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On I-radicals

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Abstract. In this paper I-radicals are studied. Rings are characterized with the help of I-radicals. For example, each I-radical over a left perfect ring splits if and only if this ring is a direct sum of finitely many left perfect rings, the Jacobson radicals of which are maximal ideals of them.

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As usual, all rings are associative with $1 \neq 0$, all modules are unitary, $J(R)$ denotes the Jacobson radical of a ring R. The category of all left R-modules (right R-modules) will be denoted by $R-\text{Mod}$ ($\text{Mod}-R$).

A subset I of a ring R is left (right) T-nilpotent whenever for every sequence $a_1, a_2, \ldots$ in I there is an n such that $a_n \ldots a_2 a_1 = 0$ ($a_1 a_2 \ldots a_n = 0$).

A ring R is said to be left (right) perfect if $J(R)$ is right (left) T-nilpotent and $R/J(R)$ is semisimple.

A preradical r is said to be a hereditary preradical in case r is a left exact preradical.

A preradical r is said to be a hereditary torsion in case r is a left exact radical.

A hereditary torsion r of $R-\text{Mod}$ is an S-torsion if there exists a left ideal H of R satisfying the following condition $\{I \text{ is a left ideal of } R \mid I + H = R\} = \{I \text{ is a left ideal of } R \mid r(R/I) = R/I\}$ (see [8]).

It is well known that for each left (right) ideal D of R r_D is an idempotent radical of $R-\text{Mod}$ ($\text{Mod}-R$), where

$$r_D(M) = \sum\{N \mid N \text{ is a submodule of } M, DN = N\}$$

$$(r_D(M) = \sum\{N \mid N \text{ is a submodule of } M, ND = N\})$$

for every left (right) R-module M [6].

A preradical r is said to be an I-radical if $r = r_D$ for some left (right) ideal D of R.

If R is a ring, then the lattice of all I-radicals of $R-\text{Mod}$ is denoted by $\text{Ir}(l, R)$ [6].

We shall say that a preradical r of $R-\text{Mod}$ splits if for each left R-module M $r(M)$ is a direct summand of M.

Let R be a ring and let M be a right R-module. For each $m \in M$ we define the following subset of R

$$\text{Ann}_r(m) = \{x \in R \mid mx = 0\}.$$

Lemma 1. *Let I be a two-sided ideal of a ring R. Then the set of right ideals $E_I = \{T \mid T+I = R\}$ is a radical filter if and only if the set $S_I = \{a \mid a \in R,\ aR+I = R\}$ satisfies the following conditions:*

1) S_I is multiplicatively closed;

2) if $s \in S_I$ and $a \in R$ then there exist $s' \in S_I$ and $a' \in R$ such that $sa' = as'$.

Proof. E_I has a basis consisting of principal right ideals (for example, $\{aR \mid a \in S_I\}$ is a basis). Now we consider the conditions S1 − S4 [3, Proposition 15.1]. S2 − S3 are clear. To verify S1 we take into account that $1 \in S_I$. The property S4 is immediate from the fact that $st \in S_I$ implies that $s \in S_I$ [5]. □

Theorem 1. *Let I be a two-sided ideal of R and $S_I = \{a \mid a \in R, aR + I = R\}$. Then r_I is a hereditary torsion if and only if the following conditions are fulfilled:*

1) S_I is multiplicatively closed;

2) if $s \in S_I$ and $a \in R$ then there exist $s' \in S_I$ and $a' \in R$ such that $sa' = as'$;

3) for every sequence $\{a_i\}_{i=1}^{\infty}$ (where $a_i \in I$ for each $i = 1, 2, \dots$)

$$\bigcup_{i=1}^{\infty} Ann_r(a_i a_{i-1} \dots a_1) + I = R.$$

Proof. (⇒) Let I be a two-sided ideal and r_I be a hereditary torsion. Then the radical filter for r_I is the set $E_I = \{T \mid T \text{ is a right ideal of } R, T + I = R\}$. In accordance with Lemma 1 conditions 1 − 2 are fulfilled. Suppose that condition 3 does not hold true. Then there exists a sequence $\{a_i\}_{i=1}^{\infty}$ (where $a_i \in I$ for each $i = 1, 2, \dots$) such that $\bigcup\limits_{i=1}^{\infty} \mathrm{Ann}(a_i a_{i-1} \dots a_1) + I \neq R$. Let F be a free module with free basis $\{x_i\}_{i=1}^{\infty}$ and P be a submodule of F spanned by $\{x_i - x_{i+1}a_i\}_{i=1}^{\infty}$. Then $r_I(F/P) = F/P$ but the submodule $\overline{x}_1 R$ of F/P does not belong to $T(r_I)$. This contradicts the assumption that r_I is a hereditary torsion.

(⇐) Let I be a two-sided ideal of R satisfying conditions 1–3 of the Theorem. Then in accordance with Lemma 1 $E_I = \{T \mid T \text{ is a right ideal of } R, T+I = R\}$ is a radical filter. Let α is a hereditary torsion corresponding to the radical filter E_I. If $\alpha \neq r_I$ then there exists a right module N such that $r_I(N) = N$ and $\alpha(N) \neq N$. Put $M = N/\alpha(N)$. Then $M \in T(r_I)$ and $M \in F(\alpha)$. The last relation means that for every $m \in M \setminus \{0\}$ $\mathrm{Ann}_r(m)+I \neq R$. On the other hand since $M \in T(r_I)$, for every element $x \in M \setminus \{0\}$ there exist $x_i^{(1)} \in M$ and $a_i^{(1)} \in I$ $(i = 1, \dots, n_1)$ such that $x = \sum\limits_{i=1}^{n_1} x_i^{(1)} a_i^{(1)}$. At least one of the elements $x_i^{(1)} a_i^{(1)}$ $(i = 1, \dots, n_1)$ is non-zero. Suppose that $x_1^{(1)} a_1^{(1)} \neq 0$. Reasoning similarly we have that $x_1^{(1)} = \sum\limits_{i=1}^{n_2} x_i^{(2)} a_i^{(2)} \neq 0$. Hence $x_1^{(1)} a_1^{(1)} = \sum\limits_{i=1}^{n_2} x_i^{(2)} a_i^{(2)} a_1^{(1)} \neq 0$. Therefore there exists i, for example $i = 1$, such that $x_1^{(2)} a_1^{(2)} a_1^{(1)} \neq 0$. Going on we obtain the sequence $\{x_1^{(i)} a_1^{(i)} a_1^{(i-1)} \dots a_1^{(1)}\}_{i=1}^{\infty}$ of non-zero elements belonging to M, where $a_1^{(i)} \in I$ for each $i = 1, 2, \dots$. Property 3 shows that for the sequence $\{a_1^{(i)}\}_{i=1}^{\infty}$ there exists k such that $\mathrm{Ann}_r(a_1^{(k)}a_1^{(k-1)}\dots a_1^{(1)})+I=R$. Since $\mathrm{Ann}_r(x_1^{(k)}a_1^{(k)}a_1^{(k-1)}\dots a_1^{(1)})\supseteq \mathrm{Ann}_r(a_1^{(k)}a_1^{(k-1)}\dots a_1^{(1)})$, $\mathrm{Ann}_r(y)+I=R$, where $y=x_1^{(k)}a_1^{(k)}a_1^{(k-1)}\dots a_1^{(1)}\neq 0$. Thus, $0\neq y\in\alpha(M)$. It means that $M\notin F(\alpha)$. But $M\in F(\alpha)$. We have a contradiction. □

Theorem 2. *Let R be a commutative ring. Then each I-radical is a hereditary torsion if and only if $R/J(R)$ is a von Neumann regular ring and $J(R)$ is left T-nilpotent.*

Proof. ($\Leftarrow$) Let $J(R)$ be left T-nilpotent and $R/J(R)$ be a von Neumann regular ring. Since conditions 1–2 of Theorem 1 are satisfied for every commutative ring, we have to verify condition 3 of Theorem 1 for an arbitrary two-sided ideal $I\neq R$.

Let $\{a_i\}_{i=1}^{\infty}$ be any sequence such that $a_i\in I$ for each $i=1,2,\dots$. Suppose that there exist infinitely many elements a_i belonging to $J(R)$. Then taking into consideration that R is commutative and $J(R)$ is left T-nilpotent, it is obvious that $\bigcup_{i=1}^{\infty}\mathrm{Ann}_r(a_na_{n-1}\dots a_1)=R$. Hence $\bigcup_{n=1}^{\infty}\mathrm{Ann}_r(a_na_{n-1}\dots a_1)+I=R$. Therefore assume that $a_i\notin J(R)$ for any $i\geq k$, where $k\in\mathbb{N}$. Since $R/J(R)$ is a von Neumann regular ring, there exist elements $x_i\in R$, $g_i\in J(R)$ for each $i\geq k$ such that $a_i(x_ia_i-1)=g_i$. There exists $m\in N$ such that $g_m\dots g_k=0$ $(m\geq k)$ because $J(R)$ is left T-nilpotent. Hence $(g_m\dots g_k)(x_ma_m-1)\dots(x_ka_k-1)=0$. It is clear that $(x_ma_m-1)\dots(x_ka_k-1)=a\pm 1$ for some $a\in I$. Thus $\mathrm{Ann}_r(a_m\dots a_1)+I=R$.

($\Rightarrow$) Suppose that every I-radical is a hereditary torsion. Since every idempotent radical over a commutative ring corresponding torsion theory to which is cogenerated by a simple module is an I-radical ([4, Proposition 2]), every such an idempotent radical is a hereditary torsion. Therefore the idempotent radical r corresponding torsion theory to which is cogenerated by the class of all simple modules is also a hereditary torsion because it is an intersection of hereditary torsions [1, p.51]. For each maximal ideal M of R $R/M\in F(r)$. Therefore $\{R\}$ is a radical filter for r. This means that $T(r)=\{0\}$. Hence for each non-zero R-module N there exists a simple module P such that $\mathrm{Hom}_R(N,P)\neq 0$. Therefore N contains a maximal submodule. Thus every non-zero module N contains a maximal submodule. Now apply Theorem 1.8 [7]. Therefore $J(R)$ is left T-nilpotent and $R/J(R)$ is a von Neumann regular ring. □

Theorem 3. *Let R be a ring. Then the following statements are equivalent:*

(1) Every preradical of $\mathrm{Mod}-R$ *is an I-radical;*
(2) Every hereditary preradical of $\mathrm{Mod}-R$ *is an I-radical;*
(3) soc *of* $\mathrm{Mod}-R$ *is an I-radical;*
(4) R is semisimple.

Proof. (3) $\Rightarrow$ (4) Let soc of $\mathrm{Mod}-R$ be an I-radical. Then $\mathrm{soc}=r_S$ for some two-sided ideal S of R. Then $r_S(R/M)=\mathrm{soc}(R/M)=R/M$ for any maximal right ideal M of R. It follows from this that $(R/M)S=R/M$ for any maximal right ideal M of R. Hence $(S+M)/M=R/M$, i.e. $S+M=R$ for any maximal right ideal M of R. Thus $S=R$. Then $RS=RR=R$. Therefore $\mathrm{soc}(R)=R$.

(4) ⇒ (1) Let R be semisimple. Then every right R-module M is projective. Now apply Proposition 1.4.4 [1] and we have that $r(M) = Mr(R)$ for every right R-module M, where r is an arbitrary preradical of $\mathrm{Mod}-R$. It follows from this that every preradical of $\mathrm{Mod}-R$ is an I-radical.

(1) ⇒ (2). This is clear.

(2) ⇒ (3). This is clear. □

Theorem 4. *Let R be a ring. If every hereditary torsion of* $\mathrm{Mod}-R$ *is an I-radical then R is left perfect.*

Proof. If a hereditary torsion is an I-radical then it is an S-torsion [8]. Now apply Corollary 3 [8]. □

Theorem 5. *Let R be a ring satisfying the following conditions:*

$$R/J(R) \cong T_1 \times \ldots \times T_n \text{ for some simple rings}$$

$$T_1, \ldots, T_n \text{ and } J(R) \text{ is right } T\text{-nilpotent.}$$

Then the following statements are equivalent:

(A) Each I-radical splits;

(B) Each atom of the lattice $\mathrm{Ir}(l, R)$ *splits;*

(C) $R = R_1 \dot{\to} + \ldots \dot{\to} + R_n$, where $R_i/J(R_i)$ is simple for every $i \in \{1, \ldots, n\}$.

Proof. $(A) \Rightarrow (B)$ This is clear.

$(B) \Rightarrow (C)$ Assume that each atom of $\mathrm{Ir}(l, R)$ splits. By Theorems 4–5 [6], the lattice $\mathrm{Ir}(l, R)$ has n atoms $r_1, \ldots, r_n$. Then $r_i = r_{I_i}$ for every $i \in \{1, \ldots, n\}$, where I_i is an idempotent ideal (see Theorem 9 [6]). Let $i \in \{1, \ldots, n\}$. Then

$$R = r_i(R) \oplus H_i, \tag{1}$$

where H_i is a left ideal of R. By Proposition 2 [6], $r_i(R) = I_i R = I_i$. Taking into consideration (1), we have that $I_i \oplus H_i = R$. This implies

$$I_i = Re_i, \tag{2}$$

where e_i is an idempotent of R.

Therefore $\{e_1, \ldots, e_n\}$ is a set of idempotents of the ring. Let's show that all these idempotents are pairwise orthogonal. To prove this we shall show that $I_i I_j = 0$ for $i \neq j$, $i, j \in \{1, \ldots, n\}$. Really, in view of splitingness we have

$$I_j = r_i(I_j) \oplus L_{ij}, \tag{3}$$

where L_{ij} is a left ideal of R. By Proposition 2 [6]

$$r_i(I_j) = I_i I_j. \tag{4}$$

By (3)–(4),

$$I_j = I_i I_j \oplus L_{ij}. \tag{5}$$

It follows from (1), (2), (5) that

$$R = I_i I_j \oplus L_{ij} \oplus H_j. \tag{6}$$

By (6),

$$I_i I_j = Re_{ij}, \tag{7}$$

where e_{ij} is an idempotent of R.

Since r_i and r_j are atoms, $n_R = r_i \wedge r_j$, where n_R is 0 in $\mathrm{Ir}(l, R)$ [6].

Taking into account the proof of Theorem 1 [6],

$$r_i \wedge r_j = r_{I_i} \wedge r_{I_j} = r_{I_i I_j}.$$

Therefore $n_R = r_{I_i I_j}$. By Proposition 1 [6] $I_i I_j$ is right T-nilpotent. By (7), $e_{ij} \in I_i I_j$. Since $I_i I_j$ is right T-nilpotent, $e_{ij}^s = 0$ for some $s \in \mathbb{N}$. Since e_{ij} is an idempotent, $e_{ij} = e_{ij}^s$. Hence $e_{ij} = 0$. It follows from (7) that $I_i I_j = 0$. Since $e_i e_j \in I_i I_j$, $e_i e_j = 0$. We shall show that $R = I_1 + \ldots + I_n$. Since $\{r_1, \ldots, r_n\}$ is the set of atoms of $\mathrm{Ir}(l, R)$ (see [6]),

$$r_R = u_R = r_1 \vee \ldots \vee r_n = r_{I_1} \vee \ldots \vee r_{I_n} = r_{I_1 + \ldots + I_n}$$

(see proof of Theorem 1 [6]).

By Proposition 1 [6], $R = I_1 + \ldots + I_n$, i.e. $R = Re_1 + \ldots + Re_n$. Thus, since idempotents $e_1, \ldots, e_n$ are pairwise orthogonal, the set $\{e_1, \ldots, e_n\}$ is complete. Therefore we have the ring decomposition $R = I_1 \oplus \ldots \oplus I_n$.

Then

$$R/J(R) \cong I_1/J(I_1) \times \ldots \times I_n/J(I_n). \tag{8}$$

Since $R/J(R) \cong T_1 \times \ldots \times T_n$ for some simple rings $T_1, \ldots, T_n$, $R/J(R) \cong T_1 \times \ldots \times T_n$ is an indecomposable ring decomposition. It follows from (8) that $I_i/J(I_i)$ is a simple ring for each $i \in \{1, \ldots, n\}$ (see Proposition 7.8 [2]). It means that we have proved $(B) \Rightarrow (C)$.

$(C) \Rightarrow (A)$ Assume (C). Let $r \in \mathrm{Ir}(l, R)$. Then $r = r_I$ for some ideal I of R (see Remark 1 [5]). Let $\{e_1, \ldots, e_n\}$ be the set of idempotents for the decomposition $R = R_1 \oplus \ldots \oplus R_n$. Since $R_i/J(R_i)$ is simple, either $Ie_i + J(R_i) = J(R_i)$ or $Ie_i + J(R_i) = R_i$.

Set $A = \{i \in \{1, \ldots, n\} \mid Ie_i + J(R_i) = R_i\}$, $B = \{1, \ldots, n\} \setminus A$.

By Proposition 1 [6],

$$r_{Ie_i + J(R_i)} = n_R, \ \text{ if } \ i \in B; \qquad r_{Ie_i + J(R_i)} = r_{R_i}, \ \text{ if } \ i \in A.$$

Then

$$r_I = r_{Ie_1 \oplus \ldots \oplus Ie_n} = r_{Ie_1} \vee \ldots \vee r_{Ie_n} = r_{Ie_1 + J(R_1)} \vee \ldots \vee r_{Ie_n + J(R_n)} =$$

$$= \bigvee_{i \in A} r_{Ie_i + J(R_i)} \vee \bigvee_{i \in B} r_{Ie_i + J(R_i)} = \bigvee_{i \in A} r_{R_i} \vee n_R = r_{\bigoplus_{i \in A} R_i}.$$

Since $\bigoplus_{i\in A} R_i$ is an idempotent ideal of R, it follows from Proposition 2 [6] that for each left R-module M

$$r_I(M) = r_{\bigoplus_{i\in A} R_i}(M) = \Big(\oplus_{i\in A} R_i\Big)M.$$

Hence $M = r_I(M) \oplus \Big(\bigoplus_{i\in B} R_i\Big)M$. □

Corollary 1. *Let R be a left perfect ring. Then each atom of the lattice* $\mathrm{Ir}(l,R)$ *splits if and only if the ring R is a direct sum of finitely many left perfect rings, the Jacobson radicals of which are maximal ideals of them.*

Corollary 2. *Let R be a left perfect ring. Then each I-radical of* $R-\mathrm{Mod}$ *splits if and only if the ring R is a direct sum of finitely many left perfect rings, the Jacobson radicals of which are maximal ideals of them.*

References

[1] Kashu A.I. *Radicals and torsions in modules.* Kishinev, Stiinca, 1983 (in Russian).

[2] Anderson F.W., Fuller K.R. *Rings and categories of modules.* Springer-Verlag, New York, 1973.

[3] Stenström B. *Rings and modules of quotients.* Lecture Notes in Math., **237**, Springer-Verlag, New York, 1971.

[4] Jambor P. *Hereditary tensor-orthogonal theories.* Comment math. Univ. carol., 1975, **16**, N 1, p. 139–145.

[5] Horbachuk O.L., Komarnitskiy N.Y. *I-radicals and their properties.* Ukr. Matem. Zhurnal, 1978, N 2(30), p. 212–217 (in Russian).

[6] Horbachuk O.L., Maturin Yu.P. *Rings and properties of lattices of I-radicals.* Bull. Moldavian Academy of Sci. Math., 2002, N 1(38), P. 44–52.

[7] Koifman L.A. *Rings over which every module has a maximal submodule.* Mat. Zametki, 1970, **7**, N 3, p. 359–367 (in Russian).

[8] Horbachuk O.L., Maturin Yu.P. *On S-torsion theories in $R-$*Mod. Matematychni Studii, 2001, **15**, N 2, p. 135–139.

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