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The Schauder basis in symmetrically normed ideals of operators

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Abstract. In this paper we build a basis in a separable symmetrically normed ideal.

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It is well known that every Banach space with Shauder basis is separable. Converse proposition, as P.Enflo showed in 1973 [1] is not true. In the present work the problem of the existence of a Schauder basis in separable symmetrically normed ideals is considered. It is found that all such ideals have a basis. For particular case, symmetrically normed Lorentz ideals $\Upsilon_{p,q}$, a basis was built in [2].
The terminology of the article is based on [3].

Theorem. *Let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis in a Hilbert space H. A sequence of linear continuous operators $\{A_n\}_{n=1}^{\infty}$ of the form*

$$A_{m^2+j} = \left\{ \begin{array}{ll} (\cdot, \phi_{m+1})\phi_j, & 1 \leq j \leq m+1 \\ (\cdot, \phi_{2m+2-j})\phi_{m+1}, & m+1 < j \leq 2m+1 \end{array} \right. , m = 0, 1, \ldots$$

forms a basis in every symmetrically normed ideal.

Proof. Let Υ be a separable symmetrically normed ideal. Since the ideal Υ is separable there is a symmetrically normed function $\Phi(x)$ so that $\Upsilon = \Upsilon_{\Phi}^{(0)}$. For every operator $A \in \Upsilon_{\Phi}^{(0)}$ we can write the Schmidt representation: $A = \sum_{j=1}^{\infty} s_j(A)(\cdot, x_j)y_j$. For every $\epsilon > 0$ we can choose $n_0 \in \mathbf{N}$ such that $||A - A_{n_0}|| < \epsilon/2$, where $A_{n_0} = \sum_{j=1}^{n_0} s_j(A)(\cdot, x_j)y_j$. For every $0 < \delta < 1$ and $\forall j \in \mathbf{N}$ there are $u_j, v_j \in span\{\phi_j\}_{j=1}^{\infty}$ such as $||x_j - u_j|| < \delta, ||y_j - v_j|| < \delta$. We have $||(\cdot, x_j)y_j - (\cdot, u_j)v_j||_{\Phi} \leq ||(\cdot, x_j - u_j)y_j||_{\Phi} + ||(\cdot, u_j)(v_j - y_j)||_{\Phi} \leq 3\delta.$

If we take $\delta = \frac{\epsilon}{2n_o s_1(A)}$ and $B_{n_0} = \sum_{j=1}^{n_0} s_j(A)(\cdot, u_j)v_j \in \overline{span\{A_n\}_{n=1}^{\infty}}$ we get that $||A_{n_0} - B_{n_0}|| \leq \epsilon/2.$ Thus $||A - B_{n_0}||_{\Phi} \leq ||A - A_{n_0}||_{\Phi} + ||A_{n_0} - B_{n_0}||_{\Phi} < \epsilon$. Hence, $A \in \overline{span\{A_n\}_{n=1}^{\infty}}$, in other words, the sequence $\{A_n\}_{n=1}^{\infty}$ is complete in Υ. We show that the sequence $\{A_n\}_{n=1}^{\infty}$ is minimal. To prove that it is sufficient to show that this system has a biorthogonal one.

Define $F_{m^2+j} = sp(XA^*_{m^2+j})$, where $X \in \Upsilon_{\Phi}^{(0)}$, $sp(A) = \sum_{j=1}^{\infty}(A\phi_j, \phi_j)$ and $\{\phi_j\}_{j=1}^{\infty}$ is a basis in H.

It is easy to note that F_{m^2+j} is a linear bounded operator on $\Upsilon_{\Phi}^{(0)}$ and

$$F_{m^2+j} = sp(XA^*_{m^2+j}) = \left\{ \begin{array}{ll} 1, & m = r, j = s \\ 0, & m^2 + j \neq r^2 + s \end{array} \right. .$$

It follows that $\{F_{m^2+j}\}$ and $\{A_{m^2+j}\}$ are a biorthogonal system.

We consider the sequence of projectors $\{\mathfrak{P}_n\}_{n=1}^{\infty}$ of the form

$$\mathfrak{P}_n(A) = \sum_{j=1}^{n} F_j(A) A_j \mathfrak{P}_{m^2}(A) = \sum_{k=1}^{m}\sum_{j=1}^{m} sp(A(\cdot,\phi_k)\phi_j)(\cdot,\phi_j)\phi_k =$$

$$= \sum_{k=1}^{m}\sum_{j=1}^{m}(A\phi_j,\phi_k)(\cdot,\phi_j)\phi_k = P_m A P_m,$$

where $P_m x = \sum_{j=1}^{m}(x,\phi)j)\phi_j,\ x = \sum_{j=1}^{\infty}(x,\phi_j)\phi_j$ and $||P_m|| = 1$. We therefore have $||\mathfrak{P}_{m^2}(A)|| = ||P_m A P_m||_\Phi \leq ||A||_\Phi$ Hence, $||\mathfrak{P}_{m^2}|| \leq 1$. Let $1 \leq j \leq m+1$. Then we have

$$\mathfrak{P}_{m^2+j}(A) = P_m A P_m + \sum_{r=1}^{j} sp(A(\cdot,\phi_r)\phi_{m+1})(\cdot,\phi_{m+1})\phi_r = P_m A P_m +$$

$$+ \sum_{r=1}^{j}(A\phi_{m+1},\phi_r)(\cdot,\phi_{m+1})\phi_r = P_m A P_m + P_j A(P_{m+1} - P_m).$$

So, $||\mathfrak{P}_{m^2+j}(A)|| \leq 3||A||_\Phi$, $\forall A \in \Upsilon_\Phi^{(0)}$. Let $m+2 \leq j \leq 2m+1$. Then we have

$$\mathfrak{P}_{m^2+j}(A) = P_{m+1} A P_{m+1} - \sum_{r=1}^{2m+1-j} sp(A(\cdot,\phi_r)\phi_{m+1})(\cdot,\phi_{m+1})\phi_r =$$

$$= P_{m+1} A P_{m+1} - P_{2m+1-j} A(P_{m+1} - P_m).$$

So, $||\mathfrak{P}_{m^2+j}(A)|| \leq 3||A||_\Phi$, $\forall A \in \Upsilon_\Phi^{(0)}$.

Thus, $||\mathfrak{P}_n|| \leq 3$ $(n = 1, 2...)$. By criterion of basis in the Banach space [4], we obtain that $\{A_n\}_{n=1}^{\infty}$ is a basis of the Banach space Υ.

References

[1] Enflo P. *A counterexample to the approximation property in Banach spaces.* Acta Math., 1973, **130**, p. 309–317.

[2] Fugarolas M., Cobos F. *On Schauder bases in the Lorentz operator ideal.* J. Math. Anal. Appl., 1983, **95**, p. 235–242.

[3] Gohberg I., Krein M. *Introduction to the theory of linear nonself-adjoint operators.* Moscow, Nauka, 1965 (in Russian).

[4] Singer I. *Bases in Banach spaces I.* Berlin-Heidelberg-New York, Springer, 1970.

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