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Cyclic planar random evolution with four directions

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Abstract. A four-direction cyclic random motion with constant finite speed v in the plane R^2 driven by a homogeneous Poisson process of rate $\lambda > 0$ is studied. A fourth-order hyperbolic equation with constant coefficients governing the transition law of the motion is obtained. A general solution of the Fourier transform of this equation is given. A special non-linear automodel substitution is found reducing the governing partial differential equation to the generalized fourth-order ordinary Bessel differential equation, and the fundamental system of its solutions is explicitly given.

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1 Introduction

Various models of Markovian random evolutions performed by a particle moving at chance with a constant finite speed are fairly attractive subject, which many researchers have been dealing with. Such an interest is mostly due to the fact that a great deal of practically important applied models in statistical physics, biology, transport processes and engineering (see, for instance, Tolubinsky [15], Ratanov [14], Papanicolaou [13], Brooks [1], Kolesnik [6] and the bibliography therein) can be described and studied in terms of random evolutions.

The one-dimensional motions are the most studied models in which one often managed to obtain the explicit forms of distributions (see Foong [3], Foong and Kanno [4], Orsingher [10], Ratanov [14], Kolesnik [7]) or the estimates of their normal approximations (see Brooks [1]). As far as their multidimensional counterparts are concerned, only a few particular planar random evolutions were studied so far (see Kolesnik [5], Orsingher and San Martini [12], Kolesnik and Turbin [8], Orsingher [11], Kolesnik and Orsingher [9], Di Crescenzo [2]). By this, an explicit form of the distribution was obtained only for the planar random motion with four mutually orthogonal directions without reflection (see Orsingher [11]).

The planar random evolutions performed by a particle changing the directions of its motion in a cyclic way are of a special interest because various cyclic processes are

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rather broadly used for modelling real phenomena. For example, in the well-known statistical problem of discovering a random signal in a multi-channel system the optimal strategy is just the cyclic choice of the channels. In biology the behaviour of tetramers obeys a cyclic scheme too.

The cyclic planar random evolutions have been examined by some authors. In particular, in Orsingher and San Martini [12] such a motion with three cyclically changing directions has been studied, and the explicit solutions of some initial-value problems for the governing equations have been found. The similar three-direction model have recently been investigated by Di Crescenzo [2] where, by different methods, the functional relations for the distribution of this motion have been given in terms of multidimensional convolutions.

In this paper we present a further generalization of the models mentioned above to the case of four mutually orthogonal directions changing in the cyclic way. We obtain a fourth-order hyperbolic equation governing the transition law of the motion and give its *general* solution in terms of Fourier transforms. It is important to note that the roots of corresponding characteristic equation are found *explicitly*. As an alternative approach, we were able to find a non-linear *automodel* substitution reducing governing partial differential equation to the generalized fourth-order ordinary Bessel differential equation, whose linearly independent solutions (i.e. fundamental system of solutions) are also given. It is worth to especially emphasize that we were able to find the fundamental system of solutions in an *explicit* form. This interesting fact gives us some hints for further generalizations of such types of models.

2 Description of the Motion and the Governing Equation

A particle moves with some constant finite speed v in the plane R^2. At every time instant t it can have one of the four possible directions of motion $D(t) = E_k$, where the direction E_k is orientated like the unit vector $(\cos(\pi k/2),\ \sin(\pi k/2)), k = 0, 1, 2, 3$. In other words, the particle can move parallelly to the coordinate axes OX and OY only. The motion is controlled by a homogeneous Poisson process of rate $\lambda > 0$ changing the directions according to the cyclic scheme

$$\cdots \to E_0 \to E_1 \to E_2 \to E_3 \to E_0 \to \dots$$

This means that at each Poisson-paced time moment the particle instantly changes its direction in accordance with this rule and continues its motion in the chosen direction with the same speed v until the next Poisson event occurs, then it cyclically takes on a new direction, and so on.

Denote by $Z(t) = (X(t), Y(t))$ the particle's position in the plane R^2 at some time instant $t > 0$. We are interested in studying the behaviour of the transition law of the process $Z(t)$. Introduce the joint partial densities $f_k = f_k(x, y, t),\ (x, y) \in R^2,\ t > 0$, of the particle's position and its direction as follows

$$f_k(x, y, t)\ dxdy = P\{x \le X(t) < x + dx,\ y \le Y(t) < y + dy,\ D(t) = E_k\}, \qquad (1)$$

$$k = 0, 1, 2, 3.$$

Since the random events $\{D(t) = E_k,\ k = 0, 1, 2, 3, \}$ do not intersect and form the full group of events, then the function $p = p(x, y, t),\ (x, y) \in R^2,\ t > 0$, defined as $p = f_0 + f_1 + f_2 + f_3$, represents the transition density of the motion $Z(t)$.

Our first result concerns the equation governing function p. It is given by the following theorem.

Theorem 1. *The transition density $p = p(x, y, t),\ (x, y) \in R^2,\ t > 0$, of the cyclic planar random evolution with four directions satisfies the following fourth-order hyperbolic equation with constant coefficients*

$$\left\{ \left[\left(\frac{\partial}{\partial t} + \lambda \right)^2 - v^2 \frac{\partial^2}{\partial x^2} \right] \left[\left(\frac{\partial}{\partial t} + \lambda \right)^2 - v^2 \frac{\partial^2}{\partial y^2} \right] - \lambda^4 \right\} p = 0. \tag{2}$$

Proof. The Kolmogorov equation written down for the densities (1) leads to the following hyperbolic system of four first-order PDEs

$$\frac{\partial f_k}{\partial t} = -v \cos \frac{\pi k}{2} \cdot \frac{\partial f_k}{\partial x} - v \sin \frac{\pi k}{2} \cdot \frac{\partial f_k}{\partial y} - \lambda f_k + \lambda f_{k-1},$$

$$k = 0, 1, 2, 3, \quad f_{-1} \stackrel{\text{def}}{=} f_3.$$

Computing the determinant of this system and according to Kolesnik [7], Theorem 2, we come to the conclusion that each function f_k as well as their sum satisfy hyperbolic PDE (2). $\square$

It is easy to check that the exponential substitution

$$p(x, y, t) = e^{-\lambda t} w(x, y, t) \tag{3}$$

reduces equation (2) to the equation

$$\left\{ \left(\frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial x^2} \right) \left(\frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial y^2} \right) - \lambda^4 \right\} w(x, y, t) = 0. \tag{4}$$

This equation will become the main subject of our further analysis.

The Fourier transform of the function $w = w(x, y, t)$

$$\mathcal{W}(\alpha, \beta, t) = \iint\limits_{R^2} e^{i\alpha x + i\beta y} w(x, y, t)\, dxdy$$

satisfies the ordinary differential equation

$$\frac{d^4 \mathcal{W}}{dt^4} + v^2(\alpha^2 + \beta^2) \frac{d^2 \mathcal{W}}{dt^2} + (v^4 \alpha^2 \beta^2 - \lambda^4) \mathcal{W} = 0. \tag{5}$$

Our next result concerns the general solution of equation (5). It is given by the following theorem.

Theorem 2. *The general solution of equation (5) has the form*

$$\mathcal{W}(\alpha,\beta,t) = C_0 e^{r_0 t} + C_1 e^{r_1 t} + C_2 e^{r_3 t} + C_3 e^{r_3 t}, \tag{6}$$

where C_0, C_1, C_2, C_3 are arbitrary constants, and

$$\begin{aligned} r_0 &= \sqrt{\frac{-v^2(\alpha^2+\beta^2) + \sqrt{v^4(\alpha^2-\beta^2)^2+4\lambda^4}}{2}}, \\ r_1 &= \sqrt{\frac{-v^2(\alpha^2+\beta^2) - \sqrt{v^4(\alpha^2-\beta^2)^2+4\lambda^4}}{2}}, \\ r_2 &= -\sqrt{\frac{-v^2(\alpha^2+\beta^2) + \sqrt{v^4(\alpha^2-\beta^2)^2+4\lambda^4}}{2}}, \\ r_3 &= -\sqrt{\frac{-v^2(\alpha^2+\beta^2) - \sqrt{v^4(\alpha^2-\beta^2)^2+4\lambda^4}}{2}}. \end{aligned} \tag{7}$$

Proof. The characteristic equation of the ordinary differential equation (5) is the bi-square equation

$$r^4 + v^2(\alpha^2+\beta^2)r^2 + (v^4\alpha^2\beta^2 - \lambda^4) = 0,$$

whose roots, as is easy to see, are given by (7). □

Remark. The constants C_0, C_1, C_2, C_3 (depending on α and β) can be found from the initial conditions in each particular case.

Corollary. *The general solution $\mathcal{P}(\alpha,\beta,t)$ of the Fourier transform of equation (2) has the form*

$$\mathcal{P}(\alpha,\beta,t) = C_0 e^{(-\lambda+r_0)t} + C_1 e^{(-\lambda+r_1)t} + C_2 e^{(-\lambda+r_3)t} + C_3 e^{(-\lambda+r_3)t},$$

where r_0, r_1, r_2, r_3 are given by (7). This immediately follows from (3).

3 Fundamental System of Solutions

In this section we give an alternative approach leading to the fundamental system of solutions of equation (2). One should especially emphasize that we obtain such a system in an explicit form, unlike the solutions in terms of Fourier transforms given above.

The principal result of this section is given by the following theorem.

Theorem 3. *The fundamental system of solutions of equation (2) has the form*

$$g_i(x,y,t) = e^{-\lambda t} J^{(i)}(x,y,t), \qquad i = 0,1,2,3, \tag{8}$$

where $J^{(i)}$ are the generalized Bessel functions

$$J^{(0)}(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{(k!)^4} \left(\frac{2\lambda}{v}\right)^{4k} \left(\frac{z}{4}\right)^{4k},$$

$$J^{(1)}(x,y,t) = \ln z + \sum_{k=1}^{\infty} \frac{1}{(k!)^4} \left(\frac{2\lambda}{v}\right)^{4k} \left(\ln z - 1 - \frac{1}{2} - \dots - \frac{1}{k}\right) \left(\frac{z}{4}\right)^{4k},$$

$$\begin{aligned} J^{(2)}(x,y,t) = (\ln z)^2 + \sum_{k=1}^{\infty} \frac{1}{(k!)^4} \left(\frac{2\lambda}{v}\right)^{4k} \Bigg[\left(\ln z - 1 - \frac{1}{2} - \dots - \frac{1}{k}\right)^2 \\ + \frac{1}{4}\left(1 + \frac{1}{2^2} + \dots + \frac{1}{k^2}\right)\Bigg] \left(\frac{z}{4}\right)^{4k}, \end{aligned} \tag{9}$$

$$\begin{aligned} J^{(3)}(x,y,t) = (\ln z)^3 + \sum_{k=1}^{\infty} \frac{1}{(k!)^4} \left(\frac{2\lambda}{v}\right)^{4k} \Bigg[\left(\ln z - 1 - \frac{1}{2} - \dots - \frac{1}{k}\right)^3 \\ + \frac{3}{4^2}\left(\ln z - 1 - \frac{1}{2} - \dots - \frac{1}{k}\right)\left(1 + \frac{1}{2^2} + \dots + \frac{1}{k^2}\right) \\ - \frac{2}{4^2}\left(1 + \frac{1}{2^3} + \dots + \frac{1}{k^3}\right)\Bigg] \left(\frac{z}{4}\right)^{4k}, \end{aligned}$$

and z is given by the equality

$$z = \left[(v^2t^2 - x^2)(v^2t^2 - y^2)\right]^{1/4}. \tag{10}$$

Proof. By means of simple but fairly unwieldy computations one can show that the automodel substitution (10) reduces partial differential equation (4) to the generalized fourth-order ordinary Bessel differential equation

$$\left\{ \mathbf{B}_z^4 - \left(\frac{2\lambda}{v}\right)^4 z^4 \right\} \psi(z) = 0, \tag{11}$$

where $\mathbf{B}_z^4$ is the generalized fourth-order Bessel differential operator

$$\mathbf{B}_z^4 = \left(z \frac{d}{dz}\right)^4.$$

According to Turbin and Plotkin [16], p.118, the solutions of equation (11) are given by the generalized Bessel functions (9). In order to check the linear independence of these functions one needs to show that their Wronskian is not zero at some arbitrary point. It is convenient to check that, for instance, at the point $z = 1$ or $z = 4$. Then taking into account (3) we obtain the statement of the theorem. □

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