BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(45), 2004, Pages 67–88
ISSN 1024–7696

Discrete optimal control problems on networks and dynamic games with p players

Dmitrii Lozovanu, Stefan Pickl

Abstract. We consider a special class of discrete optimal control problems on networks. The dynamics of the system is described by a directed graph of passages. An additional integral-time cost criterion is given and the starting and final states of the system are fixed. The game-theoretical models for such a class of problems are formulated, and some theoretical results connected with the existence of the optimal solution in the sense of Nash are given. A polynomial-time algorithm for determining Nash equilibria is proposed. The results are applied to decision making systems and determining the optimal strategies in positional games on networks.

Mathematics subject classification: 90C47.
Keywords and phrases: Multiobjective discrete control, Optimal strategies in the sence Nash, dynamic Netwerks, dynamic c-game.

1 Introduction and problems formulations

In this paper we study control processes for time-discrete systems with finite set of states. The main results are concerned with game-theoretical approach to the following control problem [1, 2].

Let us consider a discrete dynamical system L with the set of states $X \subseteq \mathbb{R}^n$. At every time-step $t = 0, 1, 2, \dots$ the state of the system L is $x(t) \in X$, $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$. The dynamics of the system L is described as follows

$$x(t+1) = g_t(x(t), u(t)), \quad t = 0, 1, 2, \dots, \tag{1}$$

where $x(0) = x_0$ is the starting point of the dynamical system and $u(t) = (u_1(t), u_2(t), \dots, u_m(t)) \in \mathbb{R}^m$ represents the vector of control parameters (see). For any time-step t the feasible set $U_t(x(t))$ for the vector of control parameter $u(t)$ is given, i.e.,

$$u(t) \in U_t(x(t)), \quad t = 0, 1, 2, \dots .$$

Assume that in (1) the vector-functions

$$g_t(x(t), u(t)) = (g_t^1(x(t), u(t)),\ g_t^2(x(t), u(t)), \dots, g_t^n(x(t), u(t)))$$

are determined uniquely by $x(t)$ and $u(t)$ at every time-step $t = 0, 1, 2, \dots$. So, $x(t+1)$ is determined uniquely by $x(t)$ and $u(t)$.

We consider the following discrete optimal control problem:

Problem 1. Find T and $u(0), u(1), \dots, u(T-1)$ which satisfy the conditions

$$\begin{cases} x(t+1) = g_t(x(t), u(t)), \quad t = 0, 1, 2, \dots, T-1; \\ \qquad\qquad u(t) \in U_t(x(t)), \quad t = 0, 1, \dots, T-1, \\ x(0) = x_0, \ x(T) = x_f \end{cases}$$

and minimize the objective function

$$F_{x_0 x_f}(u(t)) = \sum_{t=0}^{T-1} c_t(x(t), g_t(x(t), u(t))),$$

where $c_t(x(t), g_t(x(t), u(t))) = c_t(x(t), x(t+1))$ represents the cost of the system's passage from the state $x(t)$ to the state $x(t+1)$ at the stage $[t, t+1]$. The vectors $v(0), v(1), \dots, v(T-1)$ generate the trajectory $x_0 = x(0), x(1), x(2), \dots, x(T) = x_f$ which transfers the system L from the starting state x_0 to the final state x_f with minimal integral-time costs (cf. [1, 2]).

This problem can be applied in decision making systems where the dynamics of the systems are controlled by one person. Here, we formulate the game theoretic approach to this problem, i.e., we consider that the dynamics of the system is controlled by p actors (players) and it is described as follows

$$x(t+1) = g_t(x(t), u^1(t), u^2(t), \dots, u^p(t)), \quad t = 0, 1, 2, \dots \ , \tag{2}$$

where $x(0) = x_0$ is the starting state of system L and $u^i(t) \in \mathbb{R}^{m_i}$ represents the vector of control parameters of player i, $i = \overline{1,p}$, i.e. $i \in \{1, \dots, p\}$. The state $x(t+1)$ of the system L at time-step $t+1$ is obtained uniquely if the state $x(t)$ at time-step t is known and the players $1, 2, \dots, p$ fix their vectors of control parameters $u^1(t), u^2(t), \dots, u^p(t)$, respectively. For each $i = \overline{1,p}$, the admissible sets $U_t^i(x(t))$ for the vectors of control parameters $u^i(t)$ are given, i.e.

$$u^i(t) \in U_t^i(x(t)), \quad i = \overline{1,p}, \quad t = 0, 1, 2, \dots \ ,$$

We shall assume that the sets $U_t^i(x(t)) \quad i = \overline{1,p}$, $t = 0, 1, 2, \dots$, are non-empty and $U_t^i(x(t)) \cap U_t^j(x(t)) = \emptyset$ for $i \neq j$, $t = 0, 1, 2, \dots$.

Let us consider that the players $1, 2, \dots, p$ fix their vectors of control parameters

$$u^1(t), u^2(t), \dots, u^p(t); \ t = 0, 1, 2, \dots,$$

respectively, and the starting state $x_0 = x(0)$ and final state x_f of the system L are known. Then for a given set of vectors $u^1(t), u^2(t), \dots, u^p(t)$ either a unique trajectory

$$x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$$

from x_0 to x_f exists and $T(x_f)$ represents the time step when the state x_f is reached by L, or such a trajectory does not exist.

The game version of our control problem is the following.

Problem 2. We denote by

$$F^i_{x_0x_f}(u^1(t),u^2(t),\dots,u^p(t))=\sum_{t=0}^{T(x_f)-1} c^i_t(x(t),g_t(x(t),u^1(t),u^2(t),\dots,u^p(t)))$$

the integral-time cost of system's passage from x_0 to x_f for the players i, $i=1,2,\dots,p$, if the set of vectors $u^1(t),u^2(t),\dots,u^p(t)$ generates a trajectory

$$x_0=x(0),x(1),x(2),\dots,x(T(x_f))=x_f$$

from x_0 to x_f and

$$u^i(t)=U^i(x(t)),\ t=0,1,2,\dots,T(x_f).$$

Otherwise we put

$$F^i_{x_0x_f}(u^1(t),u^2(t),\dots,u^p(t))=\infty.$$

Here

$$c^i_t(x(t),g_t(x(t),u^1(t),u^2(t),\dots,u^p(t)))=c^i_t(x(t),x(t+1))$$

represents the cost of system's passage from the state $x(t)$ to state $x(t+1)$ at the stage $[t,t+1]$. We consider the problem of finding vectors of control parameters

$$u^{1^*}(t),u^{2^*}(t),\dots,u^{i-1^*}(t),\ u^{i^*}(t),u^{i+1^*}(t),\dots,u^{p^*}(t)$$

which satisfy the following condition

$$F^i_{x_0x_f}(u^{1^*}(t),u^{2^*}(t),\dots,u^{i-1^*}(t),\ u^{i^*}(t),u^{i+1^*}(t),\dots,u^{p^*}(t))\le$$

$$\le F^i_{x_0x_f}(u^{1^*}(t),u^{2^*}(t),\dots,u^{i-1^*}(t),\ u^{i}(t),u^{i+1^*}(t),\dots,u^{p^*}(t))$$

$$\forall\, u^i(t)\in\mathbb{R}^{m_i},\ i=\overline{1,p}.$$

So, we consider the problem of finding the solution in the sense of Nash [3].

In order to determine the existence of Nash equilibria for multiobjective control in problem 2 we assume that players i and j never actively control the system at the same state in time, although for different moments of time different players may control the system at same state. This condition in the game model corresponds to the case when for any moment of time t and for an arbitrary state $x(t)\in X$ the application in (2) depends only on one of the vectors of control parameters $u^i(t)$, $i\in\{1,2.\dots.p\}$. The multiobjective control problem with a such condition allows us to regard it as a dynamic noncooperative game on a network which consists of p interacting subnetworks controlled by different players. On this network the problem is considered to control the given initial vertex (state) toward some prescribed final state. It is assumed that the costs on edges of the network depend on time, i.e. depend on order an edge is visited in the directed path from starting state to final one. We are seeking for a Nash equilibrium in this dynamic game. Polynomial-time algorithms for determining the optimal strategies of players are proposed.

This problem generalizes the classical control problems with integral-time cost criterion by a trajectory and arose as auxiliary one when solve dynamic games in positional form [4 – 10]. The main tool we shall use for studying and solving our problem is based on dynamic programming and concept of noncooperative games in positional form. A such approach for determining the optimal strategies in dynamic games for the case with constant cost functions on edges of the network has been used in [8]. Here we extend this approach for the general case of the problem.

2 Discrete optimal control problems and dynamic games with p players on networks

In this section we consider the discrete optimal control problem on networks. We formulate the game variant of the problem when the dynamics of the system is described by a directed graph of passages [4 − 6]. The graph vertices in this problem correspond to the states of the system, where the edges identify the possibility of the system to pass from one state to another. Moreover, the cost functions are associated to the edges of the graph which depend on time and express the cost of the system's passages. The graph of passages, on which edges time-depending cost functions are defined, and in which two vertices corresponding to the starting and the final states of the system are chosen, is called a *dynamic network* [6]. First, we formulate the discrete optimal control problem on dynamic networks, and then we shall extend the model according to a game-theoretical approach.

2.1 The discrete optimal control problem on networks

Let L be a dynamical system with a finite set of states X, $|X| = N$, and at every discrete moment of time $t = 0, 1, 2, \ldots$ the state of the system L is $x(t) \in X$. Note, that we associate $x(t)$ with an abstract element (in Section 1, $x(t)$ represents a vector from $\mathbb{R}^n$). Two states x_0 and x_f are chosen in X, where x_0 is a the starting point of the system L, $x_0 = x(0)$, and x_f is the final state of the system, i.e., x_f is the state in which the system must be brought. The dynamics of the system is described by a directed graph of passages $G = (X, E), |E| = m$, an edge $e = (x, y)$ which signifies the possibility of passages of the system L from the state $x = x(t)$ to the state $y = x(t + 1)$ at any moment of time $t = 0, 1, 2, \ldots$. This means that the edges $e = (x, y) \in E$ can be regarded as the possible values of the control parameter $u(t)$ when the state of the system is $x = x(t)$, $t = 0, 1, 2, \ldots$. The next state $y = x(t + 1)$ of the system L is determined uniquely by $x = x(t)$ at the time-step t and an edge $e = (x, y) \in E(x)$, where $E(x) = \{X \,|\, (x, y) \in E\}$. So $E(x) = E\big(x(t)\big)$ corresponds to the admissible set $U_t\big(x(t)\big)$ for the control parameter $u(t)$ at every time-step t. To each edge $e = (x, y)$ a function $c_e(t)$ is assigned, which reflects the costs of system's passage from the state $x(t) = x \in X$ to the state $x(t + 1) = y \in X$ at any time-step $t = 0, 1, 2, \ldots$ We consider the discrete optimal control problem on networks [1, 2, 7] for which the sequence of system's passages $(x(0),\, x(1)),\ (x(1),\, x(2)),\ \ldots,\ (x(T(x_f) - 1),\, x(T(x_f))) \in E$, which transfers the system L from $x_0 = x(0)$ to $x_f = x(T(x_f))$ with minimal integral-time cost of the passages by a trajectory $x_0 = x(0), x(1), x(2), \ldots, x(T(x_f)) = x_f$. Here, we distinguish the following variants of the problem:

1) the number of the stages (time $T(x_f)$) is fixed, i.e. $T(x_f) = T$;
2) for $T(x_f)$ is given the restriction $T(x_f) \in [T_1, T_2]$, where T_1 and T_2 are known;
3) the parameter $T(x_f)$ is unknown and must be found.

2.2 A dynamic programming approach and computational complexity

Let us assume that $T(x_f)$ is fixed, i.e. $T(x_f) = T$ (case 1). Denote by

$$F_{x_0 x_f}(T) = \min_{x_0=x(0),x(1),\dots,x(T)=x_f} \sum_{t=0}^{T-1} c_{(x(t),x(t+1))}(t)$$

the minimal integral-time cost of system's passages from x_0 to x_f. If the state x_f couldn't be reached by using T stages, then we put $F_{x_0 x_f}(T) = \infty$. For $F_{x_0 x(t)}(t)$ the following recursive formula can be gained

$$F_{x_0 x(t)}(t) = \min_{x(t-1) \in X_G^-(x(t))} \Big\{ F_{x_0 x(t-1)}(t-1) + c_{(x(t-1),x(t))}(t-1) \Big\},$$

where $X_G^-(y) = \{x \in X \mid e = (x,y) \in E\}$. It is easy to observe that using dynamical programming methods we could tabulate the values $F_{x_0 x(t)}(t)$, $t = 1, 2, \dots, T$ $(F_{x_0 x(0)}(0) = 0)$. So, if T is fixed, then the problem can be solved in time $O(N^2 T)$ (Here we do not take in consideration the number of operations for calculations of the value of functions $c_e(t)$ for given t.)

In the case when $T(x_f) \in [T_1, T_2]$ the problem can be reduced to $T_2 - T_1 + 1$ problems with $T(x_f) = T_1$, $T(x_f) = T_1 + 1$, $T(x_f) = T_1 + 2, \dots, T(x_f) = T_2$, respectively; compare the minimal integral-costs of these problem we find the best one.

Case 3) of the problem can be reduced to case 2) if we find T_1 and T_2 such that $T(x_f) \in [T_1, T_2]$. It is obvious that for positive and non-decreasing cost functions $c_e(t)$, $e \in E$, we have $T(x_f) \in [1, N-1]$, i.e., $T_1 = 1, T_2 = N - 1$. Therefore, the problem for positive and non-decreasing functions on edges can be solved in time $O(N^3)$.

2.3 A game theoretic approach for the discrete optimal control problem on networks

Now, we consider the game-theoretical versions of the problem from Section 2.1. First we formulate the stationary case of the problem and then we extend it to nonstationary one.

2.3.1 The problem of determining the optimal stationary strategies of players in dynamic c-game

Let $G = (X, E)$ be a directed graph of system's passages, where G has the property that for any vertex $x \in X \setminus \{x_f\}$ there exists a leaving edge $e = (x, y) \in$

E. Assume that the vertex set X is divided into p disjoint subsets $X_1, X_2, \dots, X_p$ $(X = \bigcup_{i=1}^{p} X_i,\ X_i \cap X_j = \emptyset, i \neq j)$ and consider vertices $x \in X_i$ as the positions of player i, $i = \overline{1,p}$. Moreover, we consider that to each edge $e = (x, y) \in E$ of the graph of passages p functions $c_e^1(t), c_e^2(t), \dots, c_e^p(t)$ are assigned, where $c_e^i(t)$ expresses the cost of system's passage from the state $x = x(t)$ to the state $y = x(t+1)$ at the stage $[t, t+1]$ for player i. Define the stationary strategies of players $1, 2, \dots, p$ as maps $s_1, s_2, \dots, s_p$ on $X_1, X_2, \dots, X_p$, respectively:

$$\begin{array}{ll} s_1 : x \mapsto y \in X_G(x) & \text{for } x \in X_1 \setminus \{x_f\}; \\ s_2 : x \mapsto y \in X_G(x) & \text{for } x \in X_2 \setminus \{x_f\}; \\ \dots\dots\dots\dots\dots\dots & \\ s_p : x \mapsto y \in X_G(x) & \text{for } x \in X_p \setminus \{x_f\}, \end{array}$$

where $X_G(x)$ is the set of extremals of edges $e = (x, y)$, starting in x, i.e., $X_G(x) = \{y \in X \mid e = (x, y) \in E\}$. For a given set of strategies $s = (s_1, s_2, \dots, s_p)$ denote by $G_s = (X, E_s)$ the subgraph generated by the edges $e = (x, s_i(x))$ for $x \in X \setminus \{x_f\}$ and $i = \overline{1,p}$. Then, in G_s for every vertex $x \in X \setminus \{x_f\}$ there exists a unique directed edge $e = (x, y) \in E_s$, originating in x. Obviously, for fixed $s_1, s_2, \dots, s_p$ either a unique directed path $P_s(x_0, x_f)$ from x_0 to x_f exists in G_s or such a path does not exist in G_s. In the second case, if we pass through the edges from x_0 we get a unique directed cycle C_s.

For fixed strategies $s_1, s_2, \dots, s_p$ and fixed states x_0 and x_f define the quantities

$$F^1_{x_0x_f}(s_1, s_2, \dots, s_p),\ F^2_{x_0x_f}(s_1, s_2, \dots, s_p), \dots,\ F^p_{x_0x_f}(s_1, s_2, \dots, s_p)$$

in the following way. We assume that the path $P_s(x_0, x_f)$ does exist in G_s. Then it is unique and we can assign to its edges numbers $0, 1, 2, 3, \dots, k_s$, starting with the edge that begins in x_0. These numbers characterize the time steps $t_e(s_1, s_2, \dots, s_p)$ when the system passes from one state to another, if the strategies $s_1, s_2, \dots, s_p$ are applied. In this case, we put

$$F^i_{x_0x_f}(s_1, s_2, \dots, s_p) = \sum_{e \in E(P_s(x_0, x_f))} c_e^i(t_e(s_1, s_2, \dots, s_p)), \quad i = \overline{1,p},$$

where $E(P_s(x_0, x_f))$ is the set of edges of the path $P_s(x_0, x_f)$. The set of vertices $x_0 = x(0), x(1), \dots, x(k) = x_f$ in the path $P_s(x_0, x_f)$ represents the trajectory generated by the strategies $s_1, s_2, \dots, s_p$ of the players. If there are no directed paths $P_s(x_0, x_f)$ from x_0 to x_f in H_s, then we put

$$F^i_{x_0x_f}(s_1, s_2, \dots, s_p) = +\infty, \quad i = \overline{1,p}.$$

Problem formulation – the dynamic c-game. We consider the problem of finding the maps $s_1^*, s_2^*, \dots, s_p^*$ for which the following condition is satisfied

$$\begin{array}{l} F^i_{x_0x_f}(s_1^*, s_2^*, \dots, s_{i-1}^*, s_i^*, s_{i+1}^*, s_p^*) \le F^*_{x_0x_f}(s_1^*, s_2^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, s_p^*), \\ \forall s_i,\ i = \overline{1,p}. \end{array} \qquad (3)$$

So, we study the problem of finding the optimal solution in the sense of Nash [3] on $S_1 \times S_2 \times \cdots \times S_p$, where $S_i = \{s_i\colon x \mapsto y \in X_G(x) \text{ for } x \in X_i\}$, $i = \overline{1,p}$.

The functions

$$F^1_{x_0x_f}(s_1, s_2, \ldots, s_p),\ F^2_{x_0x_f}(s_1, s_2, \ldots, s_p), \ldots,\ F^p_{x_0x_f}(s_1, s_2, \ldots, s_p)$$

on $S_1 \times S_2 \times \cdots \times S_p$ define a game in the normal form with p players [3, 7].

In positional form, this game is defined by the graph G, partitions $X_1, X_2, \ldots, X_p$, vector-functions $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \ldots, c^i_{e_m}(t))$, $i = \overline{1,p}$, starting and final positions x_0, x_f. We call this game the *dynamic c-game* with p players on networks $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$.

2.3.2 The nonstationary dynamic c-game

We define the nonstationary strategies of players on network as maps

$$u_1\colon (x,t) \to (y,t+1) \in X_G(x) \times \{t+1\} \quad \text{for } x \in X_1 \setminus \{x_f\},\ t = 0,1,2,\ldots,T-1;$$
$$u_2\colon (x,t) \to (y,t+1) \in X_G(x) \times \{t+1\} \quad \text{for } x \in X_2 \setminus \{x_f\},\ t = 0,1,2,\ldots,T-1;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_p\colon (x,t) \to (y,t+1) \in X_G(x) \times \{t+1\} \quad \text{for } x \in X_p \setminus \{x_f\},\ t = 0,1,2,\ldots,T-1,$$

were T is given. Here (x,t) has the same sense as the notation $x(t)$, i.e. $(x,t) = x(t)$.

For any set of strategies $u_1, u_2, \ldots, u_p$ we define the quantities

$$\overline{F}^1_{x_0x_f}(u_1, u_2, \ldots, u_p), \overline{F}^2_{x_0x_f}(u_1, u_2, \ldots, u_p), \ldots, \overline{F}^p_{x_0x_f}(u_1, u_2, \ldots, u_p)$$

in the following way.

Let us consider that $u_1, u_2, \ldots, u_p$ generate in G a trajectory $x_0 = x(0), x(1), x(2), \ldots, x(T(x_f)) = x_f$ from x_0 to x_f where $T(x_f)$ represents the time-moment when x_f is reached. Then we set

$$\overline{F}^i_{x_0x_f}(u_1, u_2, \ldots, u_p) = \sum_{t=0}^{T(x_f)-1} c_{(x(t),x(t+1))}(t),\ i = \overline{1,p}, \quad \text{if } \ T(x_f) \le T;$$

otherwise we put $\overline{F}^i_{x_0x_f}(u_1, u_2, \ldots, u_p) = \infty$, $i = \overline{1,p}$.

We regard the problem of finding the nonstationary strategies $u_1^*, u_2^*, \ldots, u_p^*$ for which the following condition is satisfied

$$\overline{F}^i_{x_0x_f}(u_1^*, u_2^*, \ldots, u_{i-1}^*, u_i^*, u_{i+1}^*, \ldots, u_p^*) \le$$
$$\le \overline{F}^i_{x_0x_f}(u_1^*, u_2^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_p^*), \quad \forall\ u_i,\ i = \overline{1,p}.$$

So we consider the nonstationary case of determining Nash equilibria in dynamic c-game.

In the following we show that the nonstationary case of the problem can be reduced to stationary one.

3 Preliminaries and some results on determining the optimal strategies in dynamic c-games

The dynamic c-game was introduced in [9] as auxiliary problem for studying and solving a special class of positional games on networks – cyclic games [4–7]. The main results from [9–11] are related to the existence of Nash equilibria for zero-sum games on networks with constant cost functions on edges. The dynamic c-games with p players for constant cost functions on edges have been studied in [8], and the following result is given.

Theorem 1. *Let $(G,\, X_1,\, X_2,\, \dots,\, X_p,\; c^1(t), c^2(t),\, \dots,\, c^p(t),\, x_0,\, x_f)$ be a network where G has the property that the vertex x_f is attainable from x_0. If the components $c^i_{e_j}(t)$ of the vectors $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \dots, c^i_{e_m}(t)), i = \overline{1,p}$, are positive constant functions then there exists the optimal solution in the sense of Nash $s^*_1, s^*_2, \dots, s^*_p$ for stationary dynamic c-game on network $(G,\, X_1,\, X_2,\, \dots,\, X_p,\; c^1(t), c^2(t),\, \dots,\, c^p(t),\, x_0,\, x_f)$.*

On the basis of this theorem we can prove the following result.

Theorem 2. *Let $(G,\, X_1,\, X_2,\, \dots,\, X_p,\; c^1(t), c^2(t),\, \dots,\, c^p(t),\, x_0,\, x_f)$ be a network where G contains at least a directed path with not more than T edges and the cost functions on edges are positive. Then for dynamic c-game on network $(G,\, X_1,\, X_2,\, \dots,\, X_p,\; c^1(t), c^2(t),\, \dots,\, c^p(t),\, x_0,\, x_f)$ there exist the optimal nonstationary strategies in the sense of Nash $u^*_1(t), u^*_2(t), \dots, u^*_p(t)$.*

Proof. It is sufficient to show that the nonstationary dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$ can be reduced to stationary dynamic c-game on an auxiliary network $(\overline{G}, Y_1, Y_2, \dots, Y_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), \overline{y}_0, \overline{y}_f)$ with constant cost functions on edges. In this auxiliary network the graph $\overline{G} = (Y, \overline{E})$ is obtained from G as follows. The vertex set Y is obtained from X when it is doubled $T+1$ times, i.e. $Y = X \times \{0,1,2,\dots,T\} = X \times \{0\} \cup X \times \{1\} \cup \dots \cup X \times \{T\}$. Each two subsets $X \times \{t\}$ and $X \times \{t+1\}$, $t = \overline{0, T-1}$, are connected with directed edges $\overline{e} = ((x,t),(y,t+1)) \in \overline{E}$ if $e = (x,y) \in E$. In addition each subset $X \times \{t\}$, $t = \overline{0, T-1}$ is connected with the set $X \times \{T\}$ by directed edges $\overline{e} = ((x,t),(y,T)) \in \overline{E}$ if $e = (x,y) \in E$.

The partition $Y = Y_1 \cup Y_2 \cup \dots \cup Y_p$ which determine the position sets of players and the cost functions $\overline{c}_{\overline{e}}(t)$ on edges $\overline{e} = ((x,t),(y,t+1)) \in \overline{E}$, $\overline{e} = ((x,t),(y,T)) \in \overline{E}$ are defined as follows:

$$Y_i = \{(x,t) \in E \,|\, x \in X_i,\ t = \overline{0,T}\}, \quad i = \overline{1,p};$$

$$\overline{c}_{\overline{e}}(t) = c_e(t), \quad \text{if} \quad e = (x,t) \in \overline{E},\ t = \overline{0, T-1}.$$

On this network we consider the dynamic c-game with starting position $\overline{y}_0 = (x_0, 0) \in X \times \{0\}$ and final position $(y_f, T) \in X \times \{T\}$.

It is easy to observe that for each subset $X \times \{t\}$ the time t is known. Therefore the cost functions $\overline{c}_{\overline{e}}(t)$ on auxiliary acyclic network may be considered constant functions. According to Theorem 1 for dynamic c-game on network $(G, X_1, X_2, \ldots .X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$ there exists the optimal by Nash solution. This solution represents the optimal by Nash strategies for nonstationary case of the problem. □

On the basis of constructive proof of Theorem 2 we may propose the following algorithm for finding the optimal nonstationary strategies of players on network $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$. We construct the auxiliary network $(\overline{G}, Y_1, Y_2, \ldots .Y_p, \overline{c}^1(t), \overline{c}^2(t), \ldots, \overline{c}^p(t), \overline{y}_0, \overline{y}_f)$ and solve the problem of finding the optimal stationary strategies of players with constant cost functions on edges (see algorithms from [8]). The obtained solution on this network corresponds to optimal nonstationary strategies of players for dynamic c-game on $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$.

It is easy to observe also that if Nash equilibria for stationary case of dynamic c-game exist then the mentioned above construction with $T = N$ can be used for determining the optimal stationary strategies of players on network $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$ when the cost functions on edges are nondecreasing. A such approch for determining the optimal stationary strategies of players is developed in Sections 5 and 6.

For some practical problems may be useful also the following variant of the dynamic c-game with backward time-step account.

Let $P_s(x_0, x_f)$ be the directed path from x_0 to x_f in H_s generated by strategies $s_1, s_2, \ldots, s_p$ of players $1, 2, \ldots, p$. In 2.3.1 for an edge $e \in E(P_s(x_0, x_f))$ is defined the time-step $t_e(s_1, s_2, \ldots, s_p)$ as an order of the edge in the path $P_s(x_0, x_f)$ starting with o from x_0. To each edge $e \in E(P_s(x_0, x_f))$ we may associate also the backward time-step account $t'_e(s_1, s_2, \ldots, s_p)$ if we start number the edges with 0 from end position x_f in inverse order, i.e. $t'_e(s_1, s_2, \ldots, s_p) = k_s - t_e(s_1, s_2, \ldots, s_p)$. For fixed strategies $s_1, s_2, \ldots, s_p$ we define the quantities

$$\overline{F}^1_{x_0x_f}(s_1, s_2, \ldots, s_p), \overline{F}^2_{x_0x_f}(s_1, s_2, \ldots, s_p), \ldots, \overline{F}^p_{x_0x_f}(s_1, s_2, \ldots, s_p)$$

in the following way. We put

$$\overline{F}^i_{x_0x_f}(s_1^*, s_2^*, \ldots, s_p) = \sum_{e \in E(P_s(x_0, x_f))} t'_e(s_1, s_2, \ldots, s_p), \ i = \overline{1, p};$$

if in H_s there exists a path $P_s(x_0, x_f)$ from x_0 to x_f; otherwise we put

$$\overline{F}^i_{x_0x_f}(s_1, s_2, \ldots, s_p) = \infty, \ i = \overline{1, p}.$$

So, we obtain a new game on network. In the case when the costs $c_e^i(t)$ are constant this problem coincides with the problem from [8]. This game can be regarded as dual problem for the dynamic c-game from 2.3.1.

4 A discrete optimization principle for dynamic networks and an algorithm for solving the problem in the case $\mathbf{p=1}$

In this section, we consider the formulated problem on networks in the special case $p = 1$. We have introduced the problem in this case for positive and non-decreasing cost functions $c_e(t)$ on edges $e \in E$ which coincides with the discrete optimal control problem on G with starting states x_0 and final state x_f. Therefore, the optimal trajectory $x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$ corresponds in G to the directed path $P^*(x_0, x_f)$ from x_0 to x_f. We call this path the optimal path for the dynamic network. For the path $P^*(x_0, x_f)$ contains no more then $N-1$ edges, the problem can be solved in finite time by using dynamical programming techniques. We show that a more effective algorithm for solving this problem can be elaborated if the dynamical network satisfies the following conditions:

Problem formulation. Let us assume that the cost functions $c_e(t)$, $e \in E$, in the dynamic network have the following property. If $P^*(x_0, x)$ is an arbitrary optimal path from x_0 to x which can be represented as $P^*(x_0, x) = P_1^*(x_0, y) \cup P_2^*(y, x)$, where $P_1^*(x_0, y)$ and $P_2^*(y, x)$ have no common edges, then a leading part $P_1^*(x_0, y)$ of the path $P^*(x_0, x)$ is also an optimal path of the problem in G with given starting state x_0 and final state y. If such a property holds, then we say that for the dynamic network the optimization principle is satisfied. In the case, when on network the cost functions $c_e(t), e \in E$, are positive functions and the optimization principle is satisfied, the following algorithm determines all optimal paths $P^*(x_0, x)$ from x_0 to each $x \in X$, which correspond to the optimal strategies in the problem for $p = 1$.

Algorithm 1

Preliminary step (Step 0). Set $Y = \{x_0\}$, $E^* = \emptyset$. Assign to every vertex $x \in X$ two labels $t(x)$ and $F(x)$ as follows:

$$t(x_0) = 0, \quad t(x) = \infty, \quad \forall\, x \in X \setminus \{x_0\};$$
$$F(x_0) = 0, \quad F(x) = \infty, \quad \forall\, x \in X \setminus \{x_0\}.$$

General step (Step k). Find the set

$$E' = \{(x', y') \in E(Y) \,|\, F(x') + c_{(x',y')}(t(x')) = \min_{x \in Y} \min_{y \in \overline{X}(x)} \{F(x) + c_{(x,y)}(t(x))\},$$

where

$$E(Y) = \{(x, y) \in E \,|\, x \in Y,\ y \in X \setminus Y\}, \quad \overline{X}(x) = \{y \in X \setminus Y \,|\, (x, y) \in E(Y)\}.$$

Find the set of vertices $X' = \{y' \in X \setminus Y | \, (x', y') \in E'\}$. For every $y' \in X'$ select one edge $(x', y') \in E'$ and build the union $\overline{E}'$ of such edges. After that change the labels $t(y')$ and $F(y')$ for every vertex $y' \in X'$ as follows

$$t(y') = t(x') + 1, \quad F(y') = F(x') + c_{(x',y')}(t(x')), \quad \forall \ (x', y') \in \overline{E}'.$$

Replace the set Y by $Y \cup X'$ and E^* by $E^* \cup \overline{E}'$. Note $X^k = Y,\ E^k = E^*$. If $X^k \neq X$ then fix the tree $H^k = (X^k, E^k)$ and go to the next step $k+1$, otherwise fix the tree $H = (X, E^*)$ and STOP.

Note that the tree $H = (X, E^*)$ contains optimal paths from x_0 to each $x \in X$. After k steps of the algorithm the tree $H^k = (X^k, E^k)$ represents a part of H. If it is necessary to find the optimal path from x_0 to x_f, then the algorithm can be interrupted after k steps as soon as the condition $x_f \in X^k$ is satisfied, i.e., in this case the condition $X^k \neq X$ in the algorithm must be replaced by $x_f \in X^k$. The labels $F(x)$, $x \in X$, indicate the costs of optimal paths from x_0 to $x \in X$ and $t(x)$ represents the number of edges in these paths.

The correctness of the algorithm is based on the following theorem:

Theorem 3. *Let $(G, c(t), x_0, x_f)$ be a dynamic network, where the vector-function $c(t) = (c_{e_1}(t), c_{e_2}(t), \ldots, c_{e_m}(t))$ has positive and bounded components for $t \in [0, N-1]$. Moreover, let us assume that the optimization principle on the dynamic network is satisfied. Then the tree $H^k = (X^k, E^k)$ obtained after k steps of the algorithm gives the optimal paths from x_0 to every $x \in X^k$ which correspond to optimal strategies in the problem for $p = 1$.*

Proof. We prove the theorem by using the induction principle on the number of steps k of the algorithm. In the case when $k = 0$ the assertion is evident.

Let us assume that the theorem holds for any $k \leq r$ and let us show that it is true for $k = r+1$. If $H^r = (H^r, E^r)$ is the tree obtained after r steps and $H^{r+1} = (X^{r+1}, E^{r+1})$ is the tree obtained after $r+1$ steps of the algorithm, then $X^\circ = X^{r+1} \setminus X^r$ and $E^\circ = E^{r+1} \setminus E^r$ represents the vertex set and edge set obtained by the algorithm at the step $r+1$. Let us show that if y' is an arbitrary vertex of X°, then in H^{r+1} the unique directed path $P^*(x_0, y')$ from x_0 to y' is optimal. Indeed, if this is not the case, then there exists an optimal path $Q(x_0, y')$ from x_0 to x', which does not contain the edge $e = (z', y') \in E^\circ$. The path $Q(x_0, y')$ can be represented as $Q(x_0, y') = Q^1(x_0, x') \cup \{(x', y)\} \cup Q^2(y, y')$, where x' is the last vertex of the path $Q(x_0, y')$ belonging to X^r when we pass from x_0 to y'. It is easy to observe that if the conditions of the theorem hold then

$$\text{cost}\,(Q(x_0, y')) \geq \text{cost}\,(P^*(x_0, y')),$$

where

$$\text{cost}\,(Q(x_0, y')) = \sum_{t=0}^{m_Q} c_{e_t}(t),$$

$e_0, e_1, \ldots, e_{m_Q}$ are the corresponding edges of the directed path $Q(x_0, y')$ when we pass from x_0 to y' and

$$\text{cost}\,(P^*(x_0, y')) = \sum_{t=0}^{m_p} c_{e'_t}(t),$$

were $e'_0, e'_1, \ldots, e'_{m_p}$ are the corresponding edges of the directed path $P^*(x_0, y')$ when we pass from x_0 to y'. This means that the tree $H^{r+1} = (X^{r+1}, E^{r+1})$ contains an optimal path from x_0 to every $y' \in X^{r+1}$. □

Remark 1. *In analoguos way can be proposed the algorithm for solving the problem with backward time-step account in the case when the optimization principle on network is satisfied. Here the optimization principle should be defined as follows: every part* $P_2^*(y, x_f)$ *of an arbitrary optimal path* $P^*(x, x_f) = P_1^*(x, y) \cup P_2^*(y, x_f)$ $(E(P_1^*(x, y)) \cap E(P_2^*(x, y)) = \emptyset)$ *is optimal one.*

Algorithm 1 is an extension of Dijkstra's Algorithm. Furthermore, such an algorithm we develop for the dynamic c-game with p players in the case when the optimization principle is satisfied with respect to each player. In the next section, we define the optimization principle on dynamic networks $(G, X_1, X_2, \ldots, X_n, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$ with respect to player i.

5 The optimization principle for dynamic networks with p players and determining Nash equilibria for stationary case of the problem

In this section we extend the optimization principle for stationary case of the problem on dynamic networks with p players. We define the optimization principle with respect to player i, $i \in \{1, 2, \ldots, p\}$, on dynamic networks $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$.

We denote by E^i the subset of edges from E starting in verteces $x \in X_i$, i.e., $E^i = \{(x, y) \in E \,|\, x \in X_i\}$, $i = \overline{1, p}$. Hereby, the set E^i represents the admissible set of system's passages from the states $x \in X_i$ to the state $y \in X$ for the player i. Furthermore, the set E^i indicates the set of edges of player i. By E_{s_i} we denote the subset of E generated by a fixed strategy s_i of player $i, i \in \{1, 2, \ldots, p\}$, i.e., $E_{s_i} = \{(x, y) \in E^i \,|\, x \in X_i, y = s_i(x)\}$.

Let $s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p$ be the set of strategies of the players $1, 2, \ldots, i-1, i+1, \ldots, p$ and let $G_{S \setminus s_i} = (X, E_{S \setminus s_i})$ be the subgraph of G, where

$$E_{S \setminus s_i} = E_{s_1} \cup E_{s_2} \cup \cdots \cup E_{s_i - 1} \cup E^i \cup E_{s_i + 1} \cup \cdots \cup E_{s_p}.$$

The graph $G_{S \setminus s_i}$ represents the subgraph of G generated by the set of edges of player i and edges of E when the players $1, 2, \ldots, i-1, i+1, \ldots, p$ fix their strategies $s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p$, respectively. On $G_{S \setminus s_i}$ we consider the single objective control problem with respect to cost functions $c_e^i(t)$ of player i, starting vertex x_0 and final vertex $x \in X$.

Definition 1. *Let us assume that for any given set of strategies*

$$s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p$$

the cost functions $c_e^i(t)$, $e \in E_{S \setminus s_i}$ *in* $G_{S \setminus s_i}$ *have the property that if an arbitrary optimal path* $P^*(x_0, x)$ *can be represented as* $P^*(x_0, z) = P_1^*(x_0, z) \cup P_2^*(z, x)$ ($P_1^*(x_0, z)$ *and* $P_2^*(z, x)$ *have no common edges), then the leading part* $P_1^*(x_0, z)$ *of* $P^*(x_0, x)$ *is an optimal one. We call this property the optimization principle for dynamic networks with respect to player* i.

Note that if $c^i_e(t)$, $i = \overline{1,p}$, $e \in E$ are constant positive functions then the optimization principle for dynamic c-game is valid. It is easy to observe that in the case when

$$c^i_e(t) = f^i(t), \ \ i = \overline{1,p},\, e \in E, \qquad (4)$$

where $f^1(t), f^2(t), \dots, f^p(t)$ are arbitrary positive and non-decreasing functions, the optimization principle for dynamic c-game is also satisfied with respect to each player. If the dynamic network has the structure of a graph without directed cycles then $f^1(t), f^2(t), \dots, f^p(t)$ in (4) may be arbitrary non-decreasing functions. In the case when G has the structure of a k-partite directed graph without directed cycles, the optimization principle is satisfied for arbitrary positive cost functions.

Theorem 4. *Let $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$ be the dynamic network with p players for which the vertex x_f in G is attainable from x_0 and for any vertex $x \in X$ there exists an edge $e = (x, y) \in E$. Assume that the vector-functions $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \dots, c^i_{e_N}(t)), i = \overline{1,p}$, have positive and nondecreasing components. Moreover, let us assume that the optimization principle on the dynamic network is satisfied with respect to each player. Then, in the dynamic c-game on networks $(G, X_1, X_2, \dots, X_p,\ c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$ for the players $1, 2, \dots, p$ there exists an optimal solution in the sense of Nash $s^*_1, s^*_2, \dots, s^*_p$.*

Proof. Let us regard the auxiliary network $(\overline{G}, Y_1, Y_2, \dots, Y_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), \overline{y}_0, \overline{y}_f)$ from Section 3 when $T = N$ (see the proof of Theorem 2). As we have already noted for the dynamic c-game on this network there exist the optimal by Nash stationary strategies $\overline{s}^*_1, \overline{s}^*_2, \dots, \overline{s}^*_p$ which generate in $\overline{G} = (Y, \overline{E})$ a trajectory $\overline{y}_0 = (x_0, 0), (x_1, 1), (x_2, 2), \dots, (x_{T(x_f)}, T(x_f)) = \overline{y}_f$ from $\overline{y}_0$ to $\overline{y}_f$. The construction given below shows that $x_0, x_1, \dots, x_{T(x_f)} = x_f$ correspond to a trajectory generated by an optimal stationary strategies $s^*_1, s^*_2, \dots, s^*_p$ for dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$. The stationary strategies $s^*_1, s^*_2, \dots, s^*_p$ can be obtained from $\overline{s}^*_1, \overline{s}^*_2, \dots, \overline{s}^*_p$ as follows.

Algorithm 2

Preliminary step (Step 0). Set $W^0 = \{(x_0, 0), (x_1, 1), \dots, (x_{T(x_f)}, T(x_f))\}$, and $X^0 = \{x_0, x_1, \dots, x_{T(x_f)}\}$. For every $x_t \in X$, $t = \overline{0, T(x_f)}$, we put $s^*_i(x_t) = x_{t+1}$ if $x_t \in X_i$, $i \in \{1, 2, \dots, p\}$.

General step (Step k). If $X^{k-1} = X$ then STOP; otherwise we find the set

$$W_{\overline{s}^*}(X^{k-1}) = \{(x,t) \in (X \setminus X^{k-1}) \times \{1,2,\dots,N\} \,\big|\, \overline{s}^*(x,t) \in W^{k-1}$$

$$\text{for } (x,t) \in Y_i,\ i = \overline{1,p}\}.$$

If $W_{\overline{s}^*}(X^{k-1}) = \emptyset$ then for every $x \in X \setminus X^{k-1}$ we put $s^*_i(x) = z$ where $\overline{s}^*_i(x,t) = (z, t+1)$ with $(x,t) \in Y_i$ and minimal t, $i \in \{1, 2, \dots, p\}$. In the case $W_{\overline{s}^*}(X^{k-1}) \neq \emptyset$ we find a vertex $(x', t') \in W_{\overline{s}^*}(X^{k-1})$ with a minimal t' for given x'. Then we form the sets $W^k = W^{k-1} \cup \{x', t'\}$, $X^k = X^{k-1} \cup \{x'\}$ and go to next step.

It is easy to observe that if the condition of the theorem holds then the verteces $x_0, x_1, \dots, x_{T(x_f)}$ are different and the stationary strategies $s_1^*, s_2^*, \dots, s_p^*$ represent the optimal solution in the sense of Nash for dynamic c-game on $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$. □

So the optimal by Nash solution for stationary case of the problem in the a case when the conditions of Theorem 4 hold can be find in the following way. We construct the auxiliary network $(\overline{G}, Y_1, Y_2, \dots, Y_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), \overline{y}_0, \overline{y}_f)$ we find the optimal stationary strategies on this network by using the algorithm from [8]. Then we apply algorithm 2 and find optimal stationary strategies $s_1^*, s_2^*, \dots, s_p^*$.

6 Tree of optimal paths in dynamic c-game

In [11] is shown that if the cost functions c_e^i, $i = \overline{1,p}$, on edges $e \in E$ are constant and the final position x_f in G is attainable from each $x \in X$ then there exist the optimal strategies $s_1^*, s_2^*, \dots, s_p^*$ such that the graph $H_{S^*} = (X, E_{S^*})$ generated by these strategies has the structure of a directed tree with sink vertex x_f. Moreover $s_1^*, s_2^*, \dots, s_p^*$ represent the solution of the dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x, x_f)$ with an arbitrary starting position $x \in X$ and final position x_f. This means that optimal strategies of players for considered case does not depend on starting position $x_0 \in X$. In general case for arbitrary cost function on edges the optimal strategies of players depend on starting position x_0.

Let us consider the dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$ for which the optimization principle is satisfied with respect to each player and the cost function on edges are non-decreasing functions. We show that if every vertex $x \in X$ in G is attainable from x_0 then there exists a tree $H^* = (X, E^*)$ with root vertex x_0 such that H^* gives all optimal paths $P_{H^*}(x_0, x)$ from x_0 to $x \in X$. A unique directed path $P_{H^*}(x_0, x)$ from x_0 to an arbitraty $x \in X$ in H^* corresponds to a solution $s_1^*, s_2^*, \dots, s_p^*$ of the game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f)$ with starting position x_0 and final position x. But for different verteces x and y the directed paths $P'_{H^*}(x)(x_0, x)$ and $P''_{H^*}(x_0, y)$ in H^* correspond to different optimal strategies of players $\overline{s}_1^*, \overline{s}_2^*, \dots, \overline{s}_p^*$ and $\overline{\overline{s}}_1^*, \overline{\overline{s}}_2^*, \dots, \overline{\overline{s}}_p^*$ in different dynamic c-games with starting vertex x_0 and final positions x, y, respectively.

Theorem 5. *Let $(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots c^p, x_0, x_f)$ be the dynamic network with p players for which in G any vertex $x \in X$ is attainable from x_0 and vector-functions $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \dots, c^i_{e_m}(t))$, $i = \overline{1,p}$, have non-negative and non-decreasing components. Moreover, let us consider that the optimization principle for the dynamic network is satisfied with respect to each player. Then, in G there exists a tree $H^* = (X, E^*)$ for which any vertex $x \in X$ is attainable from x_0, and a unique directed path $P_{H^*}(x_0, x_f)$ from x_0 to x in H^* corresponds to an optimal strategies $s_1^*, s_2^*, \dots, s_p^*$ of players in dynamic c-games on network*

$(G, X_1, X_2, \ldots, X_p, c^1, c^2, \ldots, c^p, x_0, x)$ *with starting position* x_0 *and final position* x. *For different verteces* x *and* y *the optimal paths* $P'_{H^*}(x_0, x)$ *and* $P''_{H^*}(x_0, y)$ *correspond to different strategies of players* $\overline{s}_1^*, \overline{s}_2^*, \ldots, \overline{s}_p^*$ *and* $\overline{\overline{s}}_1^*, \overline{\overline{s}}_2^*, \ldots, \overline{\overline{s}}_p^*$ *in different games with starting vertex* x_0 *and final positions* x, y, *respectively.*

Proof. According to Theorem 4 in G for any vertex $x \in X$ there exists the optimal path $P_{s^*}(x_0, x)$ from x_0 to x which corresponds to the optimal strategies of the players in the dynamic c-game with starting position x_0 and final position x. Let us select all vertices $x \in X$ for which optimal paths in G contain not more than one edge. Obviously, the graph $H^1 = (X^1, E^1)$ generated by these paths has the structure of a directed tree with root vertex x_0. If $X^1 = X$ the assertion is proved. If $X^1 \neq X$, then we select the vertices $x \in X \setminus X^1$ for which there exist optimal paths $P_{H^*}(x_0, x)$ from x_0 to x which contain two edges. According to Lemma 1 each of the paths $P^1_{H^*}(x_0, y)$, representing the part of the optimal paths $P_{H^*}(x_0, x)$ in G is an optimal one. Therefore, each of the optimal paths $P_{H^1}(x_0, x)$ can be regarded as the path which contains one part of the paths $P^1_{H^*}(x_0, y)$. If we add to H^1 the last edges of the optimal paths $P_{H^*}(x_0, x)$ with vertices x we obtain the tree $H^2 = (X^2, E^2)$ with root vertex x_0. In H^2 any directed path $P_{H^2}(x_0, x)$ from x_0 to x is an optimal path. If $X^2 = X$, the theorem is proved. If $X^2 \neq X$, then select the vertices $x \in X$ for which there exist optimal paths $P_{H^*}(x_0, x)$ from x_0 to x containing three edges. In an analogous way, we find the tree $H^3 = (X^3, E^3)$ and so on. In a finite number of steps, we find the tree $H^q = (X^q, E^q)$ for which $X^q = X$ and for any vertex $x \in X$ the unique directed path $P_{H^q}(x_0, x)$ from x_0 to x in H^q is an optimal one. □

Note that if for the problem with backward time-step account we define optimization principle in analogous way then the optimal strategies of players in this game satisfy the same property as the optimal stationary strategies of players in the problem with constant cost functions on edges. This means that there exist the optimal stationary strategies $s_1^*, s_2^*, \ldots, s_p^*$ such that the graph $H_{s^*} = (X, E_{s^*})$ generated by these strategies has the structure of a directed graph with sink vertex x_f. So, $s_1^*, s_2^*, \ldots, s_p^*$ represent the solution of dynamic c-game on network $(G, X_1, X_2, \ldots, X_p, c^1, c^2, \ldots c^p, x, x_f)$ with an arbitrary starting position x and final position x_f.

7 The algorithm for determining the tree of optimal paths in a dynamic c-game on acyclic networks

Assumption. We regard a dynamic c-game with p players and assume that in G any vertex $x \in X$ is attainable from x_0. Moreover, let us assume that the optimization principle is satisfied with respect to each player and the functions $c_e^i(t)$, $e \in E$, $i = \overline{1,p}$, have positive and non-decreasing components.

We propose an algorithm for determining the tree of optimal paths $H^* = (X, E^*)$ when G has no directed cycles, i.e. G is an acyclic graph. We assume that the positions of the network are numbered with $0, 1, 2, \ldots, N-1$ according to partial order determined by the structure of acyclic graph G. This means that if $y > x$ then there is directed path $P(y,x)$ from x to y. The algorithm consists of N steps and construct a sequence of trees $H^k = (X^k, E^k)$, $k = \overline{0, N-1}$, such that at the final step $k = N-1$ we obtain $H^{N-1} = H^*$.

Algorithm 3

Preliminary step (step 0). Set $H^\circ = (X^\circ, E^\circ)$, where $X^\circ = \{x_0\}$, $E^\circ = \emptyset$. Assign to every vertex $x \in X$ a set of labeles $F^1(x), F^2(x), \dots, F^p(x), t(x)$ as follows:

$$\begin{array}{ll} F^i(x_0) = 0, & i = \overline{1,p}, \\ F^i(x) = \infty, & \forall x \in X \setminus \{x_0\},\ i = \overline{1,p}, \\ t(x_0) = 0, & \\ t(x) = \infty, & \forall x \in X \setminus \{x_0\}. \end{array}$$

General step (step k). Find in $X \setminus X^{k-1}$ the least vertex x^k and the set of incoming edges $E^-(x^k) = \{(x^r, x^k) \in E \,|\, x^r \in X^{k-1}\}$ for x^k. If $|E^-(x^k)| = 1$ then go to a); otherwise go to b):

a) Find a unique vertex y such that $e' = (y, x^k) \in E^-(x^k)$ and calculate

$$\begin{array}{l} F^i(x^k) = F^i(y) + c^i_{(y,x^k)}(t(y)), \quad i = \overline{1,p}; \\ t(x^k) = t(y) + 1. \end{array} \tag{5}$$

After that form the sets $X^k = X^k \cup \{x^k\}$, $E^k = E^{k-1} \cup \{x^k\}$ and put $H^k = (X^k, E^k)$. If $k < N-1$ then go to the next step $k+1$; otherwise fix $E^* = E^{N-1}$, $H^* = (X, E^*)$ and stop.

b) Select the greatest vertex $z \in X^{k-1}$ such that in graph $H^k = \Big(X^{k-1} \cup \{x^k\},\ E^{k-1} \cup E^-(x^k)\Big)$ there exist at least two parallel directed paths $P'(z, x^k)$, $P''(z, x^k)$ from z to x^k without common edges, i.e. $E(P'(z,x^k)) \cap E(P''(z,x^k)) = \emptyset$. Let $e' = (x^r, x^k)$ and $e'' = (x^s, x^k)$ be the respective edges of these paths with common end vertex in x^k. So, $e', e'' \in E^-(x^k)$. For the vertex z we determine i_z such that $z \in X_{i_z}$.

If

$$F^{i_z}(x^r) + c^{i_z}_{(x^r,x^k)}(t(x^r)) \le F^{i_z}(x^s) + c^{i_z}_{(x^s,x^k)}(t(x^s))$$

then we delete the edge $e'' = (x^s, x^k)$ from $E^-(x^k)$ and from G; otherwise we delete the edge $e' = (x^r, x^k)$ from $E^-(x^k)$ and from G. After that check again the condition $|E^-(x^k)| = 1$? If $|E^-(x^k)| = 1$ then go to a) otherwise go to b). □

Remark 2. *The values $F^i(x)$, $i = \overline{1,p}$, for $x \in X$ in the algorithm 3 express the respective costs of the players in dynamic c-game with starting position x_0 and final position x.*

Note that the version of the problem with backward time-step account can be solved using algorithm 3. This algorithm finds the optimal strategies of players and constructs the tree of optimal paths $H_{s^*} = (X, E_{s^*})$ with sink vertex x_f if the optimization principle is satisfied with respect to each players. We explain the algorithm for the case of acyclic networks.

Algorithm 4

Preliminary step (step 0). Set $\overline{H}^\circ = (\overline{X}^\circ, \overline{E}^\circ)$, where $\overline{X}^\circ = \{x_f\}$, $\overline{E}^\circ = \emptyset$. Assign to every vertex a set of labels $\overline{F}^1(x), \overline{F}^2(x), \dots, \overline{F}^p(x), t'(x)$ as follows:

$$\begin{array}{ll} \overline{F}^i(x_f) = 0, & i = \overline{1,p}, \\ \overline{F}^i(x) = \infty, & \forall x \in X \setminus \{x_f\},\ i = \overline{1,p}, \\ t'(x_f) = 0, & \\ t'(x) = \infty, & \forall x \in X \setminus \{x_f\}. \end{array}$$

General step (step k). Find a vertex $y^k \in X \setminus X^k$ which satisfies the condition

$$X^+(y^k) \subseteq X^{k-1},$$

where $X^+(y^k) = \{x \in X \,|\, (x^k, y^k) \in E\}$. Denote $E^+(y^k) = \{(y^k, x) \in E \,|\, x \in X^+(y^k)\}$ and select an edge (y^k, x^k) which satisfies the condition

$$\overline{F}^{i_k}(y^k) + c^{i_k}_{(y^k,x^k)}(t'(x^k)) = \min_{x^k \in X^+(y^k)} \{\overline{F}^{i_k}(y^k) + c^{i_k}_{(y^k,x)}(t'(x))\} \text{ if } y^k \in X_{i_k}.$$

After that we calculate

$$\begin{array}{l} \overline{F}^i(y^k) = \overline{F}^i(y^k) + c^i_{(y^k,x^k)}(t'(x^k)), \quad i = \overline{1,p}, \\ t'(y^k) = t'(x^k) + 1. \end{array}$$

Form the sets $\overline{X}^k = \overline{X}^{k-1} \cup \{y^k\}$, $\overline{E}^k = \overline{E}^{k-1} \cup \{(y^k, x^k)\}$ and put $H^* = (X^k, E^k)$. If $k < N - 1$ then go to the next step $k + 1$; otherwise fix $E^* = \overline{E}^{n-1}$, $H_{s^*} = (X, E_{s^*})$ and stop.

Remark 3. *The values $\overline{F}^i(x)$, $i = \overline{1,p}$, for $x \in X$ in the algorithm 4 express the respective costs of players in the dynamic c-game with starting position x and final position x_f.*

Example

Let us consider the stationary dynamic c-game of two players. The game is determined by the network given in Fig. 2. This network consists of directed graph $G = (X, E)$ with partition $X = X_1 \cup X_2$, $X_1 = \{0, 2, 5, 6\}$, $X_2 = \{1, 3, 4\}$, starting position $x_0 = 0$, final position $x_f = 6$ and the cost functions of players 1 and 2 given in paranthesis in Fig. 1.

It is easy to check that the optimization principle for this dynamic network is satisfied with respect to each player. Therefore if we use algorithm 2 we obtain:

Step 0: $H^\circ = (\{0\}, \emptyset)$; $X^\circ = \{0\}$; $E^\circ = \emptyset$; $F^1(0) = 0$; $F^2(0) = 0$; $t(0) = 0$; $F^i(x) = \infty$ for $x \neq 0, i = 1, 2$; $t(x) = \infty$ for $x \neq 0$.

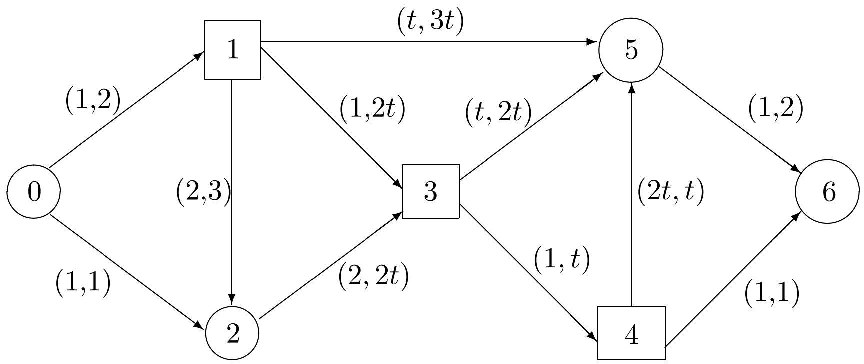

Fig. 1

Step 1: $x^1 = 1$; $E^-(1) = \{(0,1)\}$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$; $t(0) = 0$; $t(1) = 1$; $F^i(x) = 0$ for $x \neq 0,1$ and $i = 1,2$; $t(x) = \infty$ for $x \neq 0,1$.

After step 1 we have $H^1 = (\{0,1\}, \{(0,1)\})$, i.e. $X^1 = \{0,1\}$, $E^1 = \{0,1\}$.

Step 2: $x^2 = 2$; $E^-(2) = \{(0,2), (1,2)\}$. Since $E^-(2) \neq 1$ we have the case b): $z = 0$; $P'(0,2) = \{(0,2)\}$, $P''(0,2) = \{(0,1), (0,2)\}$; $e^1 = (0,2)$; $e'' = (1,2)$ and $i_z = 1$ because $0 \in X_1$. For e' and e'' the following condition holds:

$$F^1(0) + c^1_{(0,2)}(0) \leq F^1(1) + c^1_{(1,2)}(1), \quad \text{i.e.} \quad 1 \leq 2.$$

Therefore we delete (1,2) from $E^-(2)$ and obtain $E^-(2) = \{(0,2)\}$ (case a)). We calculate $F^1(2) = F^1(0) + c^1_{(0,2)}(0) = 1$; $F^2(2) = F^2(0) + c^2_{(0,2)}(0) = 1$;

$t(2) = t(0) + 1 = 1$.

After step 2 we obtain $H^2 = (\{0,1,2\}, \{(0,1), (0,2)\})$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$; $F^1(2) = 1$; $F^2(2) = 1$; $F^i(x) = \infty$ for $x \neq 0,1,2$, $i = 1,2$;

$t(0) = 0$; $t(1) = 1$; $t(2) = 1$; $t(x) = \infty$ for $x \neq 0,1,2$.

Step 3: $x^3 = 3$; $E^-(3) = \{(1,3), (2,3)\}$. So, we have $E^-(3) \neq 1$ (case b): $z = 0$; $P'(0,3) = \{(0,1), (1,3)\}$; $P''(0,3) = \{(0,2), (2,3)\}$; $e^1 = (1,3)$; $e'' = (2,3)$; $i_z = 0$. For e' and e'' we have

$$F^1(1) + c^1_{(1,3)}(1) \leq F^1(2) + c^1_{(2,3)}(1)$$

Therefore we delete (2,3) from $E^-(3)$. So, $E^-(3) = \{(1,2)\}$ and $F^1(3) = F^1(1) + c^1_{(1,3)}(1) = 1 + 1 = 2$; $F^2(3) = F^2(1) + c^2_{(1,3)}(1) = 2 + 2 = 4$; $t(3) = t(1) + 1 = 2$.

We delete the $H^3 = (\{0,1,2,3\}, \{(0,1), (0,2), (1,3)\})$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$; $F^1(2) = 1$; $F^2(2) = 1$; $F^1(3) = 2$; $F^2(3) = 4$; $F^i(x) = \infty$ for $x \in \{4,5,6\}$, $i = 1,2$;

$t(0) = 1$; $t(1) = 1$; $t(2) = 1$; $t(3) = 2$; $t(4) = t(5) = t(6) = \infty$.

Step 4: $x^4 = 4$; $E^-(3) = \{(3,4)\}$. Therefore we obtain $H^4 = (\{0,1,2,3,4\}, \{(0,1), (0,2), (1,3), (3,4)\})$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$;

$F^1(2) = 1$; $F^2(2) = 1$; $F^1(3) = 2$; $F^2(3) = 4$; $F^1(4) = 3$; $F^2(4) = 8$; $F^1(5) = \infty$; $F^2(5) = \infty$; $F^1(6) = \infty$; $F^2(6) = \infty$;

$t(0) = 0$; $t(1) = 1$; $t(2) = 1$; $t(3) = 2$; $t(4) = 3$; $t(5) = t(6) = \infty$.

Step 5: $x^5 = 5$; $E^-(3) = \{(1,5), (3,5), (4,5)\}$. Since $E^-(3) = 3$ we have case b): $z = 3$; $P'(3,5) = \{(3,5)\}$; $P''(3,5) = \{(3,4), (4,5)\}$; $e' = (3,5)$; $e'' = (4,5)$; $i_z = 2$. For e' and e'' we have

$$F^2(3) + c^2_{(3,5)}(2) \leq F^2(4) + c^2_{(4,5)}(3).$$

We delete the edge (4,5) from $E^-(4)$ and obtain $E^-(4) = \{(1,5),(3,5)\}$. For the edges $e' = (1,5)$ and $e'' = (3,5)$ we find $z = 1$, $i_z = 2$ and the paths $P'(1,5)$, $P''((1,3), (3,5))$. Since the following condition holds:

$$F^2(1) + c^2_{(1,5)}(1) \leq F^2(3) + c^2_{(3,5)}(2).$$

We delete the edge (3,5) from $E^-(4)$ and obtain $E^-(4) = \{(1,5)\}$. So

$$F^1(5) = F^1(1) + c^1_{(1,5)}(1) = 2;\ F^2(5) = F^2(1) + c^2_{(1,5)}(1) = 5;\ t(5) = t(1) + 1 = 2.$$

After step 5 we obtain $H^5 = (\{0,1,2,3,4,5\}, \{(0,1), (0,2), (1,3), (3,4), (1,5)\})$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$; $F^1(2) = 1$; $F^2(2) = 1$; $F^1(3) = 2$; $F^2(3) = 4$; $F^1(4) = 3$; $F^2(4) = 8$; $F^1(5) = 2$; $F^2(5) = 5$; $F^1(6) = \infty$; $F^2(6) = \infty$;

$t(0) = 0$; $t(1) = 1$; $t(2) = 1$; $t(3) = 2$; $t(4) = 3$; $t(5) = 2$.

Step 6: $x^6 = 6$; $E^-(6) = \{(4,6), (5,6)\}$; $E^-(6) \neq 1$ we have case b): $z = 1$; $P'(1,6) = \{(1,5), (5,6)\}$; $P''(1,6) = \{(1,3), (3,4), (4,6)\}$; $e' = (5,6)$; $e'' = (4,6)$; $i_7 = 2$. For e' and e'' the following condition holds:

$$F^2(5) + c^2_{(5,6)}(2) \leq F^2(4) + c^2_{(4,6)}(3).$$

We delete (4,6) from $E^-(4)$. Therefore we obtain

$$F^1(6) = F^1(5) + c^1_{(5,6)}(2) = 3;\ F^2(6) = F^2(5) + c^2_{(5,6)}(2) = 7;\ t(6) = t(5) + 1 = 3.$$

Finally we obtain $H^6 = H^* = (\{0,1,2,3,4,5,6\}, \{(0,1), (0,2), (1,3), (3,4), (1,5), (5,6)\})$; $F^1(0) = 0$; $F^2(0) = 0$; $F^1(1) = 1$; $F^2(1) = 2$; $F^1(2) = 1$; $F^2(2) = 1$; $F^1(3) = 2$; $F^2(3) = 4$; $F^1(4) = 3$; $F^2(4) = 8$; $F^1(5) = 2$; $F^2(5) = 5$; $F^1(6) = 3$; $F^2(6) = 7$;

$t(0) = 0$; $t(1) = 1$; $t(2) = 1$; $t(3) = 2$; $t(4) = 3$; $t(5) = 2$; $t(6) = 3$.

So, for the dynamic c-game on our network we obtain the tree of optimal paths given in Fig. 2. For the case of dynamic c-game with backward time-step account we obtain the tree of optimal paths given in Fig. 3.

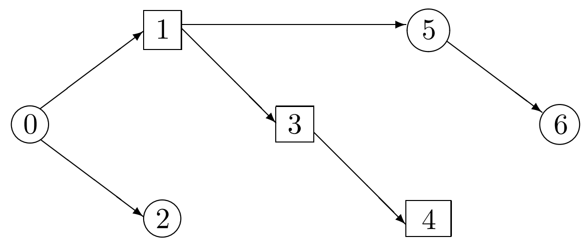

Fig. 2

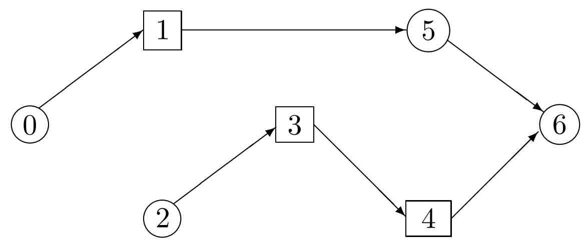

Fig. 3

7.1 Computational complexity and correctness of the algorithm

Note that if for a given $e \in E$, $t \in \{0, 1, 2, \dots, N-1\}$ and $i \in \{1, 2, \dots, p\}$ the value $c_e^i(t)$ can be calculated in time k then the algorithm determines the tree of optimal paths $H^* = (X, E^*)$ in time $O(pN^3k)$. Indeed, each general step of the algorithm requires $O(pN^2k)$ operations and the maximal number of the steps is $N-1$. So the computational complexity of the algorithm is $O(pN^3k)$.

The correctness of algorithm 2 can be proved in the same way as the correctness of algorithm 1 if we use the induction principle on number of steps k. In the case $k = 1$ the correctness of the algorithm is evident. Assume that algorithm 2 finds the tree optimal paths for $k = r$ and let us show that it finds the tree of optimal paths for the case $k = r + 1$. Denote by $H^r = (X^r, E^r)$ the tree obtained after r steps and by $H^{r+1} = (X^{r+1}, E^{r+1})$ denote the tree obtained after $r + 1$ steps. So, $X' = X^{r+1} \setminus X^r$, $E' = E^{r+1} \setminus E^r$.

Let x^{r+1} be the vertex from X' and consider the stationary dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x^{r+1})$ with starting position x_0 and final position x^{r+1}.

According to induction principle each path from x_0 to $y \in X^r$ in H^r is optimal one. This means that to reach a position $y \in X$ with the best costs for the players in the game each player should pass through the edges of the unique directed path from x_0 to y in H^r. But to reach the vertex $x^{r+1} \in X'$ from x_0 with the best costs for the players in the game each player should pass through the edge $e' = (y, x^{r+1}) \in E'$ as soon vertex y is reached. A such best solution is well-provid by conditions of the algorithm in the case when the optimization principle on network is satisfied with respect to each player.

So, a directed path $P_{H^{r+1}}(x_0, x^{r+1})$ in the $H^{r+1} = (X^{r+1}, E^{r+1})$ corresponds to a optimal solutions $s_1^*, s_2^*, \ldots, s_p^*$ in dynamic c-game.

8 Generalizations

For our problem formulated in Section 3 we have assumed that the time of passages $\tau_{x,y}$ of the system L from one state $x \in X$ to another state $y \in X$ for every $(x, y) \in E$ is equal to 1. It is easy to observe that all the results of the paper can be extended to the problem in general case where different edges (x, y) and (x', y') of the graph of passages G may have different time of passages $\tau_{xy}, \tau_{x'y'}$ and each of them may be different from 1. Theorems 1–5 for the problem in general case hold, too, and can be proved analogously. Therefore, if we replace in the algorithm the relation in (5) by the following relation

$$t(y') = t(x') + \tau_{x'y'},$$

then we obtain an algorithm for solving the problem in general form. The computational complexity of the algorithm in this general case is also $O(pN^3k)$ operations, where k is the running-time for the calculation of the value $c_e^i(t)$ for given i, e and t.

A more general mathematical model of a dynamic c-game may be obtained when the positions of the players are changing in time, i.e. for any moment of time $t = 0, 1, 2, \ldots$, the partitions $X = X_1(t) \cup X_2(t) \cup \cdots \cup X_p(t)$ $(X_i(t) \cap X_j(t) = \emptyset,\ i \neq j)$ are given. Using the dynamical decomposition of the network from [6, 12] this problem can be reduced to the problem formulated in Section 2.

All formulated problems in the paper may be studied also in the cases when the optimality criterion is considered in the sense of Pareto [12]. The results can be seen in a wider sense as a continuation of [13] regarding game-theoretical approaches on networks [14, 15] including global structures for such problems [16].

Note that some generalizations of routing and flow problems by using game-theoretical approach have been used in [17, 18]. But the generalizations from [17, 18] is not related to dynamic games in positional form.

References

[1] Bellman R., Kalaba R. *Dynamic Programming and Modern Control Theory.* Academic Press, New York and London, 1965.

[2] Krabs W., Pickl S. *Analysis, controlability and optimization of time-discrete systems and dynamical games.* Springer, 2003.

[3] Nash J.F. *Non cooperative games.* Annals of Math., 1951, 2, N 1, p. 286–295.

[4] Gurvitch V.A., Karzanov A.V., Khatchiyan L.G. *Cyclic games: Finding min-max mean cycles in digraphs.* J. Comp. Mathem. and Math., Phys., 1988, **28(9)**, p. 1407–1417.

[5] Lozovanu D.D. *Extremal-combinatorial problems and algorithms for their solving.* Kishinev, Stiinta, 1991.

[6] Lozovanu D.D. *Dynamic games with p players on networks.* Bul. Acad. de Ştiinţa a Rep. Moldova, Ser. Matematica, 2000, N 1(32), p. 41–54.

[7] Moulin H. *Theorie de Jeux pour l'Economie et la Politique.* Hermann, Paris, 1981.

[8] Boliac R., Lozovanu D., Solomon D. *Optimal paths in network games with p players.* Discrete Applied Mathematics, 2000, **99**, N 1–3, p. 339–348.

[9] Lozovanu D.D. *Algorithms for solving some network minimax problems and their applications.* Cybernetics, 1991, **1**, p. 70–75.

[10] Lozovanu D.D, Trubin V.A. *Minimax path problem on network and an algorithm for its solving.* Discrete Mathematics and Applications, 1994, **6**, p. 138–144.

[11] Lozovanu D.D. *A strongly polynomial time algorithm for finding minimax paths in networks and solving cyclic games.* Cybernetics and System Analysis, 1993, **5**, p. 145–151.

[12] Pickl S. *Convex Games and Feasible Sets in Control Theory.* Mathematical Methods of Operations Research, 2001, **53**, N 1, p. 51–66.

[13] Leitmann G. *Some extensions of a direct optimization method.* Journal of Optimization Theory and Applications, 2001, **111**, p. 1–6.

[14] Van den Nouweland A., Maschler M., Tijs S. *Monotonic Games are spanning network games.* International Journal of Game Theory, 2001, **21**, p. 419–427.

[15] Granot D., Hamers H., Tijs S. *Spanning network games.* International Journal of Game Theory, 1998, **27**, p. 467–500.

[16] Weber G.-W. *Optimal control theory: On the global structure and connection with optimization.* Journal of Computational Technologies, 1999, **4**, N 2, p. 3–25.

[17] Altman E., Basar T., Srikant R. *Nash quilibria for combined flow control and routing in network: asymptotic behavior for a large number of users.* IEEE Transactions on automatic control, 2002, **47**, N 6, p. 917–929.

[18] Boulogne T., Altman E., Kameda H., Pourtallier O. *Mixed equilibrium for multiclass routing games.* IEEE Transactions on automatic control. Special issue on control issues in telecommunication networks, 2002, **47**, N 6, p. . 903–916.

Received August 16, 2004

Dmitrii Lozovanu
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str.
Chişinău, MD−2028
Moldova
E-mail: *lozovanu@math.md*

Stefan Pickl
Institute of Mathematics
Center of Applied Computer Science
University of Cologne
E-mail: *pickl@zpr.uni-koeln.de*