral cost $F(x)$ of dynamic flow on N is defined as follows:

$$F(x) = \sum_{e\in E} \sum_{t\in\mathbb{T}} \varphi_e(x_e(t), t). \tag{4}$$

Our dynamic minimum-cost flow problem is to find a flow that minimizes the objective function (4).

It is easy to observe that if $\tau_e = 0, \forall\, e \in E$ and $T = 0$ then the formulated problem becomes the classical minimum-cost flow problem on a static network.

3 Main results

We have obtained a necessary and sufficient condition for the existence of admissible flow in dynamic network N, i.e. the condition when the set of solutions of the system (1)–(3) is not empty. In this paper we propose a new approach for solving the formulated problem, which is based on its reduction to a static minimum-cost flow problem. We show that our problem on network $N = (V, E, u, \tau, d, \varphi)$ can be reduced to a static problem on auxiliary static network $N^T = (V^T, E^T, u^T, d^T, \varphi^T)$; we name it the time-expanded network. We define this network as follows:

1. $V^T\colon\ = \{v(t) \,|\, v \in V,\ t \in \mathbb{T}\}$;
2. $E^T\colon\ = \{(v(t), w(t + \tau_e)) \,|\, e = (v, w) \in E,\ 0 \le t \le T - \tau_e\}$;
3. $u^T_{e(t)}\colon\ = u_e(t)$ and $\varphi^T_{e(t)}(x_e(t))\colon\ = \varphi_e(x_e(t), t)$ for $e(t) \in E^T$;
4. $d^T_{v(t)}\colon\ = d_v(t)$ for $v(t) \in V^T$.

If we define a flow correspondence to be $x^T_{e(t)}\colon\ = x_e(t)$, the minimum-cost flow problem on dynamic networks can be solved by using the solution of the static minimum cost flow problem on the expanded network.

The essence of the time-expanded network is that it contains a copy of the vertices of the dynamic network for each time $t \in \mathbb{T}$, and the transit times and flows are implicit in the edges linking those copies.

Now let us define a correspondence between feasible dynamic flows on the dynamic network N and feasible static flows on the time-expanded network N^T. A feasible static flow on N^T is a function $x^T_{e(t)}$ that satisfies the following conditions:

$$\sum_{e(t) \in E^+(v(t))} x^T_{e(t)} - \sum_{e(t) \in E^-(v(t))} x^T_{e(t)} = d^T_{v(t)},\ \forall\, v(t) \in V^T;$$

$$0 \le x^T_{e(t)} \le u^T_{e(t)},\ \forall\, e(t) \in E^T;$$

$$x^T_{e(t)} = 0,\ \forall\, e(t) \in E^T,\ t = \overline{T - \tau_e + 1, T}.$$

Let $e(t) = (v(t), w(t + \tau_e)) \in E^T$ and let $x_e(t)$ be a flow on the dynamic network N. The corresponding function $x^T_{e(t)}$ on the time-expanded network N^T is defined as follows:

$$x^T_{e(t)} = x(v(t), w(t + \tau_e)) = x_e(t),\ \forall\, e(t) \in E^T. \tag{5}$$

Lemma 1. *The correspondence (5) is a bijection from the set of feasible flows on the dynamic network N onto the set of feasible flows on the time-expanded network N^T.*

Proof. It is obvious that the correspondence above is a bijection from the set of $\mathbb{T}$-horizon functions on the dynamic network N onto the set of functions on the time-expanded network N^T. It is also easy to observe that a feasible flow on the dynamic network N is a feasible flow on the time-expanded network N^T and vice-versa. Indeed,

$$0 \leq x^T_{e(t)} = x_e(t) \leq d_e(t) = d^T_{e(t)}, \ \forall\, e \in E, \quad 0 \leq t < T.$$

Therefore it is enough to show that each dynamic flow on the dynamic network N is put into the correspondence with a static flow on the time-expanded network N^T and vice-versa.

Let $x_e(t)$ be a dynamic flow on N and let $x^T_{e(t)}$ be a corresponding function on N^T. Let's prove that $x^T_{e(t)}$ satisfies the conservation constraints on its static network. Let $v \in V$ be an arbitrary node in N and t: $0 \leq t < T$ an arbitrary moment of time:

$$d_v(t) \overset{(i)}{=} \sum_{\substack{e\in E^+(v) \\ t-\tau_e \geq 0}} x_e(t-\tau_e) - \sum_{e \in E^-(v)} x_e(t) =$$

$$= \sum_{e(t-\tau_e)\in E^+(v(t))} x^T_{e(t-\tau(e))} - \sum_{e(t)\in E^-(v(t))} x^T_{e(t)} \overset{(ii)}{=} d^T_{v(t)}. \tag{6}$$

Note that according to the definition of the time-expanded network the set of edges $\{e(t-\tau_e)|e(t-\tau_e) \in E^+(v(t))\}$ consists of all edges that enter $v(t)$, while the set of edges $\{e(t)|e(t) \in E^-(v(t))\}$ consists of all edges that originate from $v(t)$. Therefore, all necessary conditions are satisfied for each node $v(t) \in V^T$. Hence, $x^T_{e(t)}$ is a flow on the time-expanded network N^T.

Let $x^T_{e(t)}$ be a static flow on the time-expanded network N^T and let $x_e(t)$ be a corresponding function on the dynamic network N. Let $v(t) \in V^T$ be an arbitrary node in N^T. The conservation constraints for this node in the static network are expressed by equality (ii) from (6), which holds for all $v(t) \in V^T$ at all times t: $0 \leq t < T$. Therefore, equality (i) holds for all $v \in V$ at all times t: $0 \leq t < T$ and $x_e(t)$ is a flow on the dynamic network N. □

The total cost of the static flow in the time-expanded network N^T is denoted as follows:

$$F^T(x) = \sum_{e(t)\in E} \sum_{t \in \mathbb{T}} \varphi^T_{e(t)}(x_e(t)).$$

Lemma 2. *If $x_e(t)$ is a flow on the dynamic network N and $x^T_{e(t)}$ is a corresponding flow on the time-expanded network N^T, then*

$$F(x_e(t)) = F^T(x_e(t)).$$

Proof. The proof is straightforward:

$$F(x_e(t)) = \sum_{e \in E} \sum_{t \in \mathbb{T}} \varphi_e(x_e(t), t) = \sum_{e(t) \in E} \sum_{t \in \mathbb{T}} \varphi^T_{e(t)}(x_e(t)) = F^T(x_e(t)). \qquad \square$$

The above lemmas imply the validity of the following theorem:

Theorem 1. *For each minimum-cost flow in the dynamic network there is a corresponding minimum-cost flow in the static network.*

Therefore, we can solve the dynamic minimum-cost flow problem by reducing it to the minimum-cost flow problem on static networks.

4 Algorithm

Let a dynamic network N be given. The minimum-cost flow problem is to be solved on N. Proceedings are following:

1. Building the time-expanded network N^T for the given dynamic network N.
2. Solving the classical minimum-cost flow problem on the static network N^T.
3. Reconstructing the solution of the static problem on N^T to the dynamic problem on N. $\square$

5 Generalization

Now let us study some general cases of the dynamic networks. First of all, we assume that only a part of the flow is dumped into the considered network at the time 0, i.e. the condition b) in the definition of the demand function $d_v(t)$ doesn't hold. Using the following, this case can be reduced to the one considered above.

Let us consider an arbitrary dynamic network N defined above and let the flow be dumped into the network from the node $v \in V$ at an arbitrary moment of time $\mathtt{t}$, different from the ordinary moment. We can reduce this problem to the problem in which all of the flow is dumped into the network at the initial time by introducing loops in all nodes from V, except the node v from which the flow is dumped into the network at the time $\mathtt{t}$. For such loops we attribute capacities $u_e(t)$ and transit times which are equal to the time $\mathtt{t}$. The cost functions are equal to 0 on these loops. So, we can consider that all the flow is dumped in the network at the time $\mathtt{t}$, which we define as the initial time.

The argumentation is the same when the flow is dumped in the network from different nodes at different moments of time. Let `t` be the maximum of those moments. In this case we take `t` as the initial time and attribute capacities $u_e(t)$ and transit times to loops constructed from all the nodes, except those that dump the flow in the network at time `t`. The transit times are equal to the difference between time `t` and the time when the flow from those nodes that generate loops is dumped in the network. We consider the cost functions that are zero on such loops. So, we reduce this problem to the one considered above where the whole flow is dumped into the network at the initial moment of time.

Further we consider the variation of the dynamic network when the condition c) in the definition of the demand function $d_v(t)$ doesn't hold. We assume that after time $t = T$ there still is flow in the network, i.e. the following condition is true:

$$\sum_{t \in \mathbb{T}} \sum_{v \in V} d_v(t) \geq 0.$$

We also can reduce this case to the initial one, using the following argumentation.

Let us consider an arbitrary dynamic network N defined above and let the flow exist in the network after time $t = T$. We can reduce this problem to the problem without flow in the network after an upper bound of time by the introduction of an additional node $v \notin V$ and additional edges which are not contained in E. The rest of the flow in the network is sent to the node v through the arcs which we just introduced. We consider that these arcs have capacities $u_e(t)$ and specified limited transit times and that the cost functions on these loops are zero. In such a way we obtain the initial model of the dynamic network.

The next model of the dynamic network is the one when we allow flow storage at the nodes. In this case we can reduce this dynamic network to the initial one by introducing the loops in those nodes in which there is flow storage. For these loops we attribute capacities $u_e(t)$, specified limited transit times, and zero cost functions. The flow which was stored at the nodes passes through these loops. Accordingly, we reduce this problem to the initial one.

The other variation of the dynamic network is the one when the cost functions also depend on the flow at the nodes. In this case we can reduce this model of the dynamic network to the initial one by introducing new arcs and attributing the cost functions, which were defined in the nodes, capacities $u_e(t)$, and fixed transit times to these arcs. Consequently, we obtain the initial model of the dynamic network.

The same reasoning to solve the minimum-cost flow network problem on the dynamic networks and its generalization can be held in the case when, instead of the condition (2) in the definition of the feasible dynamic flow, the following condition takes place:

$$u_e^1(t) \leq x_e(t) \leq u_e^2(t), \quad \forall\, t \in \mathbb{T},\ \forall\, e \in E,$$

where $u_e^1(t)$ and $u_e^2(t)$ are lower and upper boundaries of the capacity of the edge e, respectively.

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