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The commutative Moufang loops with minimum conditions for subloops II

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Abstract. It is proved that the following conditions are equivalent for an infinite nonassociative commutative Moufang loop Q: 1) Q satisfies the minimum condition for subloops; 2) if the loop Q contains a centrally solvable subloop of class s, then it satisfies the minimum condition for centrally solvable subloops of class s; 3) if the loop Q contains a centrally nilpotent subloop of class n, then it satisfies the minimum condition for centrally nilpotent subloops of class n; 4) Q satisfies the minimum condition for noninvariant associative subloops. The structure of the commutative Moufang loops, whose infinite nonassociative subloops are normal is examined.

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This paper is the continuation of the article [1], where the construction of the commutative Moufang loops (abbreviated CML) with the minimum condition for subloops is examined. A normal weakening for this condition is the minimum condition for the centrally solvable (centrally nilpotent) subloops of a given class. A broader question regarding these conditions is examined in Section 2, and namely, the existence in a CML of infinite centrally solvable (centrally nilpotent) subloops, possessing a property, which, by analogy with the group theory [2], will be called steady central solvability (steady central nilpotence). We will say that an infinite centrally solvable (centrally nilpotent) of the class of the loop Q is *steadily centrally solvable* (*steadily centrally nilpotent*) if any infinite centrally solvable (centrally nilpotent) subloop of the class n of loop Q contains a proper subloop of central solvability (central nilpotence) of class n. It turned out that the existence of steadily centrally solvable (centrally nilpotent) subloop of a certain given class n in CML is equivalent to the existence of an infinite decreasing series of subloops in CML. In particular it follows from here that for a CML, possessing a centrally solvable (centrally nilpotent) subloop of a certain class n, the minimum condition for subloops is equivalent to the minimum condition for subloops which have the same class of central solvability (central nilpotence) n.

It is shown in Section 3 that the minimum condition for subloops and for noninvariant associative subloops are equivalent in an infinite nonassociative CML. The infinite nonassociative CML which do not have proper infinite nonassociative subloops are described in Section 2. A weakening of the last condition is the condition for infinite nonassociative CML, when all infinite subloops are normal in them. The construction of such CML is given in Section 4.

1 Preliminaries

A *multiplicative group* $\mathfrak{M}(Q)$ of a CML Q is a group generated by all *translations* $L(x)$, where $L(x)y = xy$. The subgroup $I(Q)$ of the group $\mathfrak{M}(Q)$, generated by all the *inner mappings* $L(x,y) = L(xy)^{-1}L(x)L(y)$, is called an *inner mapping group* of the CML Q. The subloop H of the CML Q is called *normal* (*invariant*) in Q if $I(Q)H = H$.

Lemma 1.1 [3]. *The inner mappings are automorphisms in the commutative Moufang loops.*

Further we will denote by $< M >$ the subloop of the loop Q, generated by the set $M \subseteq Q$.

Lemma 1.2 [3]. *Let H and K be such loop's subloops that K is normal in $< H, K >$. Then $HK = KH =< H, K >$.*

The *associator* (a, b, c) of the elements a, b, c of the CML Q is defined by the equality $ab \cdot c = (a \cdot bc)(a, b, c)$. The identities:

$$L(x, y)z = z(z, y, x), \tag{1.1}$$

$$(x, y, z) = (y^{-1}, x, z) = (y, x, z)^{-1} = (y, z, x), \tag{1.2}$$

$$(x^p, y^r, z^s) = (x, y, z)^{prs}, \tag{1.3}$$

$$(x, y, z)^3 = 1, \tag{1.4}$$

$$(xy, u, v) = (x, u, v)((x, u, v), x, y)(y, u, v)((y, u, v).y, x) \tag{1.5}$$

hold in a CML [3].

Lemma 1.3 [3]. *The periodic commutative Moufang loop is locally finite.*

Lemma 1.4 [4]. *The periodic commutative Moufang loop Q decomposes into a direct product of its maximal p-subloops Q_p, and besides, Q_p belongs to the centre $Z(Q) = \{x \in Q | (x, y, z) = 1 \forall y, z \in Q\}$ of CML Q for $p \neq 3$.*

We denote by Q_i (respect. $Q^{(i)}$) the subloop of the CML Q, generated by all associators of the form $(x_1, x_2, \ldots, x_{2i+1})$ (respect. $(x_1, \ldots, x_{3^i})^{(i)}$) where $(x_1, \ldots, x_{2i-1}, x_{2i}, x_{2i+1}) = ((x_1, \ldots, x_{2i-1}), x_{2i}, x_{2i+1})$ (respect. $(x_1, \ldots, x_{3^i})^{(i)} = ((x_1, \ldots, x_{3^{i-1}})^{(i-1)}, (x_{3^{i-1}+1}, \ldots, x_{2 \cdot 3^{i-1}})^{(i-1)}, (x_{2 \cdot 3^{i-1}+1}, \ldots, x_{3^i})^{(i-1)})$). The series of normal subloops $1 = Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_i \subseteq \ldots$ (respect. $1 = Q^{(o)} \subseteq Q^{(1)} \subseteq \ldots \subseteq Q^{(i)} \subseteq \ldots$) is called the *lower central series* (respect. *derived series*) of the CML Q. We will also use for associator loop the designation $Q^{(1)} = Q'$.

The CML Q is *centrally nilpotent* (respect. *centrally solvable*) *of class* n if and only if its lower central series (respect. derived series) has the form $1 \subset Q_1 \subset \ldots \subset Q_n = Q$ (respect. $1 \subset Q^{(1)} \subset \ldots \subset Q^{(n)} = Q$).

Lemma 1.5 (Bruck-Slaby Theorem) [3]. *Let n be a positive integer, $n \geq 3$. Then every commutative Moufang loop Q which can be generated by n elements is centrally nilpotent of class at most $n - 1$.*

Let M be a subset, H be a subloop of the CML Q. The subloop

$$Z_H(M) = \{x \in H | (x, u, v) = 1 \forall u, v \in M\}$$

is called the *centralizer* of the set M in the subloop H.

Lemma 1.6 [1]. *If M is a normal subloop of the subloop H of the commutative Moufang loop Q then for $a, b \in H$ $aZ_H(M) = bZ_H(M)$ if and only if $L(a, b)(a, u, v) = (b, u, v)$ for any $u, v \in M$.*

The *upper central series* of the CML Q is the series

$$1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots \subseteq Z_\alpha \subseteq \ldots$$

of the normal subloops of the CML Q, satisfying the conditions: 1) $Z_\alpha = \sum_{\beta<\alpha} Z_\beta$ for the limit ordinal and 2) $Z_{\alpha+1}/Z\alpha = Z(Q/Z_\alpha)$ for any α.

Lemma 1.7 [3]. *The statements: 1) $x^3 \in Q$ for any $x \in Q$; 2) the quotient loop $Q/Z(Q)$ has the index 3 hold for a commutative Moufang loop Q.*

A CML Q is called *divisible* it the equation $x^n = a$ has at least one solution in Q, for any $n > 0$ and any element $a \in Q$.

Lemma 1.8 [1]. *The following conditions are equivalent for a commutative Moufang loop D: 1) D is a divisible loop; 2) D is a direct factor for any commutative Moufang loop that contains it.*

Lemma 1.9 [1]. *The following conditions are equivalent for a commutative Moufang loop Q: 1) Q satisfies the minimum condition for subloops; 2) Q is a direct product of a finite number of quasicyclic groups, lying in the centre $Z(Q)$, and a finite loop.*

2 Steadily centrally solvable (centrally nilpotent) commutative Moufang loops

Lemma 2.1. *A infinite centrally solvable (centrally nilpotent) commutative Moufang loop Q of class n contains a proper centrally solvable (centrally nilpotent) subloop of class n.*

Proof. Let us suppose the contrary, i.e., all proper subloops of the centrally solvable CML Q of class n have a class of central solvability less than n. Let us prove that in such a case the CML is finite.

As the CML Q is centrally solvable of the class n, there are such elements $a_1, \ldots, a_{3^{n-1}}$ in Q that $(a_1, \ldots, a_{3^{n-1}})^{(n-1)} \neq 1$. Due to the fact that all proper subloops of the CML Q are centrally solvable of class less than n, the elements

$a_1, \ldots, a_{3^{n-1}}$ generate the CML Q. Without violating the generality, we will suppose that all the elements $a_1, \ldots, a_{3^{n-1}}$ are different. For example, let an element a_1 have an infinite order. Then the subloop $< a_1^4, \ldots, a_{3^{n-1}} >$ is proper in Q. Now, by the identities (1.3), (1.4) we calculate

$$(a_1^4, \ldots, a_{3(n-1)})^{(n-1)} = ((a_1, \ldots, a_{3(n-1)})^{(n-1)})^4 =$$

$$= (a_1, \ldots, a_{3(n-1)})^{(n-1)} \neq 1.$$

We have obtained that the proper subloop H is centrally solvable of the class n. Contradiction. Consequently, the generators of the CML Q have a finite orders. Basing on Lemma 1.3, we conclude that the CML Q is finite. Contradiction. The second case is proved by analogy.

Corollary 2.2. *The centrally solvable (centrally nilpotent) commutative Moufang loop of class n whose proper subloops have a class of central solvability (central nilpotence) less that n is a finite loop.*

Lemma 2.3. *If a non-periodic commutative Moufang loop Q contains a finite centrally solvable (centrally nilpotent) subloop H of class n, then it contains a steadily centrally solvable (centrally nilpotent) subloop of class n.*

Proof. If a is an element of an infinite order, then by Lemma 1.6 $a^{3^k} \in Z(Q)$, where $k = 1, 2, \ldots, Z(Q)$ is the centre of the CML Q. Then the subloop $< a^{3^k}, H >$ is steadily centrally solvable (centrally nilpotent) of the class n.

Lemma 2.4. *Let a commutative Moufang loop Q, which does not satisfy the minimum condition for subloops be centrally solvable (centrally nilpotent) of the class n. Then Q possesses a proper infinite centrally solvable (centrally nilpotent) subloop of the class n.*

Proof. Let the infinite CML Q be centrally solvable of the class n and all its proper centrally solvable subloops of the class n, be finite. By Lemma 2.1 there exists a finite proper centrally solvable subloop K of the class n of the order m in the CML Q.

If L is an arbitrary normal subloop of a finite index of the CML Q, then by Lemma 1.2 LK is an infinite centrally solvable subloop of the class n and therefore $LK = Q$. By the relation

$$Q/L = LK/L \cong K(K \cap L)$$

the index of the normal subloop L is not greater than m in the CML Q. Then in the CML Q there exists a normal subloop H of a finite index. The subloop H does not possess proper normal subloops of finite index, it means that H/H' is infinite. Therefore $H'K$ is a finite subloop, and then the associator loop H' is also finite. Let us show that the subloop H is associative. Indeed, by Lemma 1.5 $aZ(H) \neq bZ(H)$ $(a, b \in H)$ if and only if there exist such elements $u, v \in H$ that $(a, u, v) \neq (b, u, v)$. Therefore the centre $Z(Q)$ has a finite index in H. The subloop H is normal in the CML Q, i.e. it is invariant regarding the inner mapping group which consists of automorphisms (Lemma 1.1). Then it is obvious that the subloop $Z(H)$ is normal in Q. We have obtained that the CML H contains a normal in Q subloop of finite index. But it contradicts the choice of subloop H. Consequently, $Z(H) = H$. Further, the set S of the elements of the group H, having simple orders, is finite. It follows from the fact that the subloop $< S > K$ (the subloop $< S >$ is normal in Q) is finite as a proper centrally solvable subloop of the class n of the CML Q. It follows from here that H is an abelian group with the minimum condition for subgroups. The second case is proved by analogy.

Corollary 2.5. *For an infinite centrally solvable (centrally nilpotent) commutative Moufang loop to be steadily centrally solvable (steadily centrally nilpotent), it is enough that it does not contain divisible subloops different from the unitary element.*

Corollary 2.6. *For an infinite periodic centrally solvable (centrally nilpotent) commutative Moufang loop Q of the class n to be steadily centrally solvable (steadily centrally nilpotent), it is necessary and sufficient that Q does not contain divisible subloops different from the unitary element.*

Proof. If Q does not contain non-trivial divisible subloops, then the necessity follows from Corollary 2.5. Conversely, for example, let the CML Q be steadily centrally solvable and let H be the maximal divisible subloop of the CML Q. By Lemma 1.7 $H \subseteq Z(Q)$. If L is a finite centrally solvable subloop of the class n, K is a quasicyclic group from H, then the subloop $< L, K >$ is centrally solvable of the class n and satisfies the minimum condition for subloops. By the mentioned above and by Lemma 2.1 it is easy to show that there exists an infinite centrally solvable subloop of the class n in the $< L, K >$ whose all subloop's proper centrally solvable subloops of the class n are finite. But it contradicts the fact that Q is steadily centrally solvable. The second case is proved by analogy. This completes the proof of Corollary 2.6.

Let us remark that the request of the periodicity of the CML Q in Corollary 2.6 is essential (example: the additive group of rational numbers).

We will call a *minimal CML* of central solvability (central nilpotence) class n any centrally solvable (centrally nilpotent) CML whose all proper subloops have a class of central solvability (central nilpotence) less than n. It follows from Lemmas 2.1 and 1.4 that these are commutative Moufang 3-loops.

Corollary 2.7. *For a commutative Moufang loop Q to be infinite centrally solvable (centrally nilpotent) of the class n, and all its proper centrally solvable (centrally nilpotent) subloops of the class n to be finite, it is necessary and sufficient that the loop Q is a direct product of quasicyclic groups and the minimal CML of the central solvability (central nilpotence) class n.*

Proof. We will examine only the case of central solvability. If an infinite CML Q is centrally solvable of class n and all its proper centrally solvable class n are finite, then by Lemma 2.4 Q satisfies the minimum condition for subloops. By Lemma 1.9 Q decomposes into a direct product of finite number of quasicyclic groups and a finite CML. Obviously, if K is a quasicyclic group and L is a minimal subloop of central solvability class n, then $Q = K \times L$. The inverse is obvious.

Lemma 2.8. *Let a commutative Moufang loop Q which does not satisfy the minimum condition for subloops be centrally solvable (centrally nilpotent) of the class n. Then Q possesses a steadily centrally solvable (steadily centrally nilpotent) subloop of class n.*

Proof. Let $Q^{(t)}$ be the last member of derived series (lower central series)

$$Q = Q^{(0)} \supset Q^{(1)} \supset \ldots \supset Q^{(t)} \supset \ldots \supset Q^{(n)} = 1$$

of the CML Q that does not satisfy the minimum condition for subloops. If there are no steadily centrally solvable (steadily centrally nilpotent) subloops of class n in the CML Q, then by Lemma 2.1 there exists a finite centrally solvable (centrally nilpotent) subloop of the class n in it. We denote it by H.

If Q is a non-periodic CML, then the statement follows from Lemma 2.3.

Let now the subloop $Q^{(t)}$ have no elements of infinite order. By (1.4) the subloop $Q^{(t+1)}$ has the degree three and by the supposition it satisfies the minimum condition for subloops. Then by Lemma 1.9 $Q^{(t+1)}$ is finite. We denote by $L/Q^{(t+1)}$ the subgroup of the abelian group $Q^{(t)}/Q^{(t+1)}$, generated by all elements of prime orders. It cannot be finite, as the group $Q^{(t)}/Q^{(t+1)}$, and then the CML $Q^{(t)}$ would also satisfy the minimum condition for subloops. We denote by Z the centralizer of the normal subloop $Q^{(t+1)}$ in the CML L. By Lemma 1.5, if $a, b \in L$, then $aZ \neq bZ$ if and only if there exist such elements u, v from $Q^{(t+1)}$ that $L(a, b)(a, u, v) \neq (b, u, v)$. The subloop $Q^{(t+1)}$ is normal in Q, then $(a, u, v) \in Q^{(t+1)}$. As $Q^{(t+1)}$ is finite, L/Z is finite. So, the subloop Z does not satisfy the minimum condition for subloops. Now it follows from the relations

$$Z/(Z \cap Q^{(t+1)}) \cong Q^{(t+1)}Z/Q^{(t+1)} \subseteq L/Q^{(t+1)}$$

that $Z/(Z \cap Q^{(t+1)})$ is an infinite abelian group. The subloop $Z \cap Q^{(t+1)}$ is contained in the centre of the CML Z, then Z is a centrally nilpotent CML of the class 2. It follows from here that the associator loop Z' is an abelian group of the exponent three. If the associator loop Z' is infinite, then $Z'H$ is an unknown subloop (the product $Z'H$ is a subloop by Lemma 1.2, as the normality of Z' in Q follows from the normality of Z in Q). But if the associator loop Z' is finite, then the subgroup K/Z' of the group Z/Z', generated by all elements of prime orders, should be infinite, as Z does not satisfy the minimum condition for subgroups. The subloop K is normal in Q as Z is normal in Q and, obviously, K contains no divisible subloops different from the unitary element. Consequently, by Corollary 2.6 HK is a steadily centrally solvable (steadily centrally nilpotent) subloop of the class n.

Lemma 2.9. *An arbitrary centrally solvable (centrally nilpotent) commutative Moufang loop Q of class n that does not satisfy the minimum condition for subloops possesses a steadily centrally solvable (steadily centrally nilpotent) subloops of central solvability (central nilpotence) class t for any $t = 1, 2, \ldots, n$.*

Proof. Let Q be a centrally nilpotent CML of class n and let $a_1, a_2, \ldots, a_{2n+1}$ be such elements from Q that $((a_1, \ldots, a_{2i+1}), a_{2i+2}, \ldots, a_{2n-1}, a_{2n}, a_{2n+1}) = 1$, but $((a_1, \dots, a_{2i+1}), a_{2i+2}, \dots, a_{2n-1}) \neq 1$. Then the subloop $< (a_1, \dots, a_{2i+1}), a_{2i+2}, \dots, a_{2n+1} >= H$ is centrally nilpotent of class $n-1 = t$. In the case of central solvability we will examine the $(n-i)$-th member of the derived series $Q^{(n-i)}$ instead of H.

If the subloop H is not steadily centrally solvable (steadily centrally nilpotent) of class t, then by Lemma 2.1 the subloop H is finite. Let the CML Q not be periodic. Then by Lemma 2.3 Q contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class t.

Let us suppose that Q is a periodic CML. Let $Q^{(i)}$ be the last member of the derived series (of the lower central series)

$$Q = Q^{(0)} \supset Q^{(1)} \supset \dots \supset Q^{(i)} \supset \dots \supset Q^{(n)} = 1$$

of the CML Q that does not satisfy the minimum condition for subloops. The subloop $Q^{(i+1)}$ satisfies the minimum condition for subloops and by (1.4) it has the index three. Then by Lemma 1.9 it is finite. We denote by $K/Q^{(i-1)}$ the subgroup of the abelian group $Q^{(i)}/Q^{(i+1)}$ generated by all elements of prime orders. The group $K/Q^{(i+1)}$ is infinite, as the CML $Q^{(i)}$ does not satisfy the minimum condition for subloops. Let us suppose that $L = KH, L_0 = Q^{(i+1)}H$. We remind that in the case of central solvability $Q^{(t)} = H$. But if $Q^{(i+1)}$ is a member of the lower central series, then the subloop L_0 is normal in L. Indeed, for that it is enough to show that if $x \in L_0, y, z \in L$, then $(x, y, z) \in L_0$. Any element from L_0 has the form ah, where $a \in Q^{(i+1)}, h \in H$, and any element from L has the form uh, where $u \in Q^{(i)}, h \in H$. If $a \in Q^{(i+1)}, u, v \in Q^{(i)}, h_1, h_2, h_3 \in H$, then by the identity (1.5) the associator (ah_1, uh_2, vh_3) may be presented as a product of the factors of the form $(a, x, y), (h_1, h_2, h_3), (u, x, y)$, where $x, y \in L$. As the subloop $Q^{(i+1)}$ is normal in Q, $(a, x, y) \in Q^{(i+1)}$. Further, it is obvious that $(h_1, h_2, h_3) \in H$. If $a \in Q^{(i)}$, then it follows from the relation $Q^{(i)}/Q^{(i+1)} \subseteq Z(Q/Q^{(i+1)})$ that $(u, x, y) \in Q^{(i+1)}$. Consequently, the subloop L_0 is normal in L.

We have already constructed such a series of elements of the CML L

$$g_1, g_2, \dots, g_r \tag{2.1}$$

that the normal subloops $L_i =< L_0, g_1, \dots, g_i >$ form s strictly ascending series $L_0 \subset L_1 \subset \dots \subset L_r$ and for any $i = 1, 2, \dots, r$ the element g_i is bound by an associative law with all elements of the CML L_{i+1}. Let us show that the series (2.1) can be unlimitedly continued. We denote by Z the centralizer of the subloop L_r in L. By Lemma 1.9 if $a, b \in L$, then $aZ \neq bZ$ if and only if there exist such elements u, v from L_r that $L(a, b)(a, u, v) \neq (b, u, v)$. The CML L_r is finite and normal in L, therefore it is easy to see that L/Z is a finite CML. Then $Z/(Z \cap L_r)$ is an infinite CML. Let $g_{r+1} \in Z \backslash (Z \cap L_r)$. Then $L_r \neq< L_r, g_{r+1} >= L_{r+1}$ and the element g_{r+1} is bound by an associative law with all elements of the subloop L_r. So, the series (2.1) can be unlimitedly continued. The subloop $< H, g_1, g_2, \dots >$ is centrally solvable (centrally nilpotent) of class n and does not satisfy the minimum condition for subloops. Indeed, according to the choice of the element g_i, the quotient loop

$L_0 < g_1, \dots, g_i, \dots > /L_0$ is infinite, and therefore it does not satisfy the minimum condition for subloops. Consequently, the quotient loop

$$< g_1, \dots, g_i, \dots > /(< g_1, \dots, g_i, \dots > \cap L_0)$$

does not satisfy the minimum condition for subloops as well, and as L_0 is a finite CML, the subloop $< H, g_1, \dots, g_i, \dots >$ does not satisfy the minimum condition for subloops. It follows from Lemma 2.8 that on $< H, g_1, \dots, g_i, \dots >$ there exists an unknown steadily centrally solvable (steadily centrally nilpotent) subloop of class n.

Corollary 2.10. *For all centrally solvable (centrally nilpotent) of class n ($n \geq 2$) subloops of the commutative Moufang loop Q, that has such a subloop to be steadily centrally solvable (steadily centrally nilpotent) it is enough that all its infinite centrally solvable (centrally nilpotent) of class $n-1$ are steadily centrally solvable (steadily centrally nilpotent).*

Proof. Let L be an arbitrary infinite centrally solvable (centrally nilpotent) of class n subloop of the CML Q. If L is not steadily centrally solvable (steadily centrally nilpotent), then in the CML L there exists an infinite centrally solvable (centrally nilpotent) subloop H of class n whose all proper subloops of central solvability (central nilpotence) class n are finite. By Lemma 2.9 the CML H satisfies the minimum condition for subloops, and by Lemma 1.9 $H = D \times K$, where D is a divisible CML, lying in the centre $Z(H)$ and K is a finite CML. The CML K is centrally solvable (centrally nilpotent) of class n. Then it has a proper subloop T of central solvability (central nilpotence) class $n-1$. The subloop $T \times D$ is an infinite centrally solvable (centrally nilpotent) subloop of class $n-1$, satisfying the minimum condition for subloops. It follows from Lemma 2.9 $T \times D$ is not steadily centrally solvable (steadily centrally nilpotent). Contradiction.

Corollary 2.11. *For all infinite centrally solvable (centrally nilpotent) subloops of the commutative Moufang loop Q to be steadily centrally solvable (steadily centrally nilpotent) is necessary and sufficient that Q has no quasicyclic groups.*

The statement follows from the fact that an infinite abelian group is steadily centrally solvable if and only if it has no quasicyclic groups, as well as from Corollary 2.10.

Theorem 2.12. *If the commutative Moufang loop Q possesses a centrally solvable (centrally nilpotent) subloop S of class n (maybe finite), then the loop Q either contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class n, or satisfies the minimum condition for subloops.*

Proof. Let us first suppose that CML Q is a countable p-loop and it is not centrally solvable (centrally nilpotent). In such a case, Q is the union of the countable series of finite subloops (by Lemma 1.3 the commutative Moufang p-loop is locally finite)

$$H_1 \subset H_2 \subset \dots \subset H_k \subset \dots,$$

where H_i is a centrally solvable (centrally nilpotent) subloop of class n. We denote by L_k the lower layer of the centre of the CML H_k. (The *lower layer* of the p-group G is the set $\{x \in Q | x^p = 1\}$). Let us now examine the CML $R =< H_1, L_2, \ldots, L_k, \ldots >$. The CML R is centrally solvable (centrally nilpotent) of class n. If the CML R is infinite, then is obvious that R does not satisfy the minimum condition for subloops, and by Lemma 2.9 the CML R contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class n. But if the CML R is finite, then the CML $< L_1, L_2, \ldots, L_k, \ldots >$ is also finite. Therefore the centre $Z(Q)$ of the CML Q is different from the unitary element. The upper central series $Z_1 \subseteq Z_2, \subseteq \ldots \subseteq Z_\beta \subseteq \ldots$ of the CML Q stabilities on a certain ordinal number γ. If Z_γ is a centrally solvable (centrally nilpotent) CML, then the CML Q contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class n. Indeed, in this case the quotient loop Q/Z_γ is a countable p-loop, and is not centrally solvable (centrally nilpotent). Then by the above-mentioned reasoning, and as the Q/Z_γ is a CML without a centre, we obtain that the CML Q/Z_γ contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class n. Let it be the subloop K/Z_γ. By the definition of the derived series (of the lower central series) the subloop K is centrally solvable (centrally nilpotent) and it does not satisfy the minimum condition for subloops. Then by Lemma 2.9 the CML K contains a steadily centrally solvable (steadily centrally nilpotent) subloop of class n.

Let us now that Z_γ is not a centrally solvable (centrally nilpotent) subloop and let SZ_α be the first member of the series $SZ_1 \subset SZ_2 \subset \ldots \subset SZ_\beta \ldots$ which is not a centrally solvable (centrally nilpotent) subloop. If the CML SZ_β does not satisfy the minimum condition for at least one ordinal number β ($\beta < \alpha$), then by Lemma 2.9 the CML SZ_β contains an unknown steadily centrally solvable (steadily centrally nilpotent) subloop. Now suppose that for all β ($\beta < \alpha$) the subloops SZ_β satisfy the minimum condition for subloops, and denote by D the maximal divisible subloop of the CML SZ_α. By Lemma 1.9 $SZ_\alpha = D \times \overline{Z}_\alpha$, where $\overline{Z}_\alpha$ is a reduced CML. The subloops SZ_β ($\beta < \alpha$) satisfy the minimum condition, then by Lemmas 1.8, 1.7 $SZ_\beta = D_\beta \times \overline{Z}_\beta$, where D_β are divisible CML, $\overline{Z}_\beta$ are finite normal reduced subloops. Consequently, $\overline{Z}_\alpha$ is the union of an ascending series of finite normal subloops Z_β ($\beta < \alpha$). The maximal subloop $\overline{M}$ of the CML $\overline{Z}_\alpha$ that has the central solvability (central nilpotence) class n cannot be finite. Indeed, it follows from the finiteness of the subloop $\overline{M}$ that $\overline{M} \subset \overline{Z}_\beta$ for a certain $\beta < \alpha$. We denote by Z the centralizer of the subloop $\overline{Z}_\beta$ in the CML $\overline{Z}_\alpha$. If $a, b, \in \overline{Z}_\alpha$, then by Lemma 1.9 $aZ \neq bZ$ if and only if the exist such elements $u, v \in \overline{Z}_\beta$ that $L(a, b)(a, u, v) \neq (b, u, z)$. The subloop $\overline{Z}_\beta$ is normal in Q and it is finite, therefore the centralizer Z is infinite. So, there exists a non-unitary element $w \in Z \backslash \overline{M}$. The subloop $< \overline{M}, w >$ has the central solvability (central nilpotence) class n and is different from the subloop $\overline{M}$, that contradicts the choice of $\overline{M}$. Thus, $\overline{M}$ is an infinite CML. By the maximality of the divisible CML D, the CML $\overline{M}$ is a steadily centrally solvable (steadily centrally nilpotent) subloop of class n by Corollary 2.6.

Let now Q be an arbitrary CML satisfying the theorem's conditions. If a is an element of infinite order, then by Lemma 2.9 in the CML $< S, a >$ there exists a steadily centrally solvable (steadily centrally nilpotent) subloop of class n.

Let Q be a periodic CML, not centrally solvable (centrally nilpotent). By Lemma

1.4 Q decomposes into a direct product of its maximal p-subloops Q_p, besides, Q_p lies in the centre of the CML Q for $p \neq 3$. Then the subloop Q_3 is not centrally solvable (centrally nilpotent) and such a countable subloop can be found within it. By the above-mentioned, the latter contains an unknown steadily centrally solvable (steadily centrally nilpotent) subloop.

Corollary 2.13. *The following conditions are equivalent for a nonassociative commutative Moufang loop:*

1) the loop Q satisfies the minimum condition for subloops;

2) if the loop Q contains a centrally solvable subloop of class s, it satisfies the minimum condition for the centrally solvable subloops of class s;

3) if the loop Q contains a centrally nilpotent subloop of class n, it satisfies the minimum condition for the centrally nilpotent subloops of class n;

4) the loop Q satisfies the minimum condition for the associative subloops;

5) the loop Q satisfies the minimum condition for nonassociative subloops.

Corollary 2.14. *An infinite commutative Moufang loop Q, possessing a centrally solvable (centrally nilpotent) subloop H of class n, has also an infinite subloop of such type.*

Proof. Let $a \in Q$ be an element of infinite order. By Lemma 1.6 $a^{3^k} \in Z(Q)$, $k = 1, 2, \ldots$, therefore $< H, a^{3^k} >$ is an unknown subloop. If the periodic CML Q does not satisfy the minimum condition for centrally solvable (centrally nilpotent) subloops of class s, then it contains an infinite subloop of this type, as the CML Q is locally finite (Lemma 1.3). In the opposite case, by Corollary 2.13 and Lemma 1.9 $Q = D \times K$, where $D \subseteq Z(Q), K$ is a finite CML. In this case $D, H >$ is an unknown subloop.

Corollary 2.15. *Any infinite commutative Moufang loop possesses an infinite associative subloop.*

The statement follows from Corollary 2.14 and from the fact the CML is monoassociative.

Corollary 2.16. *A commutative Moufang loop with finite centrally solvable (centrally nilpotent) subloops of class n, $n = 1, 2, \ldots$, is finite itself.*

The statement is equivalent to Corollary 2.14.

In particular, the equivalence of the conditions 1), 5) of Corollary 2.13 means that each infinite nonassociative CML has an infinite nonassociative subloop different from itself with the exception of the case when it satisfies the minimum condition for subloops. It is clear that not any infinite CML with the minimum condition is an exception here. It holds true indeed.

Proposition 2.17. *The infinite nonassociative commutative Moufang loop Q does not contain its proper infinite nonassociative subloops if and only if it decomposes into a direct product of quasicyclic groups, contained in the centre $Z(Q)$ of the loop Q, and a finite nonassociative loop, generated by three elements.*

Proof. By Corollary 2.13 the CML Q satisfies the minimum condition for subloops, then by Lemma 1.9 $Q = D \times H$, where D is a direct product of a finite number of quasicyclic groups, $D \subseteq Z(Q)$, H is a finite CML. By the supposition about the CML Q, the group D contains only one quasicyclic group.

Obviously H is an nonassociative CML. If H_1 is an arbitrary proper subloop of the CML H, then by Lemma 1.2 the product DH_1 is a proper infinite subloop of the CML Q. But then DH_1 and H_1 are associative subloops. Consequently, all proper subloops of the CML Q are associative, and it follows from Lemma 1.5 [3] that H is generated by tree elements. Let now the CML Q have a decomposition $Q = D \times H$, possessing these qualities, and L be an arbitrary proper subloop of the CML Q. Obviously $D \subseteq L$. Then it follows from the decomposition $Q = D \times H$ that $L = D(L \cap H)$. As $L \neq Q$, then $L \cap H \neq H$. Then the subloop $L \cap H$, as a proper subloop of the CML H, is associative. Therefore it follows from the decomposition $L = D(L \cap H)$ that the subloop H is associative.

3 Infinite nonassociative commutative Moufang loops with minimum condition for noninvariant associative subloops

Lemma 3.1. *If an element a of an infinite order or of order three of a commutative Moufang loop Q generates a normal subloop, then it belongs to the centre $Z(Q)$ of loop Q.*

Proof. If the element $1 \neq a \in Q$ generates a normal subloop, then $L(u, v)a = a^k$ for a certain natural number k and for arbitrary fixed elements $u, v \in Q$. By (1.1) $a(a, v, u) = a^k, (a, v, u) = a^{k-1}$. If $k = 1$, then $(a, v, u) = 1$. Therefore $a \in Z(Q)$. Let us now suppose that $k > 1$. Let $a^3 = 1$. Then $k = 2$ and by (1.5) and Lemma 1.5 $a = (a, v, u), a = ((a, v, u), v, u) = 1$. We have obtained a contradiction, as $a \neq 1$. But if a has an infinite order, then by (1.4) $(a^{k-1})^3 = (a, v, u)^3 = 1$. We have obtained a contradiction again. Therefore the case of $k > 1$ is impossible. This completes the proof of Lemma 3.1.

Lemma 3.2. *The commutative Moufang loop Q, containing an element of an infinite order, is associative if and only if the subloop, generated by any element of an infinite order, is normal in Q.*

Proof. By Lemma 3.1 any element a of an infinite order of the CML Q belongs to the centre $Z(Q)$. Let b be an element of a finite order of the CML Q. Obviously the product ab has an infinite order. Again by Lemma 3.1 $ab \in Z(Q)$. Further, by (1.5) and (1.4) we have $1 = (ab, u, v) = L(a, b)(a, u, v) \cdot L(b, a)(b, u, v) = (b, L(b, a)u, L(b, a)v)$, for $u, v \in Q$. Consequently, $b \in Z(Q)$, but then the CML Q is associative.

Theorem 3.3. *If in an infinite commutative Moufang loop Q the infinite associative subloops are normal in Q, then Q is associative.*

Proof. It follows from Lemma 3.2 that it is sufficient to examine the case when the CML Q is periodic, and by Lemma 1.4 it is sufficient to examine the case when Q is a 3-loop.

Let us now first examine the case when the CML Q does not satisfy the minimum condition for subloops. By Corollary 4.5 from [1] none of its maximal elementary associative subloops H can be finite. Let

$$H = H_1 \times H_2 \times \ldots \times H_n \times \ldots$$

be the decomposition of the group H into a direct product of cyclic groups of order three. We denote by $Z_Q(H)$ the centralizer of the subloop H in Q. It is obvious that for any element a from $Z_Q(H)$ there is such an infinite subgroup $H(a) \subseteq H$ that $< a > \cap H(a) = 1$. Let $H(a) = H_1(a) \times H_2(a)$ be a decomposition of the group $H(a)$ into a direct product of infinite factors. As the cyclic group $< a >$ is the intersection of the infinite associative subloops $< a > H_1(a)$ and $< a > H_2(a)$, then $< a >$ is normal in Q. As the element a from $Z_Q(H)$ is arbitrary, we obtain that any subloop from $Z_Q(Q)$ is normal in Q, i.e. $Z_Q(H)$ is a hamiltonian CML. Then by [4] it is an associative subloop. Obviously, $H \subseteq Z_Q(H)$ and, as H_i are cyclic groups of order three, then by Lemma 3.1 $H_i \subseteq Z(Q)$, where $Z(Q)$ is the centre of the CML Q. Consequently, $Z(H) = Q$ is an associative CML.

If a CML Q satisfies the minimum condition for subloops, then by Lemma 1.9 its centre $Z(Q)$ is infinite. If a is an arbitrary element from Q, then the subloop $< a > Z(Q)$ is infinite and associative. From here and from the theorem's supposition we obtain that the subloop $< a >$ is normal in Q. Then by [4] the CML Q is associative.

Lemma 3.4. *A non-periodic commutative Moufang loop, satisfying the minimum condition for the noninvariant cycle groups, is associative.*

Proof. By Lemma 3.2 we suppose that the element a of an infinite order of the CML Q generates a noninvariant subloop. It follows from the condition of lemma that the series

$$< a > \supset < a_t > \supset < a^{t^2} > \supset \ldots \supset < a^{t^n} > \supset \ldots$$

should contain a normal subloop $< a^{t^n} >$ for any natural number t. Let t and p be two different prime numbers, $< a^{t^n} >$ and $< a^{p^k} >$ be two normal subloops corresponding to them, of such a type that u, v are such integer numbers that $ut^n + vp^k = 1$. Then

$$a = a^{ut^n + vp^k} = a^{ut^n} \cdot a^{vp^k}.$$

If x and y are arbitrary elements from Q, then by Lemma 1.1 the inner mapping $L(x, y)$ is an automorphism. Then, by the normality of the subloops $< a^{t^n} >, < a^{p^k} >$, we obtain $L(x, y)a = L(x, y)a^{ut^n} \cdot L(x, y)a^{vp^k} = (L(x, y)a^{t^n})^u \cdot (L(x, y)a^{p^k})^v \in < a >$. Consequently, the subloop $< a >$ is normal in Q. Contradiction. Then the CML Q is associative.

Theorem 3.5. *In a nonassociative commutative Moufang loop the minimum condition for subloops and the minimum condition for noninvariant associative subloops are equivalent.*

Proof. Let us suppose that the CML Q, satisfying the minimum condition for noninvariant associative subloops, does not satisfy the minimum condition for subloops. Then by Corollary 2.13 the CML Q does not satisfy the minimum condition for associative subloops. Let us show that in this case the CML Q is associative, i.e. we will obtain a contradiction. By Lemma 3.4 it is sufficient to examine the case when the CML Q is periodic, and by Lemma 1.4 when Q is a 3-loop.

As the CML Q does not satisfy the minimum condition for associative subloops, then by Corollary 4.5 from [1] Q contains an infinite direct product

$$H = H_1 \times H_2 \times \ldots \times H_n \times \ldots$$

of cyclic groups of order three. If a is an arbitrary element from the centralizer $Z_Q(H)$ of the subloop H in the CML Q, then there exists such a number $n = n(a)$ that

$$< a > \cap (H_{n+1} \times H_{n+2} \times \ldots) = 1.$$

As the CML Q satisfies the minimum condition for noninvariant associative subloops, then the infinitely descending series of associative subloops

$$S^k(a) \supset S^{k+1}(a) \subset \ldots$$

contains a normal subloop $S^l(a)(l = l(a))$, beginning with any natural $k \geq n$, where $S^k(a) =< a > (H_{k+1} \times H_{k+2} \times \ldots)$. As the intersection of all such normal subloops coincides with the subloop $< a >$, then the latter is normal in Q. But a is an arbitrary element from the centralizer $Z_Q(H)$, and it means that any normal subloop from $Z_Q(H)$ is normal. Then by [4] the CML $Z(H)$ is associative. Further, the subgroups H_i have the order three. Then it follows from Lemma 3.1 that they belong to the centre $Z(Q)$ of the CML Q. Then it follows from the definition of the centralizer $Z_Q(H)$ that $Z(Q) = Q$. Consequently, the CML Q is associative.

4 Infinite nonassociative commutative Moufang loops in which all infinite nonassociative subloops are normal

Lemma 4.1. *Let all infinite nonassociative subloops be normal in an infinite nonassociative commutative Moufang loop Q. If H is an infinite nonassociative subloop, then the quotient loop Q/H is a group.*

Proof. It is obvious that any subloop of the CML Q containing H, is normal in Q. Then the quotient loop Q/H is hamiltonian, consequently by [4] it is a group.

Proposition 4.2. *The commutative Moufang loop, in which all its infinite nonassociative subloops are normal has a finite associator loop Q'.*

Proof. Let us suppose the contrary, i.e., that the associator loop Q' is infinite. First we examine the case when Q' is nonassociative. Let H be a proper infinite nonassociative subloop of the CML Q'. Then by Lemma 4.1 Q/H is a group, i.e. $Q' \subseteq H$. Contradiction. Consequently, the associator loop Q' does not have its proper infinite nonassociative subloops. In this case, by Corollary 2.13 the CML Q' satisfies the minimum condition for subloops. But by (1.4) the associator loop Q' has degree three, therefore it is finite.

Let us now examine the case when the infinite associator loop Q' of the periodic CML is associative. Let H be a finite nonassociative subloop of the CML Q. We will examine the subloop $Q'H = \cup x_iQ', x_i \in H, i = 1, \dots, m$. If the infinite nonassociative subloop $Q'H$ does not contain its proper infinite nonassociative subloops, then by Corollary 2.13 it satisfies the minimum condition for subloops. Taking into account (1.4), it is easy to see that the CML $Q'H$ has a finite index. Then it is finite, therefore the CML Q' is also finite. It contradicts the fact the CML $Q'H$ does not contain its proper infinite nonassociative subloops. Let $(Q'H)_1$ be the proper infinite nonassociative subloops of the CML $Q'H$. By Lemma 4.1 $Q' \subseteq (Q'H)_1$. Then $(Q'H)_1 = \cup x_iQ', i = 1, \dots, n, n < m$. If the infinite nonassociative subloop $(Q'H)_1$ does not contain its proper infinite nonassociative subloops, then $(Q'H)_1$ is finite, as it is shown above. Contradiction. Therefore let $(Q'H)_2$ be the proper infinite nonassociative subloop of the CML $(Q'H)_1$. By Lemma 4.1 $Q' \subseteq (Q'H)_2$, therefore $(Q'H)_2 \subseteq \cup x_iQ', x_i \in H, i = 1, \dots, r, r < n$. Applying the previous reasoning to the CML $(Q'H)_2$, after a finite number of steps we will come to infinite nonassociative subloops $(Q'H)_i$ without proper infinite nonassociative subloops. But it contradicts the statement from the previous section. Consequently, the associator loop Q' of the CML Q cannot be infinite.

Finally, let us examine the case when the CML Q is non-periodic. Obviously, the subloop H of the CML Q is nonassociative if and only if the subloop $HZ(Q)$ is nonassociative, where $Z(Q)$ is the centre of the CML Q. If the infinite nonassociative subloops of the CML Q are normal, then the infinite nonassociative subloops of the CML $Q/Z(Q)$ are normal as well. By Lemma 1.9 the CML $Q/Z(Q)$ has index three, then, according to the previous case, its associator loop $(Q/Z(Q))'$ is finite. If $a \in Z(Q)$, then $(au, v, w) = (u, v, w)$, for any $u, v, w \in Q$. It is easy to see from here that the associator loop Q' is finite.

Corollary 4.3. *If in a non-periodic commutative Moufang loop Q all the infinite nonassociative subloops are normal in Q, then its associator loop is a finite associative subloop.*

Proof. Let us suppose that the finite associator loop Q' is nonassociative. Let H be one of its minimal nonassociative subloops, and a be an element of infinite order from Q. By Lemma 1.9 a^3 belongs to the centre of the CML Q. Then by Lemma 1.2, $H < a^3 >$ is an infinite nonassociative subloop. By Lemma 4.1 $Q' \subseteq H < a^3 >$, and it is impossible if $H \neq Q'$. According to the minimality of the nonassociative CML H, it can be presented in the form of the product of the normal associative subloop L and the cyclic group $< b >$. Indeed, by the Moufang theorem [3] the CML H is generated by three elements u, v, b. By Lemma 1.5 $Q' \neq H$. Then $L =< Q', u, v >$ is a normal associative subloop and $H = L\cdot < b >$. Now let us take the CML $B\cdot < a^3b >$. It is an infinite nonassociative subloop and, obviously, it does not contain Q'. However, by Lemma 4.1 $Q' \subseteq B$. Contradiction. Consequently, the associator loop Q' of the CML Q is associative.

Theorem 4.4. *If all infinite nonassociative subloops of a commutative Moufang loop Q are normal in it, then all nonassociative subloops are also normal in it.*

Proof. Let Q be a non-periodic CML and a be an element of an infinite order from Q. By Lemma 1.9 a^3 belongs to the centre of the CML Q. If H is a finite nonassociative subloop, then by Lemma 1.2 $< a^3 > H$ is an infinite nonassociative subloop from Q and, consequently, it is normal in Q. Therefore, H is normal in Q.

Let now Q be a periodic CML and let us suppose that the finite nonassociative subloop L is not normal in Q. The associator loop Q' is a normal subloop in Q. Therefore, by Lemma 1.9 the centralizer $Z_Q(H)$ of the subloop H in Q will be normal subloop in Q. Let us examine the set

$$C(H) = \{x \in Z_Q(H) | (x, u, v) = 1 \forall u \in Z_Q(H), \forall v \in H\}.$$

Using the identity (1.5), it is easy to show that $C(H)$ is a subloop. Moreover, it follows from the normality of the subloops $H, Z_Q(H)$, and by Lemma 1.1, that $C(H)$ is normal in Q. Indeed, if $xC(H) = yC(H)$, then $xy^{-1} \in C(H), (xy^{-1}, u, v) = 1$ for all $u \in Z_Q(H), v \in H$. Now we will use the identities (1.5), (1.1) and (1.3). We have $1 = (xy^{-1}, u, v) = L(x, y^{-1})(x, u, v) \cdot L(y^{-1}, x)(y^{-1}, u, v) = (x, L(x, y^{-1})u, L(x, y^{-1})v)(y^{-1}, L(y^{-1}, x)u, L(y^{-1}, x)v) \equiv (x, \overline{u}, \overline{v})(y^{-1}, \overline{u}, \overline{v}) = (x, \overline{u}, \overline{v})(y, \overline{u}, \overline{v})^{-1}$, $(x, \overline{u}, \overline{v}) = (y, \overline{u}, \overline{v})$ for all $u \in Z_Q(H), v \in H$. It can be proved by analogy that it follows from the equality $(x, \overline{u}, \overline{v}) = (y, \overline{u}, \overline{v})$ from all $u \in Z_Q(H), v \in H$ that $xC(H) = yC(H)$. By Proposition 4.2 the associator loop Q' is finite. Then the normal subloop $C(H)$ has a finite index in Q.

Let us show that the CML Q satisfies the minimum condition for subloops. Let us suppose the contrary. Then the subloop $C(H)$, possessing a finite index in Q, does not satisfy this condition as well. Therefore, the CML $C(H)$ has an infinite associative subloop K which decomposes into a direct product of cyclic groups of prime orders. Otherwise, by Corollary 2.13 and regardless the supposition, the CML Q would satisfy the minimum condition for subloops. It is obvious that an infinite subgroup R can be found, that intersects with L on the unitary element. Let $R = R_1 \times R_2$ be the decomposition of R into a direct product of two infinite subgroups R_1, R_2.

If S is an arbitrary associative subloop of the CML $C(H)$, then the product SL is a subloop. Indeed, by Lemma 1.2, the subloop S is normal in the CML $< S, L >$. The CML $< S, L >$ consists of all "words", composed of the elements of the set $S \cup L$. A word of the length 1 is an element of the set $S \cup L$. If u, v are words of length m, n respectively, then $u^{\epsilon_1} v^{\epsilon_2}$, where $\epsilon_1, \epsilon_2 = \pm 1$, is a word of length $\leq m + n$. It follows from the definition of the subloop $C(H)$ that if 1) $a \in S, u \in L$; 2) $a, u \in S, v \in L$, then $(a, u, v) = 1$. If $a \in S, u, v \in < S, L >$ then, using (1.2), (1.5) and the associativity of the subloop S, it can be proved by the induction on the sum of the length of the words u, v that $(a, u, v) = 1$. Then by (1.1) $L(v, u)a = a$, i.e. the subloop S is normal in $< S, L >$. Therefore $< S, L >= SL$.

By the above proved fact, the products $R_1 L, R_2 L$ are subloops. As they are infinite and nonassociative, they are normal in the CML Q. Then their intersection L is also a normal subloop in Q. We have obtained a contradiction despite the supposition of the noninvariance of the subloop L. In this case, by Lemma 1.8 the CML Q decomposes into a direct product of the divisible group D, lying in the centre $Z(Q)$ of the CML Q, and the finite CML M. If $L \neq M$, then the product DL is an infinite nonassociative subloop of the CML Q, therefore the subloop L is also normal in Q. We have obtained a contradiction to the fact that L is not normal in Q. This completes the proof of Theorem 4.4.

By Corollary 4.3 a non-periodic CML whose infinite nonassociative subloops are normal in it has a finite associative associator loop. The following statement holds true for the general case.

Corollary 4.5. *If all (infinite) nonassociative subloops of an (infinite) nonassociative commutative Moufang loop Q are normal in it, then its associator loop Q' is centrally nilpotent, and the loop Q itself is centrally solvable of a class not greater than three.*

Proof. By Proposition 4.2, the associator loop Q' is finite. Then by Lemma 1.5 Q' is centrally nilpotent.

Let us suppose that the second associator loop $Q^{(2)}$ of the CML Q is nonassociative. Then any subloop that contains $Q^{(2)}$ is non-assiciative, and by Theorem 4.4, it is normal in Q. Obviously, the CML $Q/Q^{(2)}$ is hamiltonian, when it is an abelian group, by [4]. Therefore, $Q' \subseteq Q^{(2)}$, i.e. $Q' = Q^{(2)}$. But the associator loop Q' is centrally nilpotent, therefore $Q' \neq Q^{(2)}$. Contradiction. Consequently, $Q^{(2)}$ is an associative subloop, and the CML Q is centrally solvable of step not greater than three.

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