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Kojalovich Method and Studying Abel's Equation with the one known solution

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Abstract. The problem of constructing a general solution for the Abel's equation of the special kind with a known partial solution is considered.

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1 Introduction

It is known that there are some classes of differential equations that are not integrable in quadratures but become integrable if some its partial solution has been found. As an example, let us consider the following Abel's equation of the second kind [1, 2]

$$yy' - y = r(x), \tag{1}$$

where $y = y(x)$ is an unknown function and $r(x)$ is some known function which will be determined below. Equation (1) is connected closely with many problems of physics, mechanics, chemistry, biology, ecology and other [1]. Some differential equations which are reduced to Abel equation are considered in [1–3].

In order to solve this equation we shall use the following special method which was developed in the textbook of Kojalovich [4].

2 Kojalovich's method

Let a function f be an integrating function of equation (1). Then, according to [4], it satisfies the following equality

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\ \frac{y+r}{y} + \frac{\partial f}{\partial \alpha_i} + \frac{\alpha_i + r}{\alpha_i} = \psi(x,y)\ \omega(x,\alpha_i), \tag{2}$$

where α_i $(i = \overline{1,n})$ are partial solutions of equation (1).

Instead of f let us consider new integrating function $F(x,y,\alpha_i)$ of the form

$$F(x,y,\alpha_i) = f(x,y,\alpha_i) + \lambda_1(x)\ \frac{\partial f}{\partial \alpha_i} + \lambda_2(x)\ \frac{\partial^2 f}{\partial \alpha_i^2}, \tag{3}$$

where $\lambda_1(x),\ \lambda_2(x)$ are unknown functions. The function F is obviously a superposition of functions of x. So its derivative may be written as

$$\frac{dF}{dx}=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\,\frac{d\,y}{d\,x}+\frac{\partial F}{\partial \alpha_i}\,\frac{d\,\alpha_i}{d\,x}=$$

$$=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\,\frac{y+r(x)}{y}+\frac{\partial F}{\partial \alpha_i}\,\frac{\alpha_i+r(x)}{\alpha_i}=$$

$$=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\,\frac{y+r(x)}{y}+\frac{\partial f}{\partial \alpha_i}\,\frac{\alpha_i+r(x)}{\alpha_i}\;+\lambda_1'(x)\,\frac{\partial f}{\partial \alpha_i}+$$

$$+\lambda_1'(x)\left(\frac{\partial^2 f}{\partial x\,\partial \alpha_i}+\frac{\partial^2 f}{\partial y\,\partial \alpha_i}\,\frac{y+r(x)}{y}+\frac{\partial^2 f}{\partial {\alpha_i}^2}\,\frac{\alpha_i+r(x)}{\alpha_i}\right)+$$

$$+\lambda_2'(x)\,\frac{\partial^2 f}{\partial {\alpha_i}^2}+\lambda_2(x)\left(\frac{\partial^3 f}{\partial x\,\partial {\alpha_i}^2}+\frac{\partial^3 f}{\partial y\,\partial {\alpha_i}^2}\,\frac{y+r(x)}{y}+\frac{\partial^3 f}{\partial {\alpha_i}^3}\,\frac{\alpha_i+r(x)}{\alpha_i}\right),\tag{4}$$

because of (1)

$$y'=\frac{y+r(x)}{y},\quad \alpha_i'=\frac{\alpha_i+r(x)}{\alpha_i}.$$

Differentiating equation (2) by α_i we write

$$\frac{\partial^2 f}{\partial x\,\partial \alpha_i}+\frac{\partial^2 f}{\partial y\,\partial \alpha_i}\,\frac{y+r(x)}{y}+\frac{\partial^2 f}{\partial {\alpha_i}^2}\,\frac{\alpha_i+r(x)}{\alpha_i}=\psi(x,y)\,\frac{\partial \omega(x,\alpha_i)}{\partial \alpha_i}+\frac{r(x)}{{\alpha_i}^2}\,\frac{\partial f}{\partial \alpha_i},\tag{5}$$

$$\frac{\partial^3 f}{\partial x\,\partial {\alpha_i}^2}+\frac{\partial^3 f}{\partial y\,\partial {\alpha_i}^2}\,\frac{y+r(x)}{y}+\frac{\partial^3 f}{\partial {\alpha_i}^3}\,\frac{\alpha_i+r(x)}{\alpha_i}=$$

$$=\psi(x,y)\,\frac{\partial^2 \omega(x,\alpha_i)}{\partial {\alpha_i}^2}+2\frac{r(x)}{{\alpha_i}^2}\,\frac{\partial^2 f}{\partial {\alpha_i}^2}-2\frac{r(x)}{{\alpha_i}^3}\,\frac{\partial f}{\partial \alpha_i}.\tag{6}$$

Using relations (5) and (6) we rewrite (4) in the form

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\,\frac{y+r(x)}{y}+\frac{\partial F}{\partial \alpha_i}\,\frac{\alpha_i+r(x)}{\alpha_i}=\psi(x,y)\;\omega(x,\alpha_i)+\lambda_1'(x)\frac{\partial f}{\partial \alpha_i}+$$

$$+\lambda_1(x)\left(\psi(x,y)\frac{\partial \omega(x,\alpha_i)}{\partial \alpha_i}+\frac{r(x)}{{\alpha_i}^2}\,\frac{\partial f}{\partial \alpha_i}\right)+$$

$$+\lambda_2'(x)\frac{\partial^2 f}{\partial {\alpha_i}^2}+\lambda_2(x)\left(\psi(x,y)\,\frac{\partial^2 \omega(x,\alpha_i)}{\partial {\alpha_i}^2}+2\frac{r(x)}{{\alpha_i}^2}\,\frac{\partial^2 f}{\partial {\alpha_i}^2}-2\frac{r(x)}{{\alpha_i}^3}\,\frac{\partial f}{\partial \alpha_i}\right)=$$

$$=\psi(x,y)\,\left(\omega(x,\alpha_i)+\lambda_1(x)\,\frac{\partial \omega(x,\alpha_i)}{\partial \alpha_i}+\lambda_2(x)\,\frac{\partial^2 \omega(x,\alpha_i)}{\partial {\alpha_i}^2}\right)+$$

$$+\frac{\partial f}{\partial \alpha_i}\,\left(\lambda_1'(x)+\lambda_1(x)\,\frac{r(x)}{{\alpha_i}^2}-2\lambda_2(x)\,\frac{r(x)}{{\alpha_i}^3}\right)+\frac{\partial^2 f}{\partial {\alpha_i}^2}\,\left(\lambda_2'(x)+2\lambda_2(x)\,\frac{r(x)}{{\alpha_i}^2}\right).\tag{7}$$

As the function F is an integrating function (it is easy to verify that F satisfies the criterium (2)) then the coefficients of $\dfrac{\partial f}{\partial \alpha_i}$ and $\dfrac{\partial^2 f}{\partial {\alpha_i}^2}$ must be equal to 0. Thus the functions $\lambda_1(x),\ \lambda_2(x)$ satisfy the following differential equations

$$\lambda_1'(x) + \lambda_1(x)\ \frac{r(x)}{{\alpha_i}^2} - 2\lambda_2(x)\ \frac{r(x)}{{\alpha_i}^3} = 0, \quad \lambda_2'(x) + 2\lambda_2(x)\ \frac{r(x)}{{\alpha_i}^2} = 0. \tag{8}$$

General solution of the second equation of the system (8) may be written in the form

$$\lambda_2(x) = C_1\ \exp(2\ I_1), \tag{9}$$

where $I_1 \equiv\ -\displaystyle\int\ \frac{r(x)}{{\alpha_i}^2}\ dx$ and C_1 is an arbitrary constant. Using (9) we rewrite the first equation of the system (8) in the form

$$\lambda_1'(x) + \lambda_1(x)\ \frac{r(x)}{{\alpha_i}^2} = 2C_1\ \frac{r(x)}{{\alpha_i}^3}\ \exp(2\ I_1). \tag{10}$$

Its general solution has the form

$$\lambda_1(x) = \left(C_2 + 2C_1\ \int\ \exp(I_1)\ \frac{r(x)}{{\alpha_i}^3}\ dx\right)\ \exp(I_1). \tag{11}$$

Thus, the function F satisfies the equation

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\ \frac{y + r(x)}{y} + \frac{\partial F}{\partial \alpha_i}\ \frac{\alpha_i + r(x)}{\alpha_i} = \Psi(x,y)\ \Omega(x,\alpha_i),$$

where

$$\Psi(x,y) = \psi(x,y),$$

$$\Omega(x,\alpha_i) = \omega(x,\alpha_i) + \lambda_1(x)\ \frac{\partial \omega(x,\alpha_i)}{\partial \alpha_i} + \lambda_2(x)\ \frac{\partial^2 \omega(x,\alpha_i)}{\partial {\alpha_i}^2}.$$

Hence F is the integrating function for equation (1).

3 Application of the method

Let us know one partial solution α of the equation (1). Kojalovich proved [4] that there are three types of autonomous integrating functions which may be written in the form

$$f = \int^{y-\alpha_i}\ \frac{\exp(h\ u)}{u}\ du, \tag{12}$$

$$f = \frac{1}{hy}\ \exp(h(y - \alpha_i)) - \int^{y-\alpha_i}\ \frac{\exp(h\ u)}{u}\ du,$$

$$f = -\frac{1}{h\alpha_i}\ \exp(h(y - \alpha_i)) - \int^{y-\alpha_i}\ \frac{\exp(h\ u)}{u}\ du,$$

where h is a constant. We consider here only the integrating function (12).

Substituting (12) into the equation (3) (the functions $\lambda_2(x),\ \lambda_1(x)$ are determined from equations (9), (11)) we obtain

$$\frac{\exp(h(y-\alpha)+2I(x))\ C_1\ (h(y-\alpha)-1)}{(y-\alpha)^2}+\int^x \frac{\exp(hy)}{u}\ du-$$

$$-\frac{\exp(h(y-\alpha)+I(x))\ \left(C_2+2C_1\ \int_0^x \frac{\exp(I(v_1))\ r(v_1)}{\alpha^3(v_1)}\ dv_1\right)}{y-\alpha}=0, \qquad (13)$$

where $C_1,\ C_2$ are arbitrary constants and

$$I(x)\equiv\ -\int_0^x \frac{r(v)}{\alpha^2(v)}\ dv.$$

Differentiating (13) yields

$$\frac{r(x)\ \exp(h(y-\alpha))}{\alpha^3\ y}\ (\alpha^2-C_2\alpha(1+h\ \alpha)\ e^{I(x)}+C_1(2+2h\ \alpha+h^2\alpha^2)\ e^{2I(x)}-$$

$$-2C_1\ \alpha(1+h\alpha)\ e^{I(x)}\ \int_0^x \frac{r(t)}{\alpha^3(t)}\ e^{I(t)}\ dt)=0.$$

Thus, the partial solution α must satisfy the following equation

$$\alpha^2=\alpha(1+h\alpha)\ e^{I(x)}\left(C_2+2C_1\ \int_0^x \frac{r(t)}{\alpha^3(t)}\ e^{I(t)}\ dt\right)-C_1(2+2h\ \alpha+h^2\alpha^2)\ e^{2I(x)}. \qquad (14)$$

Solving equation (14) for $C_1=3/h,\ C_2=0$ we find

$$\alpha=-\ \frac{3\ r(x)}{1+h\ r(x)}. \qquad (15)$$

Substituting (15) into equation (1) we obtain

$$x(r)=C_3+\frac{3}{h(hr+1)}+\frac{\ln(hr-2)-\ln(hr+1)}{h},$$

where C_3 is an arbitrary constant. Then equation (1) has the form

$$y(r)\ y'(r)=\frac{9(r+y(r))}{(hr-2)(hr+1)^2}. \qquad (16)$$

Substituting (15) into equation (3) (functions $\lambda_1(x),\ \lambda_2(x)$ have the form (9), (11)) we obtain the general integral of equation (16)

$$\int_0^x \frac{\exp(h\ \Phi)}{\Phi}\ dt-\frac{\exp(h\ \Phi))\ (1+hr)(6r(hr-1)+3r+(hr-2)(hr+1)y(r))}{h(hr-2)(y(r)+r(3+hy(r)))^2}=C, \qquad (17)$$

where

$$\Phi\equiv\ \frac{3r}{1+hr}+y(r)$$

and C is an arbitrary constant.

4 Conclusion

We have shown that using the integrating function (12) it is possible to build a new integrating function of the form (3), where functions $\lambda_1(x)$, $\lambda_2(x)$ are defined in (9), (11). Besides, the corresponding Abel's equation is obtained in the form (16) and its general integral has the form (17).

It is possible to consider integrating function of the form

$$F(x, y, \alpha_i) = f(x, y, \alpha_i) + \sum_{k=1}^{n} \lambda_k(x) \frac{\partial^k f}{\partial \alpha_i^k},$$

where $\lambda_k(x)$ $(k = \overline{1, n})$ are some functions of x. This functions also may be found with the help of the upper considered procedure.

This procedure may be used for solving Abel's equation of the form

$$y(r)\ y'(r) = \frac{4n^3(r + y(r))}{(hr - n)(hr + n)^2}, \tag{18}$$

where n is a natural number, h is a constant and $r = r(x)$. The corresponding partial solution of equation (18) is

$$y = -\ \frac{2nr}{n + hr}.$$

Note that equation (18) for $n = 1$ was integrated in [5].

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