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On natural classes of R-modules in the language of ring R

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Abstract. Every natural class of left R-modules is closed, i.e. is completely described by special set of left ideals of R (natural set). Some characterizations of such sets are shown. The complementation operator of sets is defined and its properties permit to transfer some results on natural classes to the lattice of left ideals of R.

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Introduction

Various types of classes of modules play an important role in the theory of radicals. For example, every idempotent radical r of R-mod can be described by each of the classes $\mathcal{R}(r)$ and $\mathcal{P}(r)$ of r-torsion or r-torsion free modules. The classes of such types are characterized by closure properties (under submodules, homomorphic images, extensions, etc.) [1–3]. In the literature numerous types of classes of modules with special properties are studied. In particular, very intensively are investigated the *natural classes* ($\equiv$ saturated classes) of R-modules, which are closed under submodules, direct sums and injective envelopes [4–8].

In the present note an attempt is made to transfer some results on natural classes of R-modules in to lattice $\mathbb{L}({}_RR)$ of left ideals of ring R, using some facts from [9–11] on the relation between classes of left R-modules and sets of left ideals of ring R. This is possible since every natural class of modules is closed, i.e. can be characterized by special set of left ideals of R. Some descriptions of such sets are indicated. The operator of complementation of natural classes is transfered in the lattice $\mathbb{L}({}_RR)$, some properties and applications are shown. In particular, the lattice R-Nat of natural sets is boolean.

1 Natural classes and natural sets

Let R be an arbitrary ring with unity and R-mod be the category of unitary left R-modules. We consider the *abstract* classes of R-modules (i.e. $M \in \mathcal{K}$, $M \cong N$ implies $N \in \mathcal{K}$).

Definition 1. *The abstract class $\mathcal{K} \subseteq R$-Mod is called* **natural** *if it is closed with respect to submodules, direct sums and injective envelopes* (*or essential extensions*).

The natural ($\equiv$ saturated) classes were studied in a series of works, in particular in [4–8]. We are interested in some aspects related to the description of such classes in the language of ring R. Firstly we remind the following known fact.

Lemma 1.1. *If the class $\mathcal{K} \subseteq R$-Mod is closed under submodules, finite direct sums and injective envelopes, than $\mathcal{K}$ is closed also under extensions* (*i.e. if in the exact sequence* $0 \to A \to B \to C \to 0$ *modules* $A, C \in \mathcal{K}$, *then* $B \in \mathcal{K}$).

Proof. Consider injective envelopes $X = E(A)$, $Y = E(C)$ and the diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "k"'] & B \arrow[r, "g"] \arrow[dl, dotted, "h"'] \arrow[d, dotted, "t"] & C \arrow[r] \arrow[d, "l"] & 0 \\
0 \arrow[r] & X \arrow[r] & X \oplus Y \arrow[r] & Y \arrow[r] & 0,
\end{tikzcd}$$

where k and l are injections. Since X is injective, there exists h: $B \to X$ with $hf = k$. Define t: $B \to X \oplus Y$ by $t(b) = \Big(h(b), lg(b)\Big)$, obtaining the commutative diagram, where k and l are mono, so t is mono. Now from $A, C \in \mathcal{K}$ we conclude that $X, Y \in \mathcal{K}$, $X \oplus Y \in \mathcal{K}$ and $B \in \mathcal{K}$. $\square$

Corollary 1.2. *Every natural class of R-modules is closed under extensions.*

In particular, the classes of the form $\mathcal{P}(r) = \{_R M \mid r(M) = 0\}$ for a torsion r can be described as classes closed under submodules, direct product and injective envelopes, but these conditions that it is closed under extensions [2, 3, 10].

Now we are going to expose some general facts on the relation between classes of modules $\mathcal{K} \subseteq R$-Mod and some sets $\mathcal{E} \subseteq \mathbb{L}(_R R)$ of left ideals of R [9–11]. As above we denote by $\mathbb{L}(_R R)$ the lattice of left ideals of R. We define the following two operators:

1) if $\mathcal{K} \subseteq R$-Mod we denote

$$\Gamma(\mathcal{K}) = \{(0 : m) \mid m \in M,\ M \in \mathcal{K}\},\ \text{where } (0 : m) = \{a \in R \mid am = 0\};$$

2) if $\mathcal{E} \subseteq \mathbb{L}(_R R)$ then by definition

$$\Delta(\mathcal{E}) = \{M \in R\text{-Mod} \mid (0 : m) \in \mathcal{E}\ \ \forall\, m \in M\}.$$

Definition 2 [9, 10]. *The class $\mathcal{K} \subseteq R$-Mod is called* **closed** *if $\mathcal{K} = \Delta\Gamma(\mathcal{K})$. The set $\mathcal{E} \subseteq \mathbb{L}(_R R)$ is called* **closed** *if $\mathcal{E} = \Gamma\Delta(\mathcal{E})$.*

Lemma 1.3 [9, 10]. *The class $\mathcal{K} \subseteq R$-Mod is closed if and only if is satisfies the condition:*

(A_1) $M \in \mathcal{K} \Leftrightarrow Rm \in \mathcal{K}\ \ \forall\, m \in M$;

(or: (A_1') $M \in \mathcal{K} \Rightarrow m \in M$ and (A_1'') $Rm \in \mathcal{K}\ \forall\, m \in M \Rightarrow M \in \mathcal{K}$).

The set $\mathcal{E} \subseteq \mathbb{L}(_R R)$ is closed if and only if is satisfied the condition:

(a_1) $I \in \mathcal{E} \Rightarrow (I : a) \in \mathcal{E}\ \ \forall a \in R$ (where $(I : a) = \{b \in R \mid ba \in I\}$).

Lemma 1.4 [9, 10]. *The operators Γ and Δ define a bijection (preserving inclusions) between closed classes of R-Mod and closed sets of left ideals of $\mathbb{L}({}_RR)$. If $\mathcal{K}$ and $\mathcal{E}$ correspond each other then: $I \in \mathcal{E} \Leftrightarrow R/I \in \mathcal{K}$.*

For the pair $(\mathcal{K}, \mathcal{E})$ with $\mathcal{K} = \Delta(\mathcal{E})$ and $\mathcal{E} = \Gamma(\mathcal{K})$ a series of closure properties of $\mathcal{K}$ can be translated in the language of R as properties of the set $\mathcal{E}$. The most important examples of such properties of $\mathcal{K}$ are the closeness under:

(A_2) homomorphic images;
(A_3) direct sums;
(A_4) direct products;
(A_5) extensions;
(A_6) essential extensions ($\equiv$ stability of $\mathcal{K}$).

In parallels the following conditions of the set $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ are considered:

(a_2) $I \in \mathcal{E},\ J \in \mathbb{L}({}_RR),\ J \supseteq I \Rightarrow J \in \mathcal{E}$;
(a_3) $I, J \in \mathcal{E} \Rightarrow I \cap J \in \mathcal{E}$;
(a_4) $I_\alpha \in \mathcal{E}(\alpha \in \mathfrak{A}) \Rightarrow \bigcap\limits_{\alpha \in \mathfrak{A}} I_\alpha \in \mathcal{E}$;
(a_5) $I \in \mathcal{E},\ J \in \mathbb{L}({}_RR),\ J \subseteq I, (J : i) \in \mathcal{E}\ \ \forall i \in I \Rightarrow J \in \mathcal{E}$;
(a_6) $J \in I,\ (J : i) \in \mathcal{E}\ \ \forall i \in I,\ I/J \subseteq^* R/J \Rightarrow J \in \mathcal{E}$
(where $\subseteq^*$ is the essential inclusion).

Theorem 1.5 [9, 10]. *Let $(\mathcal{K}, \mathcal{E})$ be a pair with $\mathcal{K} = \Delta(\mathcal{E})$ and $\mathcal{E} = \Gamma(\mathcal{K})$. The class $\mathcal{K}$ satisfies the condition (A_n) if and only if the set $\mathcal{E}$ satisfies the condition (a_n) for $n = 2, 3, 4, 5, 6$.*

Proof. These statements are proved in [9] and [10]. For convenience we verify the case $n = 6$, which presents here a special interest.

$(\Rightarrow)$ Let $\mathcal{K}$ satisfy (A_6) and consider the situation of condition (a_6). From $(J : i) \in \mathcal{E}$ for every $i \in I$ follows $I/J \in \mathcal{K}$ and now the condition $I/J \subseteq^* R/J$ implies $R/J \in \mathcal{K}$, i.e. $J \in \mathcal{E}$.

$(\Leftarrow)$ Let $\mathcal{E}$ satisfy (a_6), $M \in \mathcal{K}$ and $M \subseteq^* N$. For an element $n \in N \setminus M$ we have $0 \neq Rn \subseteq N$ and $M \cap Rn \neq 0$. Denote: $I = (M : n)$, $J = (0 : n)$. Then $In = M \cap Rn \neq 0$ and $I \supseteq J$. Moreover, for every $i \in I$ we obtain $in \in M$, $M \in \mathcal{K}$, therefore $(0 : in) = \big((0 : n) : i\big) = (J : i) \in \mathcal{E}$. It is easy to verify that $I/J \subseteq^* R/J$. Now we are in the situation of condition (a_6) and so $J \in \mathcal{E}$, i.e. $(0 : n) \in \mathcal{E}$ for every $n \in N$, therefore $N \in \mathcal{K}$ and $\mathcal{K}$ is stable (condition (A_6)). $\square$

In continuation we will apply these results for the investigation of natural classes of modules, taking into account that is true

Proposition 1.6. *Every natural class of modules is closed.*

Proof. Let $\mathcal{K}$ be an arbitrary natural class of R-Mod. Since $\mathcal{K}$ is hereditary, we have (A_1') and now we verify (A_1''). Let $M \in R$-Mod and $Rm \in \mathcal{K}$ for every $m \in M$. We consider the family $\mathfrak{P}$ of independent sets of submodules of M. Then $\mathfrak{P} \neq \emptyset$ and is inductive, therefore by Zorn's lemma it possesses a maximal element $\mathcal{F} = \{M_\alpha \,|\, \alpha \in \mathfrak{A}\}$. We denote $N = E\Big(\bigoplus\limits_{\alpha \in \mathfrak{A}} M_\alpha\Big) \bigcap M$. Since $M_\alpha \in \mathcal{K}$ for every

$\alpha \in \mathfrak{A}$, we obtain $N \in \mathcal{K}$. If $N \cap P = 0$ for $0 \neq P \subseteq M$, then from the choice of M follows the existence of $0 \neq M' \subseteq P$ with $M' \in \mathcal{K}$. But then $\mathcal{F} \cup \{M'\}$ is independent, in contradiction with maximality of $\mathcal{F}$. This shows that $N \subseteq^{*} M$, therefore $M \in \mathcal{K}$. □

So every natural class $\mathcal{K}$ is completely described by the corresponding set $\mathcal{E} = \Gamma(\mathcal{K}) \subseteq \mathbb{L}({}_R R)$. We will show some characterizations of such sets of left ideals.

Definition 3. *The set $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ is called* **natural** *if it satisfies the conditions $(a_1), (a_3)$ and (a_6).*

Theorem 1.7. *The operators Γ and Δ define a bijection (preserving inclusions) between the natural classes of R-Mod and natural sets of $\mathbb{L}({}_R R)$.*

Proof. Let $\mathcal{K}$ be a natural class and $\mathcal{E} = \Gamma(\mathcal{K})$. Then $\mathcal{K}$ is closed (Prop. 1.6), therefore $\mathcal{E}$ is closed (condition (a_1)). So we can apply Theorem 1.5: since $\mathcal{K}$ satisfies (A_3) and (A_6), the set $\mathcal{E}$ satisfies (a_3) and (a_6), i.e. $\mathcal{E}$ is a natural set.

Let now $\mathcal{E}$ be a natural set and $\mathcal{K} = \Delta(\mathcal{E})$. Then $\mathcal{E}$ is closed, therefore $\mathcal{K}$ is closed (condition (A_1)) and by Theorem 1.5 $\mathcal{K}$ satisfies (A_3) and (A_6), i.e. $\mathcal{K}$ is a natural class.

From Lemma 1.4 it is clear that the indicated correspondences define a bijection which preserves the inclusions. □

Now we show two important examples of natural classes, related to the theory of radicals in R-Mod.

Example 1. For every torsion ($\equiv$ hereditary radical) r the class $\mathcal{P}(r) = \{M \in R\text{-Mod} \mid r(M) = 0\}$ of r-torsion free modules, as we remark above, is characterized by properties (A_1), (A_4) and (A_6) (which imply (A_5)). Since from (A_1) and (A_4) follows (A_3), the class $\mathcal{P}(r)$ is natural. The corresponding set of left ideals $\mathcal{E} = \Gamma\Big(\mathcal{P}(r)\Big)$ is described by the conditions (a_1), (a_4) and (a_6) (which imply (a_5)). Such sets were called *cofilters* [9, 10] (dual to the filters which describe the class $\mathcal{R}(r)$), or *torsion-free sets* [7].

Example 2. For every stable torsion (i.e. $\mathcal{R}(r)$ is stable) the class $\mathcal{R}(r) = \{M \in R\text{-Mod} \mid r(M) = M\}$ of r-torsion modules is described by properties (A_1), (A_2), (A_3) and (A_6) (which implies (A_5)). Therefore $\mathcal{R}(r)$ in this case is a natural class. The corresponding set $\mathcal{E} = \Gamma\Big(\mathcal{R}(r)\Big)$ is characterized by conditions (a_1), (a_2), (a_3) and (a_6) (which imply (a_5)).

We will call such sets *stable filters* ($\equiv$ Gabriel filters with (a_6), which translates the stability of $\mathcal{R}(r)$). Obviouly, Γ and Δ determine a bijection between stable torsions and stable filters.

2 Operators of complementation

For investigations of natural classes of R-Mod the operator of complementation plays an essential role. It is defined as follows:

$$\mathcal{K}^{\perp} = \{M \in R\text{-Mod} \mid M \text{ is without nonzero submodules from } \mathcal{K}\}.$$

The set R-nat of all natural classes of R-Mod is a complete lattice with respect to the operations $\bigwedge$ and $\bigvee$, naturally defined ([1, 2], etc.). Moreover, R-nat by operator $(\ )^{\perp}$ becomes a *boolean lattice* [1].

The double complement of the class $\mathcal{K}$ is:

$$\mathcal{K}^{\perp\perp} = \{M \in R\text{-Mod} \mid \forall\, 0 \neq N \subseteq M,\ \exists\, 0 \neq P \subseteq N,\ P \in \mathcal{K}\},$$

i.e. every nonzero submodule of M contains a nonzero submodule from $\mathcal{K}$.

Let $\mathcal{K}$ be a hereditary class of R-modules. Then $\mathcal{K}^{\perp}$ is closed: it is hereditary and if $M \in R$-Mod, $Rm \in \mathcal{K}^{\perp}$ for every $m \in M$, then $M \in \mathcal{K}^{\perp}$ (if not, then M contains a nonzero submodule from $\mathcal{K}$ and by hereditarity it contains some cyclic submodule from $\mathcal{K}$, contradiction). Moreover, $\mathcal{K}^{\perp}$ is in that case a natural class ([8, Theorem 6]), the class $\mathcal{K}^{\perp\perp}$ is the least natural class containing $\mathcal{K}$, so the relation $\mathcal{K} = \mathcal{K}^{\perp\perp}$ is true if and only if $\mathcal{K}$ is natural [1, 8].

In continuation the operator $(\ )^{\perp}$ for classes of R-modules will be translated in the language of ring R (for sets of left ideals) and some properties of this operator will be shown.

For the set $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ we define:

$$\mathcal{E}^{\perp} = \{I \in \mathbb{L}({}_R R) \mid (I : a) \notin \mathcal{E}\ \ \forall\, a \notin I\}$$

(i.e. this set contains the left ideals without nontrivial quotients from $\mathcal{E}$). Now we verify that the operators of complementation (for classes and for sets) are perfectly concordant with the mappings Γ and Δ.

Proposition 2.1. *Let* $(\mathcal{K}, \mathcal{E})$ *be a pair with* $\mathcal{E} = \Gamma(\mathcal{K})$ *and* $\mathcal{K} = \Delta(\mathcal{E})$ (*in this case* $I \in \mathcal{E} \Leftrightarrow R/I \in \mathcal{K}$, *Lemma* 1.4). *Then the following relations are true:*

$$\Gamma(\mathcal{K}^{\perp}) = \mathcal{E}^{\perp} \qquad (1)$$

$$\Delta(\mathcal{E}^{\perp}) = \mathcal{K}^{\perp} \qquad (2)$$

$$\begin{tikzcd}
\mathcal{K} \arrow[r, shift left, "\Gamma"] \arrow[d] & \mathcal{E} \arrow[l, shift left, "\Delta"] \arrow[d] \\
\mathcal{K}^{\perp} \arrow[r, shift left, dotted, "\Gamma"] & \mathcal{E}^{\perp} \arrow[l, shift left, dotted, "\Delta"]
\end{tikzcd}$$

Proof. (1) ($\subseteq$) Firstly we verify the inclusion $\Gamma(\mathcal{K}^{\perp}) \subseteq \mathcal{E}^{\perp}$. Let $I \in \Gamma(\mathcal{K}^{\perp})$, i.e. there exists $M \in \mathcal{K}^{\perp}$ and $m \in M$ such that $(0 : m) = I$. We must show that $(I : a) \notin \mathcal{E}$ for every $a \notin I$.

Suppose the contrary: there exists $a \notin I = (0 : m)$ such that $(I : a) \in \mathcal{E}$. Then $am \notin I$ and $R/(I : a) \cong Ram \in \mathcal{K}$, in contradiction with $M \in \mathcal{K}^{\perp}$. This proves that $I \in \mathcal{E}^{\perp}$.

($\supseteq$) To prove the inverse inclusion of (1), let $I \in \mathcal{E}^{\perp}$. We must verify that there exists $M \in \mathcal{K}^{\perp}$ and $m \in M$ with $I = (0 : m)$. For that we consider $M = R/I$ and $m = 1 + I$, where $(0 : m) = I$. It remains to show that $R/I \in \mathcal{K}^{\perp}$.

If $R/I \notin \mathcal{K}^{\perp}$, then there exists $0 \neq J/I \subseteq R/I$ with $J/I \in \mathcal{K}$. Then for $\overline{0} \neq a + I \in J/I$ from $J/I \in \mathcal{K}$ follows $R(a + I) \cong R/(I : a) \in \mathcal{K}$, i.e. $(I : a) \in \mathcal{E} (a \notin I)$, in contradiction with $I \in \mathcal{E}^{\perp}$. So we have $R/I \in \mathcal{K}^{\perp}$.

(2) ($\subseteq$) Let $M \in \Delta(\mathcal{E}^{\perp})$, i.e. $(0:m) \in \mathcal{E}^{\perp}$ for every $m \in M$. Then M is without nonzero submodules from $\mathcal{K}$: in the contrary we have $0 \neq Rm \subseteq M$, $Rm \in \mathcal{K}$ and from $Rm \cong R/(0:m)$ follows $R/(0:m) \in \mathcal{K}$, i.e. $(0:m) \in \mathcal{E}$, $m = 0$, contradiction.

($\supseteq$) Let $M \in \mathcal{K}^{\perp}$. We must verify that $M \in \Delta(\mathcal{E}^{\perp})$, i.e. for every $m \in M$ we have $(0:m) \in \mathcal{E}^{\perp}$. Suppose the contrary: there exists $m \in M$ such that $(0:m) \notin \mathcal{E}^{\perp}$. Then there exists $a \notin (0:m)$ (i.e. $am \neq 0$) such that $((0:m):a) = (0:am) \in \mathcal{E}$. Therefore $R/(0:am) \cong Ram \in \mathcal{K}$, contradiction with $M \in \mathcal{K}^{\perp}$. So $(0:m) \in \mathcal{E}^{\perp}$ for every $m \in M$, i.e. $M \in \Delta(\mathcal{E}^{\perp})$. □

This proposition permits us to transfer in $\mathbb{L}({}_RR)$ some results on classes of modules avoiding the direct proofs.

It is obvious that in conditions of Prop. 2.1 $\mathcal{E}^{\perp}$ and $\mathcal{K}^{\perp}$ are closed. Moreover, is true

Corollary 2.2. *If the set $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ is closed then $\mathcal{E}^{\perp}$ is a natural set.*

Proof. If $\mathcal{E}$ is closed, then the class $\mathcal{K} = \Delta(\mathcal{E})$ is closed, therefore $\mathcal{K}^{\perp}$ is natural, so the set $\Gamma(\mathcal{K}^{\perp}) = \mathcal{E}^{\perp}$ is natural (Theorem 1.7). □

Corollary 2.3. *If $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ is a closed set, then $\mathcal{E}^{\perp\perp}$ is the least natural set containing $\mathcal{E}$, therefore the relation $\mathcal{E} = \mathcal{E}^{\perp\perp}$ is true if and only if $\mathcal{E}$ is a natural set.*

Proof. Follows from the known fact: $\mathcal{K}^{\perp\perp}$ is the least natural class containing $\mathcal{K} = \Delta(\mathcal{E})$. □

We denote by R-Nat the family of natural sets of left ideals of R. It can be transformed in a complete lattice with order relation $\subseteq$ (inclusion) and with lattice operations $\bigwedge$ and $\bigvee$, defined as follows:

$$\bigwedge_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha = \bigcap_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha, \quad \bigvee_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha = \bigcap \big\{ \mathcal{F} \in R\text{-Nat} \,\big|\, \mathcal{F} \supseteq \mathcal{E}_\alpha \ \ \forall\, \alpha \in \mathfrak{A} \big\}.$$

Since the mappings Γ and Δ define a bijection (preserving order) between R-nat and R-Nat (Theorem 1.7) we have

Corollary 2.4. *The lattices R-nat and R-Nat are isomorphic, therefore R-Nat is a boolean lattice, where $\mathcal{E}^{\perp}$ is a complement of $\mathcal{E} \in R$-Nat.*

Using the fact that for the natural class $\mathcal{K}$ the relation $\mathcal{K} = \mathcal{K}^{\perp\perp}$ is true, in the article [7] a characterization of natural sets is shown by the following condition:

(a_6') If $I \notin \mathcal{E}$ than there exists $J \in \mathbb{L}({}_RR)$, $J \supsetneqq I$ such that $(I:a) \notin \mathcal{E}$ for every $a \in J \setminus I$.

It is formulated in $\mathbb{L}({}_RR) \setminus \mathcal{E}$ and translating them for $\mathcal{E}$ we obtain:

(a_6'') If $I \in \mathbb{L}({}_RR)$ and for every $J \supsetneqq I$ there exists $a \in J \setminus I$ with $(I:a) \in \mathcal{E}$, then $I \in \mathcal{E}$.

From the definition of the operator $(\ )^{\perp}$ we have:

$$\mathcal{E}^{\perp\perp} = \{ I \in \mathbb{L}({}_RR) \,|\ \forall\, a \notin I,\ (I:a) \notin \mathcal{E}^{\perp} \} =$$

$$= \{ I \in \mathbb{L}({}_RR) \,|\ \forall\, a \notin I,\ \exists b \notin (I:a),\ \Big((I:a):b\Big) \in \mathcal{E} \}.$$

If $\mathcal{E}$ is a closed set, then to be natural the inclusion $\mathcal{E}^{\perp\perp} \subseteq \mathcal{E}$ which is necessary can be expressed as follows:

(a_6''') If $I \in \mathbb{L}({}_RR)$ and for every $a \notin I$ there exists $b \notin (I : a)$ such that $(I : ba) \in \mathcal{E}$, then $I \in \mathcal{E}$.

Now we can supplement Proposition 2.2 of [7] by

Proposition 2.5. *The following conditions on the set $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ are equivalent:*

1) *$\mathcal{E}$ is a natural set;*
2) *$\mathcal{E}$ satisfies the conditions (a_1), (a_2) and (a_6') [7];*
3) *$\mathcal{E}$ satisfies the conditions (a_1), (a_2) and (a_6'');*
4) *$\mathcal{E}$ satisfies the conditions (a_1), (a_2) and (a_6''').*

In conclusion we remark that in the definitions of cofilters and of stable filters (see examples 1 and 2) the condition (a_6) can be replaced by each of the conditions (a_6'), (a_6'') or (a_6''').

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