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About one explicit-difference scheme for solving the plane problem for two-component medium

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Abstract. The finite-difference scheme for plane dynamical problem of the theory of elasticity of two-component medium in displacements is obtained. The stability of this scheme by means of Niemann conditions is studied. Is found the maximal time step in dependence on the space step for which the stability is kept.

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The research of wave processes in many components continuous media represents a great interest for seismology, construction, the research of the dynamic behavior of the various mixtures of the soils etc. The works [1–6] are dedicated to the construction of mathematical models of such media. M.A. Biot in his works [1–3] proposed a rather general approach in the linear mechanics of deformation and distribution of acoustic waves in porous two-component medium.

The non-numerous works [7–13] devoted to the solution of concrete problems on the basis of M.A. Biot's equations refer exclusively to the simplest kinds of two-components media (mixture of two isotropically solid bodies, isotropically solid body and liquids, a liquid and a gas), the first stage of the solution of the problem doesn't provoke any difficulties being the determination of the speeds of the wave types appeared.

The purpose of the present article is the estimation of the time step providing the stability of one explicit finite-difference scheme for the plane dynamical problem of the theory of elasticity of two-component medium. Non-stationary processes in every layer are described by equations of the theory of elasticity: the equations of motion, the Hooke's law and the Cauchy relations.

The relations between stresses and deformations in conditions of plane deformation are the following:

$$\sigma_{xx} = -\alpha_2 + (\lambda_1 + 2\mu_1)\,\varepsilon_{xx} + \lambda_1\varepsilon_{yy} + (\lambda_3 + 2\mu_3)\,q_{xx} + \lambda_3 q_{yy};$$

$$\sigma_{yy} = -\alpha_2 + \lambda_1\varepsilon_{xx} + (\lambda_1 + 2\mu_1)\,\varepsilon_{yy} + \lambda_3 q_{xx} + (\lambda_3 + 2\mu_3)\,q_{yy};$$

$$\pi_{xx} = \alpha_2 + (\lambda_2 + 2\mu_2)\,q_{xx} + \lambda_2 q_{yy} + (\lambda_4 + 2\mu_3)\,\varepsilon_{xx} + \lambda_4\varepsilon_{yy};$$

$$\pi_{yy} = \alpha_2 + \lambda_2 q_{xx} + (\lambda_2 + 2\mu_2)\,q_{yy} + \lambda_4\varepsilon_{xx} + (\lambda_4 + 2\mu_3)\,\varepsilon_{yy};$$

$$\sigma_{xy} = 2\,(\mu_1\varepsilon_{xy} + \mu_3 q_{xy}) - \lambda_5\,(h_{xy} - h_{yx})\,;$$

$$\begin{aligned}
\sigma_{yx} &= 2\left(\mu_1\varepsilon_{xy} + \mu_3 q_{xy}\right) + \lambda_5\left(h_{xy} - h_{yx}\right);\\
\pi_{xy} &= 2\left(\mu_2 q_{xy} + \mu_3\varepsilon_{xy}\right) - \lambda_5\left(h_{xy} - h_{yx}\right);\\
\pi_{xy} &= 2\left(\mu_2 q_{xy} + \mu_3\varepsilon_{xy}\right) + \lambda_5\left(h_{xy} - h_{yx}\right).
\end{aligned} \tag{1}$$

The behavior of the elastic system is described by the equations of motion:

$$\begin{aligned}
&\frac{\partial\sigma_{x\,x}}{\partial\,x} + \frac{\partial\sigma_{x\,y}}{\partial\,y} - \frac{\partial\pi_0}{\partial\,x} = \rho_{11}\frac{\partial^2 u_1}{\partial\,t^2} + \rho_{12}\frac{\partial^2 u_2}{\partial\,t^2} + b\left(\frac{\partial u_1}{\partial\,t} - \frac{\partial u_2}{\partial\,t}\right);\\
&\frac{\partial\sigma_{y\,x}}{\partial\,x} + \frac{\partial\sigma_{y\,y}}{\partial\,y} - \frac{\partial\pi_0}{\partial\,y} = \rho_{11}\frac{\partial^2 v_1}{\partial\,t^2} + \rho_{12}\frac{\partial^2 v_2}{\partial\,t^2} + b\left(\frac{\partial v_1}{\partial\,t} - \frac{\partial v_2}{\partial\,t}\right);\\
&\frac{\partial\pi_{x\,x}}{\partial\,x} + \frac{\partial\pi_{x\,y}}{\partial\,y} + \frac{\partial\pi_0}{\partial\,x} = \rho_{12}\frac{\partial^2 u_1}{\partial\,t^2} + \rho_{22}\frac{\partial^2 u_2}{\partial\,t^2} - b\left(\frac{\partial u_1}{\partial\,t} - \frac{\partial u_2}{\partial\,t}\right);\\
&\frac{\partial\pi_{y\,x}}{\partial\,x} + \frac{\partial\pi_{y\,y}}{\partial\,y} + \frac{\partial\pi_0}{\partial\,y} = \rho_{12}\frac{\partial^2 v_1}{\partial\,t^2} + \rho_{22}\frac{\partial^2 v_2}{\partial\,t^2} - b\left(\frac{\partial v_1}{\partial\,t} - \frac{\partial v_2}{\partial\,t}\right),
\end{aligned} \tag{2}$$

where $u_i, v_i \ \ (i = 1, 2)$ are the components of the displacement vector of firm phases; $\sigma_{xx}, \sigma_{xy}, \sigma_{yx}, \sigma_{yy}, \pi_{xx}, \pi_{xy}, \pi_{yx}, \pi_{yy}$ are the components of the stress tensor; ε_{xx}, ε_{xy}, h_{yx}, ε_{yy}, q_{xx}, q_{xy}, h_{yx}, q_{yy} are the components of deformation; ρ_{11}, ρ_{22} are the effective component masses at their relative motion; $\rho_{11} + \rho_{12} = \rho_1, \rho_{22} + \rho_{12} = \rho_2, \rho_{12}$ is the ,,connecting parameter" between the components of the mixture or the additional apparent mass; $\alpha_2 = \lambda_3 - \lambda_4$is the constant with the dimension of stress; $\lambda_j \ \ \mu_j, (j = \overline{1,5})$ are the Lame constants; $\rho_1, \rho_2$are the densities of phases; b is the diffusion coefficient

$$\pi_0 = \rho_1/\rho\alpha_2(q_x + q_y) + \rho_1/\rho\alpha_2(\varepsilon_x + \varepsilon_y).$$

The relations between the deformations and displacements are the following

$$\begin{aligned}
&\varepsilon_{xx} = \frac{\partial u_1}{\partial x}, \quad \varepsilon_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v_1}{\partial y};\\
&q_{xx} = \frac{\partial u_2}{\partial x}, \quad q_{xy} = \frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}, \quad q_{yy} = \frac{\partial v_2}{\partial y};\\
&h_{xy} = \frac{\partial v_1}{\partial x} + \frac{\partial u_2}{\partial y}, \quad h_{yx} = \frac{\partial u_1}{\partial y} + \frac{\partial v_2}{\partial x}.
\end{aligned} \tag{3}$$

Let us consider the formulation of the problem in displacements. To obtain this formulation we substitute the relations (1) and (3) in the equations of motion. After some simple transformations the obtained equations can be presented in the form:

$$A_{11}\frac{\partial^2 u_1}{\partial x^2} + A_{12}\frac{\partial^2 u_1}{\partial y^2} + (A_{11} - A_{12})\frac{\partial^2 v_1}{\partial x\partial y} + B_{11}\frac{\partial^2 u_2}{\partial x^2} + B_{12}\frac{\partial^2 u_2}{\partial y^2} + (B_{11} - B_{12})\frac{\partial^2 v_2}{\partial x\partial y} =$$

$$= \rho_{11}\frac{\partial^2 u_1}{\partial t^2} + \rho_{12}\frac{\partial^2 u_2}{\partial t^2} + b\left(\frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial t}\right);$$

$$A_{21}\frac{\partial^2 u_1}{\partial x^2}+A_{22}\frac{\partial^2 u_1}{\partial y^2}+(A_{21}-A_{22})\frac{\partial^2 v_1}{\partial x\partial y}+B_{21}\frac{\partial^2 u_2}{\partial x^2}+B_{22}\frac{\partial^2 u_2}{\partial y^2}+(B_{21}-B_{22})\frac{\partial^2 v_2}{\partial x\partial y}=$$
$$=\rho_{12}\frac{\partial^2 u_1}{\partial t^2}+\rho_{22}\frac{\partial^2 u_2}{\partial t^2}-b\left(\frac{\partial u_1}{\partial t}+\frac{\partial u_2}{\partial t}\right);$$
$$A_{11}\frac{\partial^2 v_1}{\partial x^2}+A_{12}\frac{\partial^2 v_1}{\partial y^2}+(A_{11}-A_{12})\frac{\partial^2 u_1}{\partial x\partial y}+B_{11}\frac{\partial^2 v_2}{\partial x^2}+B_{12}\frac{\partial^2 v_2}{\partial y^2}+(B_{11}-B_{12})\frac{\partial^2 u_2}{\partial x\partial y}=$$
$$=\rho_{11}\frac{\partial^2 v_1}{\partial t^2}+\rho_{12}\frac{\partial^2 v_2}{\partial t^2}+b\left(\frac{\partial v_1}{\partial t}+\frac{\partial v_2}{\partial t}\right); \tag{4}$$
$$A_{21}\frac{\partial^2 v_1}{\partial x^2}+A_{22}\frac{\partial^2 v_1}{\partial y^2}+(A_{21}-A_{22})\frac{\partial^2 u_1}{\partial x\partial y}+B_{21}\frac{\partial^2 v_2}{\partial x^2}+B_{22}\frac{\partial^2 v_2}{\partial y^2}+(B_{21}-B_{22})\frac{\partial^2 u_2}{\partial x\partial y}=$$
$$=\rho_{12}\frac{\partial^2 v_1}{\partial t^2}+\rho_{22}\frac{\partial^2 v_2}{\partial t^2}-b\left(\frac{\partial v_1}{\partial t}+\frac{\partial v_2}{\partial t}\right),$$

where $A_{11}=\lambda_1+2\mu_1-\rho_2\alpha_2/\rho$; $A_{12}=\mu_1-\lambda_5$; $A_{21}=\lambda_2+2\mu_2+\rho_1\alpha_2/\rho$; $A_{22}=\mu_2-\lambda_5$; $B_{11}=\lambda_3+2\mu_3-\rho_1\alpha_2/\rho$; $B_{12}=\mu_3+\lambda_5$; $B_{21}=\lambda_4+2\mu_3+\rho_2\alpha_2/\rho$; $B_{22}=\mu_3+\lambda_5$.

Further it will be convenient to split this system into two systems. The first system describes the processes connected with elastic properties of the medium. The second system describes the dissipative properties of the medium. So the systems differ only in right-hand parts. The first system contains the second derivatives with respect to time and the second system contains the first derivatives with respect to time.

Let us consider the following explicit finite-difference scheme for numerical solving the first system of equations.

Let us consider the rectangular grid with the steps Δx with respect to variable x, Δy with respect to time variable. We'll denote by $f_{ij}^k=f(i\Delta x,\ j\Delta y,\ k\Delta t)$ the values of function f in the nodes of the grid and approximate the derivatives with finite-difference relations

$$\frac{\partial^2 f}{\partial t^2}\sim\frac{f_{n,m}^{k+1}-2f_{n,\,m}^{k}+f_{n,\,m}^{k-1}}{\Delta t^2}=\left(f_{m,\,n}^k\right)_{\bar{t}t};$$
$$\frac{\partial^2 f}{\partial y^2}\sim\frac{f_{n,\,m+1}-2f_{n,\,m}+f_{n,\,m-1}}{\Delta y^2}=\left(f_{m,\,n}^k\right)_{\bar{y}y};$$
$$\frac{\partial^2 f}{\partial x\partial y}\sim\frac{f_{n+1,\,m+1}-f_{n-1,\,m+1}-f_{n+1,\,m-1}+f_{n-1,\,m-1}}{4\,\Delta\,x\;\Delta\,y}=\left(f_{m,\,n}^k\right)_{\bar{x}y},$$

as a result we obtain the following discrete system of equations:

$$A_{11}\left(u_{1m,n}^k\right)_{\bar{x}x}+A_{12}\left(u_{1m,n}^k\right)_{\bar{y}y}+(A_{11}-A_{12})\left(v_{1m,n}^k\right)_{\bar{x}y}+B_{11}\left(u_{2m,n}^k\right)_{\bar{x}x}+$$
$$+B_{12}\left(u_{2m,n}^k\right)_{\bar{y}y}+(B_{11}-B_{12})\left(v_{2m,n}^k\right)_{\bar{x}y}=\rho_{11}\left(u_{1m,n}^k\right)_{\bar{t}t}+\rho_{12}\left(u_{2m,n}^k\right)_{\bar{t}t};$$

$$A_{21}\left(u^k_{1m,n}\right)_{\bar{x}x}+A_{22}\left(u^k_{1m,n}\right)_{\bar{y}y}+(A_{21}-A_{22})\left(v^k_{1m,n}\right)_{\bar{x}y}+B_{21}\left(u^k_{2m,n}\right)_{\bar{x}x}+$$
$$+B_{22}\left(u^k_{2m,n}\right)_{\bar{y}y}+(B_{21}-B_{22})\left(v^k_{2m,n}\right)_{\bar{x}y}=\rho_{12}\left(u^k_{1m,n}\right)_{\bar{t}t}+\rho_{22}\left(u^k_{2m,n}\right)_{\bar{t}t};$$
$$A_{11}\left(v^k_{1m,n}\right)_{\bar{x}x}+A_{12}\left(v^k_{1m,n}\right)_{\bar{y}y}+(A_{11}-A_{12})\left(u^k_{1m,n}\right)_{\bar{x}y}+B_{11}\left(v^k_{2m,n}\right)_{\bar{x}x}+$$
$$+B_{12}\left(v^k_{2m,n}\right)_{\bar{y}y}+(B_{11}-B_{12})\left(u^k_{2m,n}\right)_{\bar{x}y}=\rho_{11}\left(v^k_{1m,n}\right)_{\bar{t}t}+\rho_{12}\left(v^k_{2m,n}\right)_{\bar{t}t}; \quad (5)$$
$$A_{21}\left(v^k_{1m,n}\right)_{\bar{x}x}+A_{22}\left(v^k_{1m,n}\right)_{\bar{y}y}+(A_{21}-A_{22})\left(u^k_{1m,n}\right)_{\bar{x}y}+B_{21}\left(v^k_{2m,n}\right)_{\bar{x}x}+$$
$$+B_{22}\left(v^k_{2m,n}\right)_{\bar{y}y}+(B_{21}-B_{22})\left(u^k_{2m,n}\right)_{\bar{x}y}=\rho_{12}\left(v^k_{1m,n}\right)_{\bar{t}t}+\rho_{22}\left(v^k_{2m,n}\right)_{\bar{t}t}.$$

We do not consider here the initial and boundary conditions supposing that the grid is unboundly continuous with respect to $x$and y.

Let us study the stability of finite-difference scheme (5) by means of Neumann condition [14]. We find the solution of equations (5) in the form

$$u^k_{i,\,m,\,n}\;=\;\gamma^k e^{i\,\alpha\,m}e^{i\,\beta\,n}u_{i\,0};\quad v^k_{i,\,m,\,n}\;=\;\gamma^k e^{i\,\alpha\,m}e^{i\,\beta\,n}v_{i\,0}\quad (i=1,2). \qquad (6)$$

As a result we obtain the characteristic equation. After the following notations

$$\omega\;=\;-\frac{\gamma-2+\frac{1}{\gamma}}{\Delta\,t^2}\;; \qquad (7)$$

$$\xi=-\frac{e^{i\alpha}-2+e^{-i\alpha}}{\Delta\,x^2}=\frac{2\,(1-\cos\;\alpha)}{\Delta\,x^2}=\frac{4\,\sin^2\frac{\alpha}{2}}{\Delta\,x^2}\;;\quad \zeta\;=-\frac{\sin\alpha\,\sin\beta}{\Delta x\Delta y}\;;$$
$$\eta=-\frac{e^{i\beta}-2+e^{-i\beta}}{\Delta\,y^2}=\frac{2\,(1-\cos\;\beta)}{\Delta\,y^2}=\frac{4\,\sin^2\frac{\beta}{2}}{\Delta\,y^2}\;, \qquad (8)$$

the characteristic equation can be written in the form:

$$\begin{vmatrix} a_1^{\xi\eta}-\rho_{11}\omega & b_1^{\xi\eta}-\rho_{12}\,\omega & (A_{11}-A_{12})\zeta & (B_{11}-B_{12})\zeta \\ a_2^{\xi\eta}-\rho_{12}\omega & b_2^{\xi\eta}-\rho_{22}\omega & (A_{21}-A_{22})\zeta & (B_{21}-B_{22})\zeta \\ (A_{11}-A_{12})\zeta & (B_{11}-B_{12})\zeta & a_1^{\eta\xi}-\rho_{11}\omega & b_1^{\eta\xi}-\rho_{12}\,\omega \\ (A_{21}-A_{22})\zeta & (B_{21}-B_{22})\zeta & a_2^{\eta\xi}-\rho_{12}\omega & b_2^{\eta\xi}-\rho_{22}\omega \end{vmatrix}\;=\;0,$$

where $a_i^{fg}=A_{i1}f+A_{i2}g;\quad b_i^{fg}=B_{i1}f+B_{i2}g.$

The necessary Neumann condition of stability is that $|\gamma|\le 1$ for all eight roots γ calculated from (7), where ω are the four roots of the last equation. From the last equation we obtain the equation of the fourth order with respect to ω:

$$\omega^4+a^*\omega^3+b^*\omega^2+c^*\omega+d^*=0, \qquad (9)$$

where $a^*,b^*,c^*,d^*\in R,\,a^*\neq 0.$

This equation can be written in the form:

$$(\omega^2 + a^*\omega/2)^2 = (a^{*2}/4 - b^*)\omega^2 - c^*\omega - d^*.$$

Let us add to both parts of the equation the term$(\omega^2 + a^*\omega/2)^2 y + y^2$, then

$$(\omega^2 + a^*\omega/2 + y/2)^2 = (a^{*2}/4 - b^* + y)\omega^2 + (a^* y/2 - c^*)\omega + y^2/4 - d^*.$$

We'll find y in such a way that the right-hand part of the equation would be a perfect trinomial square. After the following notations $A_*^2 = a^{*2}/4 - b^* + y$, $\quad B_*^2 = y^2/4 - d^*$, $2A_* B_* = a^* y^2/2 - c^*$, this condition can be written in the form $4A_*^2 B_*^2 = (2A_* B_*)^2$. So we've obtained the resolvable equation. If y_0 is the root of the last equation, then the solution of equation (9) reduces to the solution of the following two equations $\omega^2 + a^*\omega/2 + y_0/2 = A_*\omega + B_*$ and $\omega^2 + a^*\omega/2 + y_0/2 = -A_*\omega - B_*$. The examination of the roots of these equations by means of computer showed that all four roots are real and positive.

Let us consider the equation (7). It can be written in the form

$$\gamma^2 - \left(2 - \omega\,\Delta\,t^2\right)\ \gamma\ + 1 = 0. \tag{10}$$

It is easy to realize that in the case of real ω one of the following situations is possible:

– if the next condition is fulfilled

$$0 \le \omega\ \Delta\,t^2 \le 4, \tag{11}$$

then both roots are complex and their modules are equal to 1;

– if the condition (11) is not fulfilled, then both roots are real one of them is less than 1, but another is greater than1 (the product of the roots is equal to 1).

Thus, even if one of the values ω_i $(i = \overline{1,4})$ does not satisfy (11), then among the eight roots γ of the characteristic equation (5) there is necessarily one with module greater than 1. According to Neumann condition the finite-difference scheme (5) will be unstable. If the values ω_i satisfy the condition (11), then the modules of all eight roots will be equal to 1. Hence, the finite-difference scheme (5) without boundary conditions will be stable.

The examination of the roots of the equation (9) makes it possible to say that the maximum value of the greatest of them is achieved at the corner point of the rectangle $0 \le \xi \le 4/\Delta x^2$, $\quad 0 \le \eta \le 4/\Delta y^2$.

If the maximum value of the function $\omega\,(\xi,\ \eta)$ is achieved at the corner point $\xi = 4/\Delta x^2$, $\eta = 4/\Delta y^2$, then the following estimation is fulfilled:

$$\omega \le \left(\Omega + \sqrt{\Omega^2 - 4\Delta^*(AC - BD)}\right)/(2\Delta^*), \tag{12}$$

where $\Omega = A\rho_{22} + C\rho_{11} - (B + D)\rho_{12}$, $\quad \Delta^* = \rho_{11}\rho_{22} - \rho_{12}^2$, $\quad A = 4(A_{11}/\Delta x^2 + A_{12}/\Delta y^2)$, $\quad B = 4(B_{11}/\Delta x^2 + B_{12}/\Delta y^2)$, $\quad C = 4(B_{21}/\Delta x^2 + B_{22}/\Delta y^2)$, $\quad D = 4(A_{21}/\Delta x^2 + A_{22}/\Delta y^2)$.

It is evident that $\Omega = \Delta^*(a^2 + b^2)$, where

$$a^2 = \left(\Theta + \sqrt{\Theta^2 - 4\Delta^*(A_{11}B_{21} - B_{11}A_{21})}\right) \Big/ (2\Delta^*);$$

$$b^2 = \left(\Sigma + \sqrt{\Sigma^2 - 4\Delta^*(A_{12}B_{22} - B_{12}A_{22})}\right) \Big/ (2\Delta^*);$$

$$\Theta = A_{11}\rho_{22} + B_{21}\rho_{11} - (A_{21} + B_{11})\rho_{12}; \quad \Sigma = A_{12}\rho_{22} + B_{22}\rho_{11} - (A_{22} + B_{12})\rho_{12},$$

as a result we obtain

$$\omega \le \frac{a^2 + b^2}{2}\left(\frac{4}{\Delta x^2} + \frac{4}{\Delta y^2}\right) + \frac{a^2 - b^2}{2}\left|\frac{4}{\Delta x^2} - \frac{4}{\Delta y^2}\right|. \tag{13}$$

According to the condition (11) the stability of the finite-difference scheme without boundary conditions will take place if the step with respect to time variable will satisfy the following condition:

$$\Delta t = \frac{h}{\sqrt{a^2 + b^2}}, \quad if \;\; \Delta x = \Delta y = h; \tag{14}$$

$$\Delta t = \frac{\Delta x \Delta y}{\sqrt{a^2 \Delta y^2 + b^2 \Delta x^2}}, \quad if \quad \Delta x \le \Delta y; \tag{15}$$

$$\Delta t = \frac{\Delta x \Delta y}{\sqrt{a^2 \Delta x^2 + b^2 \Delta y^2}}, \quad if \;\; \Delta x \ge \Delta y. \tag{16}$$

Now we'll take in consideration the dissipative terms in the system (4). The right-hand parts of the second system can be approximated by the relation

$$\frac{\partial f}{\partial t} \sim \frac{f^{k+1}_{n,m} - f^k_{n,m}}{\Delta t} = (f^k_{n,m})_{\bar{t}}, \tag{17}$$

where f is one of the functions $u_i, \;\; v_i, \;\; i = 1, 2.$

The values of these additional terms (in comparison with elastic model) are taking into account in the construction of transmission formulas for the next time moment $t_{k+1} = t_k + \Delta t$.

The finite-difference scheme for system (2) in the operator form can be written in the following form

$$U^{k+1} = [E + \tau(A_I + A_{II})] \; U^k, \tag{18}$$

where $A_I, \;\; A_{II}$ are difference operators with chosen approximation of the right-hand parts.

Let τ_I and τ_{II} be the time steps that provide the stability of these systems, i.e. the conditions $\|E + \tau_I A_I\| \le 1$; $\|E + \tau_{II} A_{II}\| \le 1$ are fulfilled for some norm of the difference operator. Then, if the step τ, verifies the inequality

$$\tau\left(\frac{1}{\tau_I} + \frac{1}{\tau_{II}}\right) \le 1; \tag{19}$$

then the condition

$$\|E+\tau\,(A_I+A_{II})\|\leq 1, \tag{20}$$

is fulfilled, i.e. the stability of the finite-difference scheme for system of equations (4).

In reality, from the identity $E+\tau\ \ (A_I+A_{II})=r_I\,(E+\tau_I A_I)+r_{II}\,(E+\tau_{II}A_{II})+(1-r_I-r_{II})\,E$ (here $r_I\tau_I=\tau,\ \ r_{II}\tau_{II}=\tau$) and from the convexity of the norm follows that

$$\|E+\tau\ \ (A_I+A_{II})\|\leq r_I\|E+\tau_I A_I\|+r_{II}\|E+\tau_{II}A_{II}\|+$$

$$+|\ 1-r_I-r_{II}|\ \ \leq r_I+r_{II}+|\ 1-r_I-r_{II}|\ .$$

Hence, the inequality (20) will be fulfilled, if $1-r_I-r_{II}\geq 0$. As $r_I=\tau/\tau_I$, $r_{II}=\tau/\tau_{II}$, then the last condition consider with (19).

As it was mentioned above the stability of the finite-difference scheme for elastic model is provided by conditions (15) and (16), i.e.

$$\frac{1}{\tau_I}=\frac{\sqrt{a^2\Delta y^2+b^2\Delta x^2}}{\Delta x\Delta y}\quad if\ \ \Delta x\leq\Delta y;$$

$$\frac{1}{\tau_I}=\frac{\sqrt{a^2\Delta x^2+b^2\Delta y^2}}{\Delta x\Delta y}\quad if\ \ \Delta x\geq\Delta y. \tag{21}$$

Let us obtain the value τ_{II}, which provides the stability of the corresponding scheme.

With the help of auxiliary value $(\lambda-1)/\Delta t=-\mu$, we obtain the characteristic equation in the following form:

$$\begin{vmatrix} a_1^{\xi\eta}\ -\ \rho_{11}\mu & b_1^{\xi\eta}\ -\rho_{12}\,\mu & (A_{11}-A_{12})\zeta & (B_{11}-B_{12})\zeta \\ a_2^{\xi\eta}\ -\ \rho_{12}\mu & b_2^{\xi\eta}\ -\ \rho_{22}\mu & (A_{21}-A_{22})\zeta & (B_{21}-B_{22})\zeta \\ (A_{11}-A_{12})\zeta & (B_{11}-B_{12})\zeta & a_1^{\eta\xi}\ -\ \rho_{11}\mu & b_1^{\eta\xi}\ -\rho_{12}\,\mu \\ (A_{21}-A_{22})\zeta & (B_{21}-B_{22})\zeta & a_2^{\eta\xi}\ -\ \rho_{12}\mu & b_2^{\eta\xi}\ -\ \rho_{22}\mu \end{vmatrix}\ =\ 0. \tag{22}$$

From this determinantal equation we obtain

$$(\mu-\xi-\eta)\ \left[\mu^2-\frac{4}{3}\,(\xi+\eta)\,\mu+\frac{4}{3}\left(\xi\eta-\zeta^2\right)\right]=0.$$

According to the above notations $\lambda=1-\mu\Delta\,t$ and, hence, the necessary condition of stability $|\lambda|\leq 1$ is reduced to the inequality

$$1-\mu*\Delta\,t\geq -1 \tag{23}$$

where $\mu*$ is the maximal value of the greatest root and α, β are arbitrary.

The maximal value of the greatest root for arbitrary $\alpha,\ \beta$ was studied by means of computer in rectangle $0\leq\xi\leq 4/\Delta x^2\ ,\quad 0\leq\eta\leq 4/\Delta y^2$. The maximal value of the greatest root is achieved at a corner point.

If the maximal value of the function $\mu(\xi,\ \eta)$ is achieved at the corner point $\xi = 4/\Delta x^2$, $\eta = 4/\Delta y^2$, then the following estimation is fulfilled:

$$\mu^* \leq \frac{AC - BD}{b(A + C + B + D)}. \tag{24}$$

Hence, from (19) we obtain:

$$\Delta t \leq 2/\mu*; \quad \tau_{II} = \frac{(AC - BD)}{2b(A + B + C + D)}. \tag{25}$$

So, from the condition of stability (19) for equal grid steps $\Delta\, x = \Delta\, y = h$, we obtain

$$\tau = \frac{\tau_I + \tau_{II}}{\tau_I \tau_{II}}. \tag{26}$$

Thus, in comparison with "pure" elastic model the calculations of the dissipative problem by means of explicit finite-difference scheme must be effectuated with a smaller time step.

It is obvious that the application of the explicit difference scheme is expedient only in rather narrow range of dissipative coefficient, when the ratio $b/h$is small. We shall notice that in the case of small values of b, the attributing of the dissipative terms in finite-difference equations loses sense as the coefficients of difference viscosity of this scheme are values of the order h^2.

In the case when $b >> 1$ it is expedient to consider independently a dissipative system of equations instead of system (4) The elasticity will play a role of small correction for the solution.

The carried out research allows to hope that the stability of calculations with the time step verifying condition (26) will take place. However, as the research was carried out without taking into account boundary conditions, it requires experimental examination. Such examination was carried out. As an example the problem of impact of the rectangular domain on a rigid barrier was considered. The calculated formulas were received in the boundary nodes of the grid. This explicit finite-difference scheme was successfully approved under test problems. The acceptable coordination of the compared results and obtaining the converging solutions by reducing the grid step testify their reliability and closeness to the exact solution. The realization of numerical experiments with different grids (when $h \to 0$) has allowed to estimate the actual speed of convergence of the difference scheme and to optimize the number of nodes of integration to achieve the acceptable accuracy by the minimal expenses of computer time and operative memory resources.

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