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The radical theory of convolution rings

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Abstract. Convolution rings have been defined as a unifying approach to a number of ring constructions, e.g. polynomials, matrices, necklace rings and incidence algebras. Here the radical theory of convolution rings will be investigated.

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Convolution rings were defined and studied in [22] as a unifying procedure to describe a wide variety of ring constructions. Every convolution ring is determined by a convolution type. This type is then imposed on a ring A to give the corresponding convolution ring $C(A)$. For example, a polynomial convolution type is defined which leads to the polynomial rings $A[x]$. Other examples include the direct product of a ring with itself, matrices (finite, infinite or structural), incidence algebras, necklace rings, quaternion rings, etc.

Here we will study the radical theory of convolution rings. The language of convolution rings will enable us to formulate that which is common to all the ring constructions under consideration. But it will also enable us to isolate those properties of convolution types which will enforce certain properties on the radicals of the convolution rings.

1 Introduction

Convolution types have been defined for classes of R-algebras (R any ring), but here we restrict ourselves to the class of all rings ($\mathbb{Z}$-algebras). We recall from [22]:

Definition 1. *A convolution type $\mathcal{T}$ is a quadruple $\mathcal{T} = (X, \mathcal{S}, \sigma, \tau)$ where X is a non-empty set, $\mathcal{S}$ is a non-empty set of subsets of X with $\mathcal{S} \neq \{X\}$, for every $x \in X$, $\sigma(x)$ is a non-empty subset of $X \times X$ and τ is a function $\tau : X \times X \to \mathbb{Z}$ subject to:*

(C1) $Y_1, Y_2 \in \mathcal{S}$ implies there exist a $Y \in \mathcal{S}$ with $Y \subseteq Y_1 \cap Y_2$.

(C2) $Y_1, Y_2 \in \mathcal{S}$ implies there exist a $Y \in \mathcal{S}$ such that for all $(s,t) \in \sigma(y)$, $y \in Y$, either $s \in Y_1$ or $t \in Y_2$.

(C3) For all $Y_1, Y_2 \in \mathcal{S}$, $x \in X$, the set $\{(s,t) \in \sigma(x) \mid s \in X \smallsetminus Y_1 \text{ and } t \in X \smallsetminus Y_2\}$ is finite.

(A1) For all $(s,t) \in \sigma(x), (p,q) \in \sigma(s)$ there exists a unique $v \in X$ with $(p,v) \in \sigma(x), (q,t) \in \sigma(v)$ and such that $\tau(s,t)\tau(p,q) = \tau(p,v)\tau(q,t)$.

(A2) For all $(s,t) \in \sigma(x), (p,q) \in \sigma(t)$ there exists a unique $u \in X$ with $(u,q) \in \sigma(x), (s,p) \in \sigma(u)$ and such that $\tau(s,t)\tau(p,q) = \tau(u,q)\tau(s,p)$.

Let $\mathcal{T}$ be a convolution type and let A be a ring. Let $C(A,\mathcal{T}) = \{f : X \to A \mid$there exists a $Y \in \mathcal{S}$, in general depending on f, such that $f(y) = 0$ for all $y \in Y\}$. This set Y associated with $f \in C(A,\mathcal{T})$ is called a *zero-set* for f (it need not be unique) and when necessary denoted by Y_f. On the set $C(A,\mathcal{T})$ we define the following operations: For $f, g \in C(A,\mathcal{T})$ and $x \in X$,

componentwise addition $(f+g)(x) = f(x) + g(x)$ and

convolution product $(fg)(x) = \sum_{(s,t)\in\sigma(x)} \tau(s,t)f(s)g(t)$.

Then $C(A,\mathcal{T})$ is a ring with respect to these operations. Usually we will write $C(A)$ for $C(A,\mathcal{T})$. Many examples were given in [22], as well as some first results on ideals and homomorphisms of convolution rings. We recall one which will often be used. Let I be an ideal of a ring A and let $\theta : A \to A/I$ be the corresponding surjective homomorphism. Then $C(\theta) : C(A) \to C(A/I)$, defined by $(C(\theta))(f) := \theta \circ f$ for all $f \in C(A)$, is a surjective homomorphism with $\ker C(\theta) = C(I)$, i.e. $C(A/I) \cong C(A)/C(I)$.

For a given convolution type and a radical, the single most important problem is to determine the radical of the convolution ring $C(A)$. Preferably one would like to express it in terms of the radical of the underlying ring A. For this to be possible, some connection between A and $C(A)$ will be required. To ensure this, we will impose further conditions on the convolution type. These conditions (except Example 3.1 for infinite sets X) will be in force for the remainder of this paper and all the examples discussed below will satisfy these conditions.

Let $T = \{t \in X \mid (t,t) \in \sigma(t)$ and $\tau(t,x) = 1 = \tau(x,t)$ for all $x \in X\}$. This set could be empty, but the first of the next three conditions, which we require the convolution type to satisfy, will ensure that $T \neq \varnothing$.

(T1) For every $x \in X$, there exists unique $l_x \in T$ and $r_x \in T$ such that $(l_x, x) \in \sigma(x)$ and $(x, r_x) \in \sigma(x)$.

(T2) If $(p,q) \in \sigma(x)$ and $p \in T$ (respt. $q \in T$), then $q = x$ (respt. $p = x$).

(T3) There exists $Y_T \in \mathcal{S}$ such that $T \subseteq X \setminus Y_T$.

It then follows from [22] that the mapping $\iota : A \to C(A)$ defined by

$$\iota(a) = \iota_a : X \to A \text{ with}$$
$$\iota_a(x) = \begin{cases} a \text{ if } x \in T \\ 0 \text{ if } x \notin T, \end{cases}$$

is a well-defined injective ring homomorphism. If the ring A has an identity 1_A, then $C(A)$ has an identity $e := \iota_{1_A}$ and the ideal in $C(A)$ generated by A coincides with $C(A)$ since $e \in A$.

We should point out that the embedding of A in $C(A)$ need not be unique. Suppose $T \neq \varnothing$ and choose $t_0 \in T$ fixed. The mapping $\varsigma : A \to C(A)$ defined by

$$\varsigma(a) = \varsigma_a : X \to A \text{ with}$$
$$\varsigma_a(x) = \begin{cases} a \text{ if } x = t_0 \\ 0 \text{ if } x \neq t_0 \end{cases}$$

is also an embedding of A into $C(A)$. In this case, however, an identity in A need not ensure that $C(A)$ has an identity. When $|T| = 1$, this distinction falls away, since then $\iota = \varsigma$. Without further notice we will regard ι, as defined above, as our canonical embedding of A into $C(A)$.

2 Radical theory

Throughout this section, $\mathcal{T} = (X, \mathcal{S}, \sigma, \tau)$ is a convolution type. Unless mentioned explicitly otherwise, all radicals will be in the sense of Kurosh-Amitsur and when we say that α is a radical, α will denote both the class of radical rings as well as the radical map which assigns to a ring A its radical $\alpha(A)$. For any class of rings $\mathcal{A}$, $\mathcal{SA}$ will denote the class $\mathcal{SA} = \{A \mid 0 \neq I \lhd A \Rightarrow I \notin \mathcal{A}\}$. In particular, if α is a radical class, $\mathcal{S}\alpha$ is the *semisimple class* of α.

For a given convolution type, the best possible scenario is $\alpha(C(A)) = C(\alpha(A))$ for all rings A and all radicals α. This can be realized, but only in a few very special cases. For example, for any non-empty set X, let $C(A) = \oplus_{x \in X} A$, the discrete direct sum of $|X|$-copies of A (see Example 3.1 below). However, for most convolution types one could have some radical α for which $\alpha(C(A)) = C(\alpha(A))$ holds for all rings A, but for some other radicals these two subsets of $C(A)$ need not even be comparable. For a given radical α, it could also happen that $\alpha(C(A)) = C(\alpha(A))$ for a certain convolution type, but for another convolution type, this equality need no longer be true.

A radical α is said to be $\mathcal{T}$-*invariant* if $\alpha(C(A)) = C(\alpha(A))$ for all rings A. There are two (trivial) $\mathcal{T}$-invariant radicals, namely $\alpha = \{0\}$ and α the class of all rings. In general, invariance will depend on the convolution type as well as the properties of the radical.

Recall, an ideal K of a convolution ring $C(A)$ is called $\mathcal{T}-$*homogeneous* if there is an ideal I of the ring A such that $K = C(I)$. This is equivalent to requiring the equality $C(K \cap A) = K$. Although homogeneity provides a useful link between the ideals of $C(A)$ and those of A, its real value only comes to the fore if an explicit description of the ideal $K \cap A$ of A is known. We also have a need for the following: The ideal K of $C(A)$ is called $\mathcal{T}-$ *weakly homogeneous* if $C(K \cap A) \subseteq K$. The motivation for these notions comes from the work of Amitsur [1] and subsequently Krempa [5] on the radicals of polynomial rings. For polynomial rings, the homogeneity of $\alpha(A[x])$, i.e. $\alpha(A[x]) = (\alpha(A[x]) \cap a)[x]$, is sometimes referred to as the *Amitsur Condition*. We will say the radical α is $\mathcal{T}-$*homogeneous* if $\alpha(C(A)) = C(\alpha(C(A)) \cap A)$ for all rings A and $\mathcal{T}-$ *weakly homogeneous* if $C(\alpha(C(A)) \cap A) \subseteq \alpha(C(A))$ for all rings A. Usually we will drop the reference to the convolution type.

Let P be a function which assigns to each ring A and $f \in C(A)$ a subset $P(f, A)$ of A subject to $P(0, A) = \{0\}$. In most cases we write $P(f)$ for $P(f, A)$. The most frequent definition of P is: Let $\emptyset \neq W \subseteq X$ and let $P(f) = f(W)$, but other choices will also be of some significance. When $P(f) = f(W)$ for all f, we sometimes write P as P_W. For $I \lhd A$, let $(I : P)_{C(A)} = \{f \in C(A) \mid P(f) \subseteq I\}$. When $P = P_W$ for some W, we use the usual notation $(I : W)_{C(A)} = \{f \in C(A) \mid f(W) \subseteq I\}$ in stead of $(I:P_W)_{C(A)}$. To ensure that $(I:P)_{C(A)}$ is an ideal of $C(A)$, it is sufficient to require:

(i) For all $f,g \in (I:P)_{C(A)}$, $P(f-g) \subseteq \{a-b \mid a \in P(f), b \in P(g)\}$.

(ii) For all $f \in (I:P)_{C(A)}$ and $h \in C(A), P(fh) \subseteq P(f)P(h)$ and $P(hf) \subseteq P(h)P(f)$.

In particular, $(I:W)_{C(A)}$ will be an ideal of $C(A)$ provided $\sigma(w) \subseteq W \times W$ for all $w \in W$. If $W = X$, then this condition is trivially fulfilled and $(I:X)_{C(A)} = C(I)$ is an ideal of $C(A)$ as we already know.

The radical α will be called $\mathcal{T}-$*accessible* if for all rings A there is an ideal I of A and a function P such that $\alpha(C(A)) = (I:P)_{C(A)}$ for all rings A. Note that if α is accessible, $P = P_W$ for all f and $W \cap T \neq \emptyset$, then $I = \alpha(C(A)) \cap A$. Indeed, from $\alpha(C(A)) = (I:W)_{C(A)}$ it follows that $\alpha(C(A)) \cap A = (I:W)_{C(A)} \cap A = I$. When $I = \alpha(A)$ for all A, we say α is *directly* $\mathcal{T}-$*accessible*, i.e. $\alpha(C(A)) = (\alpha(A):P)_{C(A)}$ for all A. Any invariant radical α is directly accessible with $\alpha(C(A)) = (\alpha(A):X)_{C(A)}$. For a homogeneous radical α, we know that $\alpha(C(A)) = (\alpha(C(A)) \cap A : X)_{C(A)}$, but this does not necessarily mean that α is directly accessible.

We recall from [22]: Let $D = \{x \in X \mid \sigma(x) = \{(x,x)\}$. If $D \neq \emptyset$, then there is a surjective homomorphism $\theta : C(A) \to (A/\alpha(A))^D$ with $\ker\theta = (\alpha(A):D)_{C(A)}$. Since $(A/\alpha(A))^D \in \mathcal{S}\alpha$, we have $\alpha(C(A)) \subseteq (\alpha(A):D)_{C(A)}$. Moreover, for a fixed $d_0 \in D$, there is a surjective homomorphism $\gamma : C(A) \to A$ defined by $\gamma(f) = f(d_0)$. Thus we have:

Proposition 1. *Let $\mathcal{T}$ be a convolution type with $D \neq \emptyset$. For any radical α and ring A,*

(1) $\alpha(C(A)) \subseteq (\alpha(A):D)_{C(A)}$ and

(2) $C(A) \in \alpha \Rightarrow A \in \alpha$.

Sometimes it is possible to embed a ring A as an ideal in $C(A)$. More specifically

Proposition 2. *Let $\mathcal{T}$ be a convolution type which satisfies the condition:*

(T4) If $t_0 \in T$ such that $(t_0,x) \in \sigma(x)$ or $(x,t_0) \in \sigma(x)$, then $x = t_0$.

Then A can be embedded as an ideal in $C(A)$.

Proof. Define $\eta : A \to C(A)$ by $\eta(a) = \eta_a : X \to A$

with $\eta_a(x) = \begin{cases} a \text{ if } x = t_0 \\ 0 \text{ if } x \neq t_0 \end{cases}$.

Then η is an injective homomorphism. We show $\eta(A)$ is an ideal in $C(A)$. Let $a \in A$ and $f \in C(A)$. Then

$(\eta_a f)(x) = \sum_{(p,q)\in\sigma(x)} \tau(p,q)\eta_a(p)f(q)$. Let $b := f(t_0)$. Now $\eta_a(p) = 0$ for all p unless $p = t_0$. But $(t_0,q) \in \sigma(x)$ implies $q = x$ by condition (T2) and then by (T4) we have $x = t_0$. Thus

$$\begin{aligned}(\eta_a f)(x) &= \begin{cases} \tau(t_0,t_0)\eta_a(t_0)f(t_0) \text{ if } x = t_0 \\ \quad 0 \text{ otherwise} \end{cases} \\ &= \begin{cases} ab \text{ if } x = t_0 \\ 0 \text{ otherwise} \end{cases} \\ &= \eta_{ab}(x).\end{aligned}$$

Hence $\eta_a f = \eta_{ab} \in \eta(A)$. Likewise $f\eta_a \in \eta(A)$. $\square$

Since semisimple classes are hereditary, we have

Proposition 3. *Let $\mathcal{T}$ be a convolution type which satisfies condition (T4). Then $C(A) \in \mathcal{S}\alpha \Rightarrow A \in \mathcal{S}\alpha$.*

Proposition 4. *Let $\mathcal{T}$ be a convolution type and let α be a radical.*

Then:

(1) $\alpha(C(A)) \subseteq C(\alpha(A))$ *for all* A
$\Leftrightarrow (A \in \mathcal{S}\alpha \Rightarrow C(A) \in \mathcal{S}\alpha)$
$\Rightarrow (C(A) \in \alpha \Rightarrow A \in \alpha)$

and if α is homogeneous, then $(A \in \mathcal{S}\alpha \Rightarrow C(A) \in \mathcal{S}\alpha) \Leftrightarrow (C(A) \in \alpha \Rightarrow A \in \alpha)$.

(2) $C(\alpha(A)) \subseteq \alpha(C(A))$ *for all* A
$\Leftrightarrow (A \in \alpha \Rightarrow C(A) \in \alpha)$
$\Rightarrow (C(A) \in \mathcal{S}\alpha \Rightarrow A \in \mathcal{S}\alpha)$

and if α is homogeneous, then $(A \in \alpha \Rightarrow C(A) \in \alpha) \Leftrightarrow (C(A) \in \mathcal{S}\alpha \Rightarrow A \in \mathcal{S}\alpha)$.

Proof. (1) The equivalence is clear. Suppose $A \in \mathcal{S}\alpha \Rightarrow C(A) \in \mathcal{S}\alpha$. Let $C(A) \in \alpha$. Then $C(A)/C(\alpha(A)) \in \alpha$. But $A/\alpha(A) \in \mathcal{S}\alpha$ implies $C(A)/C(\alpha(A)) \cong C(A/\alpha(A)) \in \mathcal{S}\alpha$ which gives $A = \alpha(A) \in \alpha$. Suppose α is homogeneous and $C(A) \in \alpha \Rightarrow A \in \alpha$. Let $A \in \mathcal{S}\alpha$. Then $C(\alpha(C(A)) \cap A) = \alpha(C(A)) \in \alpha$ implies $\alpha(C(A)) \cap A \in \alpha$ by the assumption. Thus $\alpha(C(A)) \cap A \subseteq \alpha(A) = 0$. This means $\alpha(C(A)) = C(\alpha(C(A)) \cap A) = 0$, i.e. $C(A) \in \alpha$.

(2) Both the equivalence and implication are clear. We only show the converse of the last implication under the assumption of homogeneity. Suppose α is homogeneous and $C(A) \in \mathcal{S}\alpha \Rightarrow A \in \mathcal{S}\alpha$. Let $A \in \alpha$. Then $\alpha(C(A)) = C(\alpha(C(A)) \cap A)$ and $C(A/\alpha(C(A)) \cap A) \cong C(A)/C(\alpha(C(A)) \cap A) = C(A)/\alpha(C(A)) \in \mathcal{S}\alpha$. By our assumption $A/\alpha(C(A)) \cap A \in \mathcal{S}\alpha$ and thus $A = \alpha(A) \subseteq \alpha(C(A)) \cap A$, i.e. $A \subseteq \alpha(C(A))$. Thus $\alpha(C(A)) = C(\alpha(C(A)) \cap A) = C(A)$ and so $C(A) \in \alpha$. $\square$

Proposition 5. *Let $\mathcal{T}$ be a convolution type and let α be a radical. The following five conditions are equivalent:*

(1) *α is invariant (i.e. $\alpha(C(A)) = C(\alpha(A))$ for all A)*
(2) *(a) $\alpha(C(A)) \subseteq C(\alpha(A))$ for all A and*
(b) $C(\alpha(A)) \subseteq \alpha(C(A))$ for all A.
(3) *(a) $A \in \mathcal{S}\alpha \Rightarrow C(A) \in \mathcal{S}\alpha$ and*
(b) $A \in \alpha \Rightarrow C(A) \in \alpha$.
(4) *(a) α is homogeneous and*
(b) $A \in \alpha \Leftrightarrow C(A) \in \alpha$.
(5) *(a) α is homogeneous and*
(b) $A \in \mathcal{S}\alpha \Leftrightarrow C(A) \in \mathcal{S}\alpha$.

We next investigate the homogeneity condition. Krempa [4] has shown that for polynomial rings $A[x]$ this is equivalent to the condition $\alpha(A[x]) \cap A = 0 \Rightarrow \alpha(A[x]) = 0$. This equivalence does not extend to convolution rings in general, which necessitates more terminology: A radical α is said to satisfy the *Krempa Condition* with respect to the convolution type $\mathcal{T}$ if $\alpha(C(A)) \cap A = 0 \Rightarrow \alpha(C(A)) = 0$.

Proposition 6. *Let $\mathcal{T}$ be a convolution type and let α be a radical. The following three conditions are equivalent:*

(1) *α is homogeneous*
(2) *(a) α is weakly homogeneous and*
(b) α satisfies the Krempa Condition
(3) *(a) α is weakly homogeneous and*
(b) $C(A) \in \mathcal{S}\alpha$ for all rings A which has no non-zero ideals I . with $C(I) \in \alpha$

Proof. Suppose (1) holds. We show the validity of (3). The first part is obvious, so we only verify (b). Let A be a ring which has no non-zero ideals I with $C(I) \in \alpha$. Then $C(\alpha(C(A)) \cap A) = \alpha(C(A) \in \alpha$ implies $\alpha(C(A)) \cap A = 0$. Thus $\alpha(C(A) = 0$, i.e. $C(A) \in \mathcal{S}\alpha$. Next we show (3) $\Rightarrow$ (2). Let A be a ring with $\alpha(C(A)) \cap A = 0$. If I is an ideal of A with $C(I) \in \alpha$, then $C(I) \subseteq \alpha(C(A))$ and thus $I \subseteq C(I) \cap A \subseteq \alpha(C(A)) \cap A = 0$. From (3)$(b)$ we get $\alpha(C(A) = 0$.

(2) $\Rightarrow$ (1). Let A be a ring and let $B := \alpha(C(A)) \cap A$. Then $C(B) \subseteq \alpha(C(A))$. Let $\overline{A} = \dfrac{A}{B}$. Then $\overline{A} \hookrightarrow C(\overline{A}) \cong \dfrac{C(A)}{C(B)}$ and under this isomorphism, $\overline{A} = \dfrac{A}{B} \cong \dfrac{A + C(B)}{C(B)} \hookrightarrow \dfrac{C(A)}{C(B)}$. Since $C(B) \subseteq \alpha(C(A))$, $\alpha(C(\overline{A})) = \dfrac{\alpha(C(A))}{C(B)}$. Thus

$$\alpha(C(\overline{A})) \cap \overline{A} = \frac{\alpha(C(A))}{C(B)} \cap \frac{A + C(B)}{C(B)} =$$

$$= \frac{(\alpha(C(A)) \cap A) + C(B)}{C(B)} = \frac{B + C(B)}{C(B)} = 0.$$

From (2)(a) we get $\alpha(C(\overline{A})) = 0$ which gives $\alpha(C(A)) \subseteq C(\alpha(C(A)) \cap A)$. The converse inclusion is given by (2)(b). □

Below we shall see that weakly homogeneity is often a consequence of the properties of the convolution type. In such cases, homogeneity is equivalent to the Krempa Condition which in turn is equivalent to condition (3)(b). This latter condition has been considered by Tumurbat and Wiegandt [16] for polynomial rings.

Proposition 7. *Let $\mathcal{T}$ be a convolution type such that for every ring R with identity, all ideals of $C(R)$ are homogeneous. Then every radical α is $\mathcal{T}$-homogeneous.*

Proof. Let $D(A)$ be the Dorroh extension of the ring A (i.e. the canonical unital extension of A). By the ADS-property, $\alpha(C(A))$ is an ideal of $C(D(A))$. The assumption implies $\alpha(C(A)) = C(I)$ for some ideal I of $D(A)$. But $C(I) = \alpha(C(A)) \subseteq C(A)$ implies $I \subseteq A$. Thus $\alpha(C(A))$ is a homogeneous ideal of $C(A)$. □

The ideal $\alpha(C(A)) \cap A$ plays an important role in the homogeneous requirement, and we next explore this and related properties. Here the work of Amitsur for polynomial rings [1], Krempa for semi-group rings [5] as well as the generalization considered by Ortiz [6] serves as motivation for our considerations.

For the radical α, we define two classes of rings α^c and $\overline{\alpha}$ by

$\alpha^c := \{R \mid C(R) \in \alpha\}$ and
$\overline{\alpha} := \{R \mid R \subseteq \alpha(C(R))\}$ and three ideals of a ring R by
$\alpha^c(R) := \sum(I \lhd R \mid I \in \alpha^c) = \sum(I \lhd R \mid C(I) \in \alpha)$,
$\overline{\alpha}(R) := \sum(I \lhd R \mid I \in \overline{\alpha}) = \sum(I \lhd R \mid I \subseteq \alpha(C(I)))$ and
$\alpha^*(R) := \alpha(C(R)) \cap R$.

All these depend, of course, on the convolution type $\mathcal{T}$, so when necessary, it will be emphasized by adding a subscript $\mathcal{T}$, as for example in $\alpha^c_{\mathcal{T}}$.

It can be verified that $\alpha^c \subseteq \overline{\alpha}$ and for any ring R, $\alpha^c(R) \subseteq \overline{\alpha}(R) \subseteq \alpha^*(R)$. If A is a ring such that the ideal generated by A coincides with $C(A)$, then $A \in \alpha^c \Leftrightarrow A \in \overline{\alpha}$. In particular, as we know from [22], if A is a ring with identity, then $A \in \alpha^c \Leftrightarrow A \in \overline{\alpha}$. If $\alpha(C(A))$ is weakly homogeneous for all rings A, then $\alpha^c = \overline{\alpha}$. Indeed, let $A \in \overline{\alpha}$. Then $A = \alpha(C(A)) \cap A$ and so $C(A) = C(\alpha(C(A)) \cap A) \subseteq \alpha(C(A))$. Thus $C(A) \in \alpha$, i.e. $A \in \alpha^c$. Also note that $C(\alpha^*(A)) \in \alpha$ for all A implies that α is weakly homogeneous and the converse holds if α is hereditary. It is clear that $\alpha^c \subseteq \alpha \Leftrightarrow (A \in \alpha \Rightarrow C(A) \in \alpha)$, $\alpha \subseteq \alpha^c \Leftrightarrow (C(A) \in \alpha \Rightarrow A \in \alpha)$ and thus $\alpha = \alpha^c \Leftrightarrow (C(A) \in \alpha \Leftrightarrow A \in \alpha)$. If $D = \{x \in X \mid \sigma(x) = \{(x,x)\}\} \neq \emptyset$, then $\overline{\alpha} \subseteq \alpha$. Indeed, as mentioned earlier, if $D \neq \emptyset$, then there is a surjective homomorphism $\theta : C(A) \to \dfrac{A}{\alpha(A)}$ with $\dfrac{C(A)}{\ker\theta} \cong \dfrac{A}{\alpha(A)} \in \mathcal{S}\alpha$, $\ker\theta \cap A = \alpha(A)$ and $C(\alpha(A)) \subseteq \ker\theta$. So, if $A \in \overline{\alpha}$, then $A \subseteq C(\alpha(A)) \cap A \subseteq \ker\theta \cap A = \alpha(A)$ which gives $A \in \alpha$.

A last remark on the coincidence of the classes under discussion here, is the following. If $D \neq \emptyset$ and X is U-bounded (cf. [22]) for some finite set U with $\emptyset \neq U \subseteq X$ and $\sigma(u) \subseteq U \times U$ for all $u \in U$, then $\alpha = \alpha^c = \overline{\alpha}$ for any hypernilpotent radical α (i.e. all nilpotent rings are radical). Indeed, $(0 : U)_{C(A)}$ is a nilpotent ideal of $C(A)$ with $\dfrac{C(A)}{(0 : U)_{C(A)}} \cong A^U$ (cf. Proposition 7 in [22]). Since α is hypernilpotent, $(0 : U)_{C(A)} \in \alpha$. Thus, if $A \in \alpha$, we have $A^U \in \alpha$ (since U is finite) and hence $C(A) \in \alpha$, i.e. $A \in \alpha^c$. Since $\alpha^c \subseteq \overline{\alpha} \subseteq \alpha$, we can conclude that $\alpha = \alpha^c = \overline{\alpha}$.

An ideal K of $C(A)$ has the *Summation Property* [22] if it satisfies: Whenever I_p is an ideal of A with $C(I_p) \subseteq K$ for all $p \in \Lambda$, Λ is some index set, then $C(\sum_{p\in\Lambda} I_p) \subseteq K$. Any weakly homogeneous ideal K has the summation property. The status of the converse is not clear. What is known is that an ideal which satisfies the summation property need not be homogeneous. Indeed, if $A = 2\mathbb{Z}$, the ring of even integers, let $C(A) = M_2(A)$, the ring of 2×2 matrices over A. Then $K = \begin{bmatrix} 4\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix}$ is weakly homogeneous (and thus satisfies the summation property), but it is not homogeneous.

Proposition 8. *If α is a radical class such that $\alpha(C(A))$ has the summation property for all rings A, then α^c is a radical class. Conversely, if α^c is a radical class and α is hereditary, then $\alpha(C(A))$ has the summation property for all rings A.*

Proof. For a surjective homomorphism $\theta : A \to B$, we have a surjective homomorphism $C(\theta) : C(A) \to C(B)$ defined by $C(\theta)(f) = \theta \circ f$ for all $f \in C(A)$. From this the homomorphic closure of α^c follows.

Suppose every non-zero homomorphic image of the ring A has a non-zero ideal which is in α^c. We show $A \in \alpha^c$. Let $J := \sum(I_p \lhd A \mid C(I_p) \subseteq \alpha(C(A)))$. Then $C(J) \subseteq \alpha(C(A))$ by the summation property. If $J \neq A$, then there is a non-zero ideal $\frac{I}{J}$ of $\frac{A}{J}$ with $\frac{I}{J} \in \alpha^c$. Now $C(I) \nsubseteq \alpha(C(A))$, for if $C(I) \subseteq \alpha(C(A))$, then $I = J$, a contradiction. Hence $0 \neq \dfrac{C(I)}{C(I) \cap \alpha(C(A))} \cong \dfrac{C(I) + \alpha(C(A))}{\alpha(C(A))} \lhd \dfrac{C(A)}{\alpha(C(A))} \in \mathcal{S}\alpha$. But $C(J) \subseteq C(I) \cap \alpha(C(A))$ and
$\dfrac{C(I) \cap \alpha(C(A))}{C(J)} \lhd \dfrac{C(I)}{C(J)} \cong C\left(\dfrac{I}{J}\right) \in \alpha$ which means $\dfrac{C(I)}{C(I) \cap \alpha(C(A))} \in \alpha \cap \mathcal{S}\alpha = 0$, a contradiction. Thus $J = A$ and so $C(A) = C(J) \subseteq \alpha(C(A))$, i.e. $C(A) \in \alpha^c$.

Conversely, suppose α^c is a radical and α is hereditary. Let $I_p \lhd A$ with $C(I_p) \subseteq \alpha(C(A))$ for all $p \in \Lambda$, Λ some index set. Since α is hereditary, we get $I_p \in \alpha^c$ for all p. Thus $\sum_{p\in\Lambda} I_p \subseteq \alpha^c(A)$ and so $C(\sum_{p\in\Lambda} I_p) \subseteq C(\alpha^c(A)) \in \alpha$ since $\alpha^c(A) \in \alpha^c$. From the hereditariness of α we get $C(\sum_{p\in\Lambda} I_p) \subseteq \alpha(C(A))$ which shows that the summation property holds. □

Next we investigate when $\overline{\alpha}$ will be a radical class. From Ortiz [6] we know that if for any $I \lhd A$, $C(I) \subseteq IC(D(A))$ where $D(A)$ denotes the Dorroh extension of A, then $\overline{\alpha}$ is a radical class. We will weaken this requirement. Since $\alpha\left(\dfrac{C(A)}{C(\alpha^*(A))}\right) \lhd \dfrac{C(A)}{C(\alpha^*(A))}$, we have $\alpha\left(\dfrac{C(A)}{C(\alpha^*(A))}\right) = \dfrac{B}{C(\alpha^*(A))}$ for some $B = B_A \lhd A$. Then $\alpha^*(A) \subseteq C(\alpha^*(A)) \subseteq B$ and so $\alpha^*(A) \subseteq B \cap A$. We say that α has the *Intersection Property* if $\alpha^*(A) = B_A \cap A$ for all rings A, i.e. $\alpha(C(A)) \cap A = B_A \cap A$ for all rings A. Note that

(1) $B_A = \alpha(C(A)) \Leftrightarrow \alpha(C(A))$ is weakly homogeneous. Indeed, if $B_A = \alpha(C(A))$, then $C(\alpha(C(A)) \cap A) = C(\alpha^*(A)) \subseteq B_A = \alpha(C(A))$. Conversely, if $\alpha(C(A))$ is weakly homogeneous, then $C(\alpha^*(A)) \subseteq \alpha(C(A))$. Then $\dfrac{B_A}{C(\alpha^*(A))} = \alpha\left(\dfrac{C(A)}{C(\alpha^*(A))}\right) = \dfrac{\alpha(C(A))}{C(\alpha^*(A))}$ which gives $B_A = \alpha(C(A))$.

(2) If α satisfies the Krempa Condition and the Intersection Property, then $\alpha(C(A)) \subseteq C(\alpha^*(A))$ for all A. This follows from $\alpha\left(C\left(\dfrac{A}{\alpha^*(A)}\right)\right) \cong \alpha\left(\dfrac{C(A)}{C(\alpha^*(A))}\right)$, $\alpha\left(C\left(\dfrac{A}{\alpha^*(A)}\right)\right) \cap \dfrac{A}{\alpha^*(A)} = \dfrac{(B_A \cap A) + C(\alpha^*(A))}{C(\alpha^*(A))} = 0$ (by the Intersection Property) and the Krempa Condition.

Proposition 9. *If the radical α satisfies the Intersection Property, then $\overline{\alpha}$ is a radical class.*

Proof. Let $\theta : A \to B$ be a surjective homomorphism with $A \in \overline{\alpha}$. Then $A \subseteq \alpha(C(A))$ and since $C(\theta) : C(A) \to C(B)$ is a surjective homomorphism, $B = \theta(A) = (C(\theta))(A) \subseteq C(\theta)(\alpha(C(A))) \subseteq \alpha(C(B))$. Thus $B \in \overline{\alpha}$ which shows that $\overline{\alpha}$ is homomorphically closed.

Note that since $\dfrac{A+C(\alpha^*(A))}{C(\alpha^*(A))}$ is the isomorphic image of $\dfrac{A}{\alpha^*(A)}$ under the isomorphism $C\left(\dfrac{A}{\alpha^*(A)}\right) \cong \dfrac{C(A)}{C(\alpha^*(A))}$, we get

$$\alpha^*\left(\frac{A}{\alpha^*(A)}\right) = \alpha\left(C\left(\frac{A}{\alpha^*(A)}\right)\right) \cap \frac{A}{\alpha^*(A)} \cong \alpha\left(\frac{C(A)}{C(\alpha^*(A))}\right) \cap \frac{A}{\alpha^*(A)} =$$

$$= \frac{B_A}{C(\alpha^*(A))} \cap \frac{A+C(\alpha^*(A))}{C(\alpha^*(A))} = \frac{(B_A \cap A)+C(\alpha^*(A))}{C(\alpha^*(A))} = \frac{\alpha^*(A)+C(\alpha^*(A))}{C(\alpha^*(A))} = 0.$$

Suppose now A is a ring such that every non-zero homomorphic image of A has a non-zero ideal which is in $\overline{\alpha}$. We show $A \in \overline{\alpha}$. Suppose to the contrary that $A \notin \overline{\alpha}$. Then $\alpha^*(A) \subsetneqq A$. By assumption, there is an ideal $0 \neq \dfrac{I}{\alpha^*(A)} \lhd \dfrac{A}{\alpha^*(A)}$ with $\dfrac{I}{\alpha^*(A)} \in \overline{\alpha}$. Then $\dfrac{I}{\alpha^*(A)} \subseteq \alpha\left(C\left(\dfrac{I}{\alpha^*(A)}\right)\right) \subseteq \alpha\left(C\left(\dfrac{A}{\alpha^*(A)}\right)\right)$. Thus $\dfrac{I}{\alpha^*(A)} \subseteq \alpha\left(C\left(\dfrac{A}{\alpha^*(A)}\right)\right) \cap \dfrac{A}{\alpha^*(A)} = \alpha^*\left(\dfrac{A}{\alpha^*(A)}\right) = 0$, a contradiction. Hence $A \in \overline{\alpha}$. $\square$

Next we investigate the properties of the ideal-mapping $\alpha^*(A) = \alpha(C(A)) \cap A$.

Proposition 10. *For any radical α, α^* is a complete pre-radical. It is a Hoehnke radical if and only if α satisfies the Intersection Property and it is idempotent if and only if $\alpha^*(A) \in \overline{\alpha}$. Thus α^* is a Kurosh-Amitsur radical map if and only if $\alpha^*(A) \in \overline{\alpha}$ for all rings A and α satisfies the Intersection Property. In this case, $\alpha^*(A) = \overline{\alpha}(A)$ for all rings A.*

Proof. α^* is a pre-radical: Let $\theta : A \to B$ be a surjective homomorphism. Then $\theta(\alpha^*(A)) = \theta(\alpha(C(A)) \cap A) \subseteq \alpha(C(\theta)(C(A)) \cap B = \alpha(C(B)) \cap B = \alpha^*(B) = \alpha^*(\theta(A))$.

α^* is complete: Let $\alpha^*(I) = I \lhd A$. Then $I = \alpha(C(I)) \cap I \subseteq \alpha(C(A)) \cap A = \alpha^*(A)$.

α^* is idempotent $\Leftrightarrow \alpha^*(\alpha^*(A)) = \alpha^*(A) \Leftrightarrow \alpha(C(\alpha^*(A))) \cap \alpha^*(A) = \alpha^*(A) \Leftrightarrow \alpha^*(A) \subseteq \alpha(C(\alpha^*(A))) \Leftrightarrow \alpha^*(A) \in \overline{\alpha}$.

Next we show that $\alpha^*\left(\dfrac{A}{\alpha^*(A)}\right) = 0$ if and only if α satisfies the Intersection Property:

$$\alpha^*\left(\frac{A}{\alpha^*(A)}\right) = \alpha\left(C\left(\frac{A}{\alpha^*(A)}\right)\right) \cap \frac{A}{\alpha^*(A)} \cong \alpha\left(\frac{C(A)}{C(\alpha^*(A))}\right) \cap \frac{A}{\alpha^*(A)} =$$

$$= \frac{B_A}{C(\alpha^*(A))} \cap \frac{A+C(\alpha^*(A))}{C(\alpha^*(A))} = \frac{(B_A \cap A)+C(\alpha^*(A))}{C(\alpha^*(A))} = 0 \Leftrightarrow B_A \cap A \subseteq C(\alpha^*(A)).$$

This inclusion holds if and only if $B_A \cap A = \alpha^*(A)$ (i.e. the Intersection Property is satisfied). Indeed, suppose $B_A \cap A \subseteq C(\alpha^*(A))$. Then $B_A \cap A = (B_A \cap A) \cap A \subseteq C(\alpha^*(A)) \cap A = \alpha^*(A)$ and $C(\alpha^*(A)) \subseteq B_A$ implies $\alpha^*(A) \subseteq B_A \cap A$. Thus $B_A \cap A = \alpha^*(A)$. The converse is clear since $\alpha^*(A) \subseteq C(\alpha^*(A))$.

Lastly we show that if α^* is a Kurosh-Amitsur radical, then $\alpha^*(A) = \overline{\alpha}(A)$. Suppose thus that α^* is a Kurosh-Amitsur radical. From the above, we know that the Intersection Property is satisfied and so $\overline{\alpha}$ is a radical class (Proposition 9). Since α^* is idempotent, $\alpha^*(A) \in \overline{\alpha}$. Thus $\alpha^*(A) \subseteq \overline{\alpha}(A)$. Since $\overline{\alpha}(A) \subseteq \alpha^*(A)$ always hold, we get $\overline{\alpha}(A) = \alpha^*(A)$. □

As mentioned earlier, weakly homogeneity of α often comes for free as a consequence of properties of the convolution type. We now investigate this and related concepts. We recall from [22]:

A convolution type $\mathcal{T}$ is said to satisfy the *Ortiz Condition* if $C(N) \subseteq NC(D(A))$ for every ring A and subring N of A (remember $D(A)$ denotes the Dorroh extension of A). The origins of the Ortiz condition is to be found in [6], playing a key role in the generalization of certain radicals classes determined by the radicals of polynomial rings. $\mathcal{T}$ is said to satisfy the *Finite Complement Property* if $X \setminus Y$ is finite for all $Y \in \mathcal{S}$. It was shown in [22] that the Finite Complement Property implies the validity of the Ortiz Condition which in turn implies that every radical is weakly homogeneous.

A case that often occurs in the examples is the following: α is a radical which is weakly homogeneous and which satisfies $A \in \alpha \Leftrightarrow C(A) \in \alpha$. In such a case, α is invariant if and only if α is homogeneous if and only if α satisfies the Krempa Condition.

3 Examples

In the examples below, we will not recall or summarize all that is known about the radical theory of the particular convolution type. We will only recall or proof results which will bring certain aspects of the radical theory of convolution rings to the fore.

3.1. Discrete direct sums. Let X be any non-empty set, $\mathcal{S} = \{Y \subseteq X \mid X \setminus Y$ is finite$\}$, $\sigma(x) = \{(x,x)\}$ for all $x \in X$ and $\tau(s,t) = 1$ for all $s,t \in X$. Then $T = X = D$. The corresponding convolution ring $C(A) = \bigoplus_{x \in X} A$, the discrete direct sum of $|X|$-copies of A. For any radical α and ring A, $\alpha(C(A)) = \alpha\left(\bigoplus_{x\in X} A\right) = \bigoplus_{x\in X} \alpha(A) = C(\alpha(A))$; the best possible scenario and there is nothing further to report.

Note that for infinite sets X, conditions (T1) and (T2) are satisfid but not (T3).

3.2. Direct products. Let X be any infinite set, $\mathcal{S} = \{\varnothing\}$, $\sigma(x) = \{(x,x)\}$ for all $x \in X$ and $\tau(s,t) = 1$ for all $s,t \in X$. Then $T = X = D$ and the convolution ring $C(A)$ coincides with the direct product A^X of $|X|$-copies of the ring A.

We know that A can be embedded as an ideal in A^X and that A is a homomorphic image of A^X. Since radical classes are homomorphically closed and semisimple classes are hereditary and closed under subdirect products (and thus also direct products), we have for any radical class $\alpha : A \in \mathcal{S}\alpha \Leftrightarrow A^X \in \mathcal{S}\alpha$ and $A^X \in \alpha \Rightarrow A \in \alpha$. This means that the salient properties of the radical of a direct product depend only on the validity of the converse of the above implication. In fact, we have: $\alpha(A^X) = (\alpha(A))^X \Leftrightarrow \alpha$ is homogenous $\Leftrightarrow (A \in \alpha \Rightarrow A^X \in \alpha) \Leftrightarrow (\alpha(A))^X \in \alpha$ for all rings A. Furthermore, $\alpha^c \subseteq \overline{\alpha} \subseteq \alpha$ and for every ring A, we have $\alpha^c(A) \subseteq \overline{\alpha}(A) \subseteq \alpha^*(A) \subseteq \alpha(A)$ with equality if and only if $\alpha^c = \alpha$.

In general, A^X need not be radical even though A is radical. Also, neither α^c nor $\overline{\alpha}$ need to be Kurosh-Amitsur radicals and α^* need not even be a Hoehnke radical. The example which follows will show all these (negative) properties. In addition, it also shows that $A \in \mathcal{S}\alpha \Leftrightarrow A^X \in \mathcal{S}\alpha$ need not be equivalent to $A^X \in \alpha \Leftrightarrow A \in \alpha$. Let α be the nil radical and let R be the Zassenhaus algebra (see for example Divinsky [3], Chapter 2, Example 3). This ring R is constructed as follows. Let F be any field. The elements of R are the formal (finite) sums $\sum_t a_t x_t$ where $a_t \in F$ and $x_t \in (0,1)$. Multiplication is done according to the rule $x_t x_s = \begin{cases} x_{t+s} \text{ if } t+s<1 \\ 0 \text{ if } t+s \geq 1 \end{cases}$. As is well known, R is a nil ring. Let $X = \mathbb{N}$ be the set of positive integers. Then R^X is not radical, for the element $x = (x_{\frac{1}{2}}, x_{\frac{1}{4}}, x_{\frac{1}{8}}, ..., x_{\frac{1}{2^n}}, ...)$ of R^X is not nilpotent. Next we show that α does not satisfy the Summation Property (and thus $\overline{\alpha}$ is not Kurosh-Amitsur radical). For every $t \in (0,1)$, let I_t be the ideal in R generated by x_t. Then I_t is nilpotent with $I_t^k = 0$ for any $k \in \mathbb{N}$ with $k > \frac{1}{t}$. Thus $C(I_t) = (I_t)^X$ is nilpotent and so $C(I_t) \subseteq \alpha(C(R))$ for all $t \in (0,1)$. But $C(\sum_t I_t) = C(R) \nsubseteq \alpha(C(R))$. Also, α^c is not a Kurosh-Amitsur radical, for if it were, then $R = \sum_t I_t \supseteq \sum(I \lhd R \mid C(I) \in \alpha) = \alpha^c(R)$. This means $R \in \alpha^c$, i.e. $R^X = C(R) \in \alpha$; a contradiction. In addition, it also shows that α does not satisfy the Intersection Property. From Proposition 10 it follows that α^* is a complete pre-radical, but not a Hoehnke radical.

Examples of radicals which do satisfy the condition $A^X \in \alpha \Leftrightarrow A \in \alpha$ can be obtained from the following. Let $k, n \in \mathbb{N}$ be fixed with $1 \leq n < k$. Let $\phi(x_1, x_2, ..., x_k)$ be a fixed element from $\mathbb{Z}\{x_1, x_2, ..., x_k\}$, the ring of polynomials in k non-commuting indeterminates over the integers $\mathbb{Z}$. For a ring A, let $\phi_A : A^k \to A$ be the corresponding evaluation map. The ring A is called an $\phi-ring$ if for all $a_1, a_2, ..., a_n \in A$ there exists $a_{n+1}, a_{n+2}, ..., a_k \in A$ such that $\phi_A(a_1, a_2, ..., a_n, a_{n+1}, a_{n+2}, ..., a_k) = 0$. Let Φ be the class of all ϕ-rings and let $\pi_x : A^X \to A$ be the $x-th$ projection. We suppose that $\pi_x(\phi_{A^X}(a_1, a_2, ..., a_k)) = \phi_A(\pi_x(a_1), \pi_x(a_2), ..., \pi_x(a_k))$ for all $a_1, a_2, ..., a_k \in A^X$ and $x \in X$. Under this assumption, we get $A \in \Phi \Leftrightarrow A^x \in \Phi$. Indeed, let $A \in \Phi$. Let $a_1, a_2, ..., a_n \in A^X$. For any $x \in X$, $\pi_x(a_1), \pi_x(a_2), ..., \pi_x(a_k) \in A$ and by assumption there exist $a'_{n+1}, a'_{n+2}, ..., a'_k \in A$ such that

$\phi_A(\pi_x(a_1), \pi_x(a_2), ..., \pi_x(a_n), a'_{n+1}, a'_{n+2}, ..., a'_k) = 0.$

Each of these a'_{n+j} 's depends on x, so when we want to emphasize this, we write $a'_{n+j} = a'_{n+j}(x)$. For each $j = n+1, n+2, ..., k$, define $a_j : X \to A$ by $a_j(x) = a'_j(x)$ for all $x \in X$. Then

$\pi_x(\phi_{A^X}(a_1, a_2, ..., a_k))$
$= \phi_A(\pi_x(a_1), \pi_x(a_2), ..., \pi_x(a_k))$
$= \phi_A(\pi_x(a_1), \pi_x(a_2), ..., \pi_x(a_n), a'_{n+1}, a'_{n+2}, ..., a'_k)$

$= 0$ for all $x \in X$. Thus $\phi_{A^X}(a_1, a_2, ..., a_k) = 0$ and $A^X \in \Phi$. Conversely, suppose $A^X \in \Phi$ and let $a_1', a_2', ..., a_n' \in A$. Choose $x_0 \in X$ fixed. For each $i = 1, 2, ..., n$, define $a_i : X \to A$ by $a_i(x) = \begin{cases} a_i' \text{ if } x = x_0 \\ 0 \text{ otherwise} \end{cases}$. Since $A^X \in \Phi$, there are $a_{n+1}, a_{n+2}, ..., a_k \in A^X$ such that $\phi_{A^X}(a_1, a_2, ..., a_k) = 0$. Thus

$\phi_A(a_1', a_2', ..., a_n', \pi_{x_0}(a_{n+1}), \pi_{x_0}(a_{n+2}), ..., \pi_{x_0}(a_k))$
$= \phi_A(\pi_{x_0}(a_1), \pi_{x_0}(a_2), ..., \pi_{x_0}(a_n), \pi_{x_0}(a_{n+1}), \pi_{x_0}(a_{n+2}), ..., \pi_{x_0}(a_k))$
$= \pi_{x_0}(\phi_{A^X}(a_1, a_2, ..., a_k))$
$= 0$. Thus $A \in \Phi$.

When $\alpha := \Phi$ is a radical class, we will have $\alpha(A^X) = (\alpha(A))^X$ for all A. As examples we may mention for $n = 1$ and $k = 2$, the polynomials $\phi(x, y) = x + y - xy$ and $\phi(x, y) = x - xyx$ which give the Jacobson radical class and the von Neumann regular radical class respectively.

3.3. Polynomials. Let $X = \mathbb{N}_0 := \{0, 1, 2, 3, ...\}$, $\mathcal{S} = \{Y_k \mid k \in \mathbb{N}_0\}$ where $Y_k = \{k+1, k+2, k+3, ...\}$, $\sigma(n) = \{(i, j) \mid i, j \in \mathbb{N}_0, i+j = n\}$ and $\tau(n, m) = 1$ for all $n, m \in \mathbb{N}_0$. Here $T = D = \{0\}$ and the convolution ring $C(A)$ is the polynomial ring $A[x]$ in one indeterminate. The radical theory of this convolution type is one of the classical cases (the other being matrices which will be discussed below). The polynomial convolution type satisfies the Finite Complement Property which means that any radical α is weakly homogeneous, i.e. $(\alpha(A[x]) \cap A)[x] \subseteq \alpha(A[x])$ for any ring A. We also have $A[x] \in \alpha \Rightarrow A \in \alpha$, $\alpha(A[x]) \subseteq \{f \in A[x] \mid f(0) \in \alpha(A)\}$ and $\alpha^c = \overline{\alpha} \subseteq \alpha$. Furthermore, the radical α will be homogeneous (i.e. satisfy the Amitsur condition) if and only if it satisfies the Krempa Condition.

Some of the well-known radicals are invariant, for example the Baer (= prime) radical as well as the Levitzky (= local nilpotent) radical. Several others are homogeneous, for example the Jacobson radical, nil radical, Brown-McCoy radical, uniformly strongly prime radical and any strongly hereditary radical (i.e. a radical such that any subring of a radical ring is radical). For these homogeneous radicals, $\alpha^*(A) \in \alpha^c = \overline{\alpha}$; hence α^* is a Kurosh-Amitsur radical with $\alpha^*(A) = \alpha^c(A) = \overline{\alpha}(A) \subseteq \alpha(A)$ for all rings A and the inclusion is in general strict. By the Krempa Condition, these radicals satisfy $A \in \mathcal{S}\alpha \Rightarrow A[x] \in \mathcal{S}\alpha$. Smoktunowicz [14] has given an example of a nil ring A for which $A[x]$ is not nil. Thus for α the nilradical, which is homogeneous, we have $A[x]$ nil implies A nil, but the converse implication is not true in general. This situation can also be realized for subidempotent radicals (hereditary and all nilpotent rings are semisimple): Let ν be the von Neumann regular radical. For any ring A, $\upsilon(A[x]) = 0$ (cf. [16]) which means υ is homogeneous, $\nu^c = \overline{\upsilon} = \{0\}, \upsilon^*(A) = 0$ for all A and $A \in \upsilon$ does not necessarily imply $A[x] \in \upsilon$.

The major outstanding problem regarding the radicals of polynomial rings is to characterize the ideal $\alpha^*(A) = \alpha(A[x]) \cap A$ of A in terms of properties of the ring A without reference to $\alpha(A[x])$.

In striking contrast to most of the other convolution types, the Jacobson radical $\mathcal{J}(A[x])$ of $A[x]$ is in general not directly accessible. It is known that $\mathcal{J}(A[x]) = N[x]$

where $N := \mathcal{J}(A[x]) \cap A = \mathcal{J}^*(A)$ is a nil ideal of A. We will now describe the elements of N and start with:

3.3.1. *Let $a \in A$. If ax^k is right quasi-regular in $A[x]$ for some $k \geq 1$, then a is nilpotent. Conversely, if $a \in A$ is nilpotent, then ax^k is right quasi-regular in $A[x]$ for all $k \geq 0$.*

Proof. Let $q(x) = q_0 + q_1x + q_2x^2 + ... + q_nx^n \in A[x], q_n \neq 0$, be such that $ax^k + q(x) - ax^kq(x) = 0$, i.e. $ax^k + (q_0 + q_1x + q_2x^2 + ... + q_nx^n) - ax^k(q_0 + q_1x + q_2x^2 + ... + q_nx^n) = 0$. Comparing constant terms, we get $q_0 = 0$. If $k > n$, then the coefficient of x^k on the left hand side is a which gives $a = 0$ and we are done. Suppose thus $k \leq n$. Comparing coefficients gives $q_i = 0$ for $i = 1, 2, 3, ..., k-1$, $a + q_k = 0$, $q_{k+i} - aq_i = 0$ for $i = 1, 2, 3, ..., n-k$ and $aq_{n-i} = 0$ for $i = 0, 1, 2, ..., k-1$. Since $k \leq n$, we have $n = mk + i$ for some $m \geq 1$ and $0 \leq i < k$. Now $q_{mk} = -a^m$ and so $0 = aq_{n-i} = aq_{mk} = -a^{m+1}$. Thus a is nilpotent.

Conversely, suppose a is nilpotent, say $a^{p+1} = 0$. If $k = 0$, let $q(x) := -a^p$. If $k \neq 0$, let $q(x) = q_kx^k + q_{2k}x^{2k} + ... + q_{pk}x^{pk}$ where $q_{ik} = -a^i$ for $i = 1, 2, 3, ..., p$. Then $q(x) \in A[x]$ and $ax^k + q(x) - ax^kq(x) = 0$, i.e. ax^k is right quasi-regular in $A[x]$. □

For a ring A and elements $c_1, c_2, ..., c_k \in A, k \geq 1$, define a sequence $h = (h_1, h_2, h_3, ...)$ by $h_1 := -c_1$ and if h_{i-1} has been defined, let $h_i := \sum_{j=1}^{i-1} c_jh_{i-j} - c_i$ for $i = 2, 3, 4, ...$ where we take $c_{k+1} = c_{k+2} = c_{k+3} = ... = 0$. Since this sequence depends on the c_i's, if necessary we will denote it by $h = h[c_1, c_2, ..., c_k] = (h_1, h_2, h_3, ...)$.

An element $a \in A$ is called *rqr-nilpotent* if for any $k \geq 1$ and $b_1, b_2, ..., b_k \in A$, the sequence $h = h[ab_1, ab_2, ..., ab_k]$ is ultimately 0, i.e. there exists an $n \geq 1$ such that $h_{n+1} = h_{n+2} = h_{n+3} = ... = 0$.

It can be verified that if a is rqr-nilpotent, then a is nilpotent and so is ab (and hence also ba) for any $b \in A$. If A is commutative, then $a \in A$ is rqr-nilpotent if and only if a is nilpotent.

3.3.2. *An element $a \in A$ is rqr-nilpotent in A if and only if $a(b_1x + b_2x^2 + ... + b_kx^k)$ is right quasi-regular in $A[x]$ for all $b_1, b_2, ..., b_k \in A, k \geq 1$.*

Proof. Suppose a is rqr-nilpotent and let $b_1, b_2, ..., b_k \in A, k \geq 1$. By definition the sequence $h = h[ab_1, ab_2, ..., ab_k] = (h_1, h_2, h_3, ...)$ is ultimately 0, say $h_{n+1} = h_{n+2} = ... = 0$ for some $n \geq 1$. Then $h(x) := h_1x + h_2x^2 + ... + h_nx^n \in A[x]$ and $a(b_1x + b_2x^2 + ... + b_kx^k) + h(x) - a(b_1x + b_2x^2 + ... + b_kx^k)h(x) = 0$. Thus $a(b_1x + b_2x^2 + ... + b_kx^k)$ is right quasi-regular in $A[x]$.

Conversely, suppose $a(b_1x + b_2x^2 + ... + b_kx^k)$ is right quasi-regular in $A[x]$ for any $b_1, b_2, ..., b_k \in A, k \geq 1$. Choose $b_1, b_2, ..., b_k \in A$ and let $b(x) := b_1x + b_2x^2 + ... + b_kx^k$. Consider the sequence $h = h[ab_1, ab_2, ..., ab_k] = (h_1, h_2, h_3, ...)$. By assumption there is a $f(x) = f_0 + f_1x + f_2x^2 + ... + f_nx^n \in A[x]$ such that $ab(x) + f(x) - ab(x)f(x) = 0$. Comparing coefficients will then give $f_i = h_i$ for all $i = 1, 2, 3, ..., n$ and $h_{n+i} = 0$ for all $i = 1, 2, 3, ...$ .Thus a is rqr-nilpotent. □

3.3.3. *Let $\mathcal{J}$ denote the Jacobson radical. For any ring A,*
$\mathcal{J}(A[x]) \cap A = \{a \in A \mid a \text{ is rqr-nilpotent}\}$.

Proof. Let $a \in J(A[x]) \cap A$. Then $ab(x)$ is right quasi-regular in $A[x]$ for any $b(x) \in A[x]$. In particular, this is true for any $b(x)$ of the form $b(x) = b_1 x + b_2 x^2 + ... + b_k x^k$. From 3.3.2 above we then know that a is rqr-nilpotent. Conversely, suppose a is rqr-nilpotent. Then ac is nilpotent for any $c \in A$ and thus acx^n is right quasi-regular in $A[x]$ for all $n \geq 0$. To get a in $\mathcal{J}(A[x])$, we need to show that $ab(x)$ is right quasi-regular for any $b(x) = b_0 + b_1 x + b_2 x^2 + ... + b_k x^k \in A[x]$. For such a $b(x) \in A[x]$ we know ab_0 is nilpotent and hence right quasi-regular, say $ab_0 + u - ab_0 u = 0$ for some $u \in A$. Let $b'(x) := b_1 x + b_2 x^2 + ... + b_k x^k$. Since a is rqr-nilpotent, we know that $a(b'(x) - ub'(x))$ is right quasi-regular in $A[x]$, say $a(b'(x) - ub'(x)) + q(x) - a(b'(x) - ub'(x))q(x) = 0$ for some $q(x) \in A[x]$. Let $f(x) := u + q(x) - uq(x)$. Then $f(x) \in A[x]$ and $ab(x) + f(x) - ab(x)f(x) = ab_0 + ab'(x) + u + q(x) - uq(x) - (ab_0 + ab'(x))(u + q(x) - uq(x)) = 0$ which shows that $a \in \mathcal{J}(A[x])$. □

If a ring is called *qr-nil* if all its elements are qr-nilpotent, then the radical class $\mathcal{J}^* = \{A \mid A \text{ is qr-nil}\}$. The question whether all the elements of a nil ring are rqr-nilpotent is equivalent to the Köthe Conjecture.

The radical theory of related convolution rings like the ring of polynomials in n commuting indeterminates $C(A) = A[x_1, x_2, ..., x_n]$, the polynomial ring in n non-commuting indeterminates $C(A) = A\{x_1, x_2, ..., x_n\}$, the ring of formal power series $C(A) = A[[x]]$, the ring of Laurent series $C(A) = A\langle x \rangle$, etc., is not as well-developed as for the polynomial rings $A[x]$. This is except for $C(A) = A[x_1, x_2, ..., x_n]$ where the results of the one indeterminate case carries over mutatis mutandis. For results on the radicals of these convolution rings, one could consult Amitsur [1] and [2], Sierpińska [13] and Puczyłowski [9] and [10].

3.4. Necklace rings. Let $X = \mathbb{N}$, $\mathcal{S} = \{\varnothing\}$, $\sigma(n) = \{(i,j) \mid i, j \in \mathbb{N}, lcm(i,j) = n\}$ and $\tau(n,m) = \gcd(n,m)$ where *lcm* and gcd denote the least common multiple and greatest common divisor respectively. Then $T = D = \{1\}$ and $C(A)$ is just the necklace ring $N(A)$ over A, see for example [7]. Necklace rings can also be defined over finite subsets $X = \{1, 2, 3, ..., k\}$ of $\mathbb{N}$ with a similar convolution type as above, cf [20]. In this case the convolution ring will be denoted by $N_k(A)$. All results on the radical theory of necklace rings can be found in [20].

We will consider the radical theory of this latter case first. In this case, since X is finite the convolution type has the Finite Complement Property which means that any radical α is weakly homogeneous. We also know that $\alpha(N_k(A)) \subseteq (\alpha(A) : P)_{N_k(A)}$ where $P(f) = \{nf(n) \mid n = 1, 2, 3, ..., k\}$ for $f \in N_k(A)$. Thus $N_k(A) \in \alpha \Rightarrow A \in \alpha$, $\alpha^c = \overline{\alpha}$ is a Kurosh-Amitsur radical, and $\alpha^c(A) = \overline{\alpha}(A) \subseteq \alpha^*(A) = \alpha(N_k(A)) \cap A \subseteq (\alpha(A) : P)_{N_k(A)} \cap A \subseteq \alpha(A)$. If α is supernilpotent (i.e. α is hereditary and contains all the nilpotent rings), then $\alpha(N_k(A)) = (\alpha(A) : P)_{N_k(A)} \supseteq N_k(\alpha(A))$ which gives $N_k(A) \in \alpha \Leftrightarrow A \in \alpha$. So, for these radicals we get $\alpha = \alpha^c$ and $\alpha^c(A) = \overline{\alpha}(A) = \alpha^*(A) = \alpha(A)$ for all rings A. Furthermore, α will be homogeneous if and only if α is invariant. Indeed, if α is homogeneous, then $\alpha(N_k(A)) = N_k(\alpha(N_k(A)) \cap A) \subseteq N_k((\alpha(A) : P)_{N_k(A)} \cap A) = N_k(\alpha(A))$ and so $\alpha(N_k(A)) = N_k(\alpha(A))$. In general, a radical α need not be homogeneous: Let α

be the nilradical and let $R = \mathbb{Z}_8$, the ring of integers $mod 8$. Let $f(1) = 2, f(2) = 5, f(3) = f(5) = 0$ and $f(4) = 3$. Then $f \in (\alpha(R) : P)_{N_5(R)} = \alpha(N_5(R))$ where $\alpha(R) = \{0, 2, 4, 6\}$, but $f \notin N_5(I)$ for any proper ideal I of R. To summarize, here we have an example of a (supernilpotent) radical α which is weakly homogeneous, $C(A) \in \alpha \Leftrightarrow A \in \alpha$, $\alpha^c(A) = \overline{\alpha}(A) = \alpha^*(A) = \alpha(A)$ for all rings A (so $C(\alpha^*(A)) \in \alpha$), the radical is directly accessible, but the Krempa Condition is not satisfied (and hence the radical is not homogeneous).

For the general necklace ring (i.e. $X = \mathbb{N}$), we have the following results: For any radical α, $\alpha(N(A)) \subseteq (\alpha(A) : P)_{N(A)}$ where P is as above. If $\alpha = \mathcal{J}$ is the Jacobson radical, the results are much stronger, for $\mathcal{J}(N(A)) = (\mathcal{J}(A) : P)_{N(A)} \supseteq N(\mathcal{J}(A))$ and thus

$$\begin{aligned} N(\mathcal{J}(N(A)) \cap A) &= N((\mathcal{J}(A) : P)_{N(A)} \cap A) \\ &= N(\mathcal{J}(A)) \subseteq \mathcal{J}(N(A)) \end{aligned}$$

which shows that $\mathcal{J}$ is weakly homogeneous. But from [20] we know that $\mathcal{J}$ is not homogeneous. Furthermore, $A \in \mathcal{J} \Leftrightarrow N(A) \in \mathcal{J}$ and hence $\mathcal{J}^c(A) = \overline{\mathcal{J}}(A) = \mathcal{J}^*(A) = \mathcal{J}(A)$ for all rings A. We thus see that the Jacobson radical enjoys the same properties for the general necklace ring as any supernilpotent radical does for the finite necklace ring, except in the general case the convolution type does not satisfy the Finite Complement Property.

3.5. Matrices. Let $n \geq 1$ be fixed and let $X = \{(i, j) \mid i, j = 1, 2, 3, ..., n\}, \mathcal{S} = \{\varnothing\}$, $\sigma(i, j) = \{(i, t), (t, j) \mid t = 1, 2, 3, ..., n\}$ and $\tau((i, j), (s, t)) = 1$ for all $(i, j), (s, t) \in X$. Then $T = \{(i, i) \mid i = 1, 2, 3, ..., n\}$ and $D = \emptyset$. In this case, the convolution ring $C(A)$ is isomorphic to $M_n(A)$, the complete $n \times n$ matrix ring over A. As is well-known, if R is a ring with an identity, then every ideal of $M_n(R)$ is homogeneous. From Proposition 7 we then know that any radical α is homogeneous. The only outstanding issues regarding the radical theory of finite matrix rings is thus the validity of the following two implications:

(*i*) $A \in \alpha \Rightarrow M_n(A) \in \alpha$ or, equivalently, $M_n(A) \in \mathcal{S}\alpha \Rightarrow A \in \mathcal{S}\alpha$ and

(*ii*) $M_n(A) \in \alpha \Rightarrow A \in \alpha$ or, equivalently, $A \in \mathcal{S}\alpha \Rightarrow M_n(A) \in \mathcal{S}\alpha$.

The invariance of a radical α is thus equivalent to ($A \in \alpha \Leftrightarrow M_n(A) \in \alpha$). In case a radical does satisfy this property, it is said to be *matrix extensible.* It is known that most of the well-known radicals are invariant. There is, however, one notable exception: For α the nilradical, it is known that $M_n(A) \in \alpha \Rightarrow A \in \alpha$, but the validity of the converse is equivalent to the well-known Köthe Conjecture (which is still open).

For infinite matrix rings, there are only a few limited results for which Patterson [8] and Sands [11] can be consulted.

3.6. Structural matrix rings. Let J be a non-empty set and let ρ be a non-empty reflexive and transitive relation on J such that the set $\{z \in J \mid (x, z) \in \rho$ and $(z, y) \in \rho\}$ is finite. Put $X = J \times J, \mathcal{S} = \{X \setminus \rho\}, \sigma(i, j) = \{((i, t), (t, j)) \mid t \in J\}$ and $\tau((i, j), (s, t)) = 1$ for all $(i, j), (s, t) \in X$. It can be shown that $T = \{(a, a) \mid a \in J\}$ and $D = \emptyset$. The convolution ring for this convolution type gives the structural matrix ring $M_J(A, \rho)$ over the ring A. We restrict our attention here to the finite case, i.e. we take $J = \{1, 2, 3, ..., n\}$ and denote the corresponding structural matrix ring by $M_n(A, \rho)$. For the radical theory of structural matrix rings, van Wyk [17], Sands [12] or Veldsman [18] can be consulted. Since X is finite, every radical α is weakly homogeneous. The radicals of these types of rings have been determined successfully, albeit in most cases only for radicals α which are invariant with respect to the finite matrix convolution type (i.e. radicals which are matrix extensible).

For a hypernilpotent radical α (i.e. all nilpotent rings are radical) which is matrix extensible, $\alpha(M_n(A, \rho)) = M_n(\alpha(A), \rho_s) + M_n(A, \rho_a)$ where ρ_s is the symmetric part and ρ_a the anti-symmetric part of ρ (cf. [12]). Thus $\alpha^*(A) = (M_n(\alpha(A), \rho_s) + M_n(A, \rho_a)) \cap A = \alpha(A)$ (remember, our canonical embedding of A into $M_n(A, \rho)$ is the mapping which assigns to every $a \in A$, the structural matrix which has a in every position $(i, i), i \in J$, and 0 elsewhere). Hence $M_n(\alpha(M_n(A, \rho)) \cap A, \rho) = M_n(\alpha(A), \rho)$ which means α is invariant if and only if α is homogeneous (this is still for α hypernilpotent with the matrix extension property). And this will be the case if and only if $\rho_a = \emptyset$. Indeed, if $\rho_a = \emptyset$, then $M_n(A, \rho) = M_n(A, \rho_s)$ is a finite direct sum of complete matrix rings $\oplus_{n_t} M_{n_t}(A)$. Then $\alpha(M_n(A, \rho)) = \alpha(\oplus_{n_t} M_{n_t}(A)) = \oplus_{n_t} \alpha(M_{n_t}(A)) = \oplus_{n_t} M_{n_t}(\alpha(A)) = M_n(\alpha(A), \rho)$. Conversely, suppose

$M_n(\alpha(A), \rho) = \alpha(M_n(A, \rho)) = M_n(\alpha(A), \rho_s) + M_n(A, \rho_a)$.

If $(i, j) \in \rho_a$, then $M_n(\alpha(A), \rho)$ has an element from $\alpha(A)$ in position (i, j), while the right hand side can have any element from A in position (i, j). Choosing $0 \neq A \in \mathcal{S}\alpha$ then leads to a contradiction which means $\rho_a = \emptyset$.

Next we let α be a subidempotent radical (i.e. α is hereditary and all nilpotent rings are semisimple) which is matrix extensible. From [18] we know that $\alpha(M_n(A, \rho)) = M_n(\alpha(A), (\rho^*)_s)$ where $(\rho^*)_s = \{(i, j) \in \rho \mid$ for $t \in \{i, j\}$ and $k \in \{1, 2, 3, ..., n\}, (t, k) \in \rho \Leftrightarrow (k, t) \in \rho\}$. If $(\rho^*)_s = \emptyset$, then $\alpha(M_n(A, \rho)) = 0$ for all rings A and α is homogeneous for such a convolution type. Suppose thus $(\rho^*)_s \neq \emptyset$. Then α is homogeneous if and only if $\rho = (\rho^*)_s$. Indeed, if $\rho = (\rho^*)_s$ then the homogeneity follows as in the above case. Conversely, suppose $(i, j) \in \rho \setminus (\rho^*)_s$. By the assumption we have $M_n(\alpha(A), (\rho^*)_s) = \alpha(M_n(A, \rho)) = M_n(\alpha(M_n(A, \rho)) \cap A, \rho)$. The $(i, j) - th$ entry on the left hand side is 0, while on the right hand side it is from $\alpha(M_n(A, \rho)) \cap A$. Thus $M_n(\alpha(A), (\rho^*)_s) = \alpha(M_n(A, \rho)) \cap A = 0$. Since $(\rho^*)_s \neq \emptyset$, this means $\alpha(A) = 0$ which is not necessarily true for all rings A. Thus $\rho = (\rho^*)_s$.

3.7. Incidence algebras. Let $(J, \leq)$ be a locally finite partially ordered set (i.e., each interval $[x, y] = \{z \in J \mid x \leq z \leq y\}$ is finite). Let $X = \{(x, y) \mid x, y \in J, x \leq y\}$, $\mathcal{S} = \{\varnothing\}$, $\sigma(x, y) = \{((x, z), (z, y)) \mid x, y, z \in J$ with $x \leq z \leq y\}$ and $\tau((x, y), (s, t)) = 1$ for all $(x, y), (s, t) \in X$. Here $T = \{(x, x) \mid x \in J\}, D = \emptyset$ and $C(A) = I_J(A)$, the incidence algebra over A. For more information on incidence algebras, see [15] and for their radicals [19].

For any radical α, $\alpha(I_J(A)) \subseteq (\alpha(A) : P)_{I_J(A)}$ where $P(f) := \{f(x, x) \mid x \in J\}$. From this it follows that $I_J(A) \in \alpha \Rightarrow A \in \alpha$ and thus $\alpha^*(A) \subseteq \alpha(A)$ for all rings A. The strongest results are for $\alpha = \mathcal{J}$, the Jacobson radical. For any ring A, $\mathcal{J}$ is directly accessible since $\mathcal{J}(I_J(A)) = (\mathcal{J}(A) : P)_{I_J(A)}$ and thus $A \in \mathcal{J} \Leftrightarrow I_J(A) \in \mathcal{J}$, $\mathcal{J}^c(A) = \overline{\mathcal{J}}(A) = \mathcal{J}^*(A) = \mathcal{J}(A)$ for all rings A. Moreover, $\mathcal{J}$ is weakly homogeneous. Since incidence algebras can be regarded as infinite structural matrix rings, the next statement does not come as a surprise, namely, $\mathcal{J}$ is homogeneous if and only if $x = y$ for all $(x, y) \in X$ if and only if $I_J(A) \cong A^J$. Indeed, if $\mathcal{J}$ is homogeneous, then $(\mathcal{J}(A) : P)_{I_J(A)} = \mathcal{J}(I_J(A)) = I_J(\mathcal{J}(I_J(A)) \cap A) = I_J(\mathcal{J}(A))$ for all rings A. Choose $(x_0, y_0) \in X$ with $x_0 \neq y_0$. For any $a \in A$, define $f : X \to A$ by $f(x,y) = \begin{cases} a \text{ if } (x,y) = (x_0, y_0) \\ 0 \text{ otherwise} \end{cases}$. Then $f \in (\mathcal{J}(A) : P)_{I_J(A)} = I_J(\mathcal{J}(A))$ which means $a \in \mathcal{J}(A)$. But $A = \mathcal{J}(A)$ does not hold for all rings A; hence $x = y$ for all $x, y \in J$. The other implications are straightforward (just remember, the Jacobson radical is invariant with respect to arbitrary direct products).

3.8. Splitting extensions. Let $(G, \cdot)$ be the cyclic group with four elements $\{e, a, a^2, a^3\}$. Let $d \in \{1, -1\}$ be fixed, $X = \{e, a\}, \mathcal{S} = \{\varnothing\}$, $\sigma(x) = \{(s,t) \mid s, t \in X, st = x \text{ or } st = a^2x\}$ and $\tau(x,y) = \begin{cases} 1 \text{ if } xy \in X \\ d \text{ if } xy \notin X \end{cases}$ for all $x, y \in X$. Then $T = \{e\}$. For a ring A, this convolution type gives a splitting extension of A. Such rings and their radicals have been considered in [21]. We identify an element $f \in C(A)$ with the ordered pair $f = (f_1, f_2) = (f(e), f(a))$. This means the product of two elements f, g of $C(A)$ is given by $fg = (f_1, f_2)(g_1, g_2) = (f_1g_1 + df_2g_2, f_1g_2 + f_2g_1)$.

Let $P(f) := \{f_1 + df_2, f_1 - df_2\} = \{f_1 + f_2, f_1 - f_2\}$. For a hypernilpotent radical α, $\alpha(C(A)) = (\alpha(A) : P)_{C(A)}$ for all rings A if and only if α satisfies:

(i) $R \in \alpha \Rightarrow C(R) \in \alpha$

(ii) $\alpha(C(R)) \cap R \in \alpha$ for all rings R.

The Jacobson radical $\mathcal{J}$ satisfies these two conditions, so we have $\mathcal{J}(C(A)) = (\mathcal{J}(A) : P)_{C(A)}, A \in \mathcal{J} \Leftrightarrow C(A) \in \mathcal{J}$, $\mathcal{J}$ is homogeneous and $\mathcal{J}^c(A) = \overline{\mathcal{J}}(A) = \mathcal{J}^*(A) = \mathcal{J}(A)$ for all rings A. But $\mathcal{J}$ need not be invariant: Let $A = \mathbb{Z}_4$, the ring of integers $mod 4$. If $\mathcal{J}(C(A)) = C(\mathcal{J}(A))$, then we have $(3,1) \in (\mathcal{J}(A) : P)_{C(A)} = \mathcal{J}(C(A)) = C(\mathcal{J}(A))$. But this is not possible since $3 \notin \{0, 2\} = \mathcal{J}(A)$.

For $d = 1$, $\alpha(C(A)) \subseteq (\alpha(A) : P)_{C(A)}$ holds for all rings A and all radicals α. If α is supernilpotent, then we have equality. For hypoidempotent radicals α (i.e. all nilpotent rings are semisimple), $\alpha(C(A)) \subseteq C(\alpha(A))$ for all rings A with equality if and only if $R \in \alpha \Rightarrow C(R) \in \alpha$. If α is subidempotent, then $C(R) \in \alpha \Leftrightarrow (R \in \alpha$ and $(0 : P)_{CR)} = 0)$. This means, for subidempotent radicals α, $\alpha(C(A)) = C(\alpha(A)) \Leftrightarrow \alpha(A) \cap (0 : P)_{C(R)} = 0$.

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