Totally bounded rings and their groups of units

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Abstract. We will present here some recent results concerning totally bounded topological rings. Most results will be presented but not proved.

Mathematics subject classification: 16W80.

Keywords and phrases: Pseudo-compact space, countably compact space, precompact group, the pointwise topology, atomic boolean ring, Bohr compactification of a topological ring.

All topological spaces are assumed to be Tychonoff. Topological groups are assumed to be Hausdorff. Topological rings are assumed to be associative and Hausdorff. The Jacobson radical of a ring $R$ will be denoted $J(R)$. The symbol $R = A \oplus B$ means that the group $R$ is a topological direct sum of its subgroups $A$ and $B$.

A topological space $X$ is called:
- pseudo-compact provided each real-valued function on it is bounded;
- countably compact provided each countable open cover has a finite subcover.

The closure of a subset $A$ of a topological space $X$ will be denoted by $\overline{A}$. If $R$ is a ring and $A$ its subset, then $\langle A \rangle$ stands for the subring of $R$ generated by $A$.

We will examine endomorphisms of linear spaces over finite fields by using of pointwise topology.

Let $k$ be a finite field and $V$ be a linear $k$-space. Recall that the pointwise topology on $\text{End}V$ is given by a fundamental system of neighbourhoods of zero consisting of subsets of the form $T(K) = \{\alpha : \alpha \in \text{End}V, \alpha(K) = 0\}$, where $K$ runs all finite subsets of $V$. We will consider $\text{End}V$ as a topological ring with the pointwise topology. Below $GL(V)$ stands for the topological group of all invertible elements of $\text{End}V$ with respect to the pointwise topology.

The pointwise topology allows to study some endomorphisms of $V$.

Definition 1. An element $\alpha$ of $\text{End}V$ is called:
- topologically nilpotent provided it is a topologically nilpotent element of $\text{End}V$;
- compact provided the subring $\langle \alpha \rangle$ is compact;
- topologically unipotent provided the element $1 - \alpha$ is topologically nilpotent;
- semisimple provided the subring $\langle \alpha \rangle$ is a compact semiprimitive ring.

Recall that an element $\alpha \in \text{End}V$ is called locally finite provided $V$ is decomposed in a direct sum of $\alpha$-invariant finite-dimensional subspaces.

Remark 1. It follows from ([18], Theorem 19.4) that if $k$ is a finite field, $V$ a left vector $k$-space, then $\alpha \in \text{End}V$ is compact $\iff$ for every $v \in V$, the subset $\langle \alpha \rangle v$ is finite.
We note that every locally finite element of $\text{End} V$ is compact. The following example shows that the reverse affirmation is not true:

Let $k$ be any finite field and $V$ a linear $k$-space of countable infinite dimension. Fix a countable base $\{v_i : i \in \omega\}$. Let $\alpha \in \text{End} V, \alpha(v_0) = 0$, and $\alpha(v_i) = v_{i-1}$ for $i > 0$. Each nonzero $\alpha$-invariant linear subspace of $V$ contains $k\mathbb{V}$, hence $V$ cannot be decomposed in a direct sum of finite-dimensional $\alpha$-invariant subspaces, i.e., $\alpha$ is not locally finite. It follows from Remark 1 that $\alpha$ is compact.

**Proposition 1.** Let $V$ be a linear space over a finite field $k$ and $\alpha \in \text{End} V$ a compact element. Then:

i) there exist unique elements $\alpha_s, \alpha_n$ such that $\alpha = \alpha_s + \alpha_n$, where $\alpha_s$ is semisimple, $\alpha_n$ is topologically nilpotent, and $\alpha_s \alpha_n = \alpha_n \alpha_s$;

ii) if $\alpha \in \text{GL}(V)$, then there exist unique elements $\alpha_s, \alpha_u \in \text{GL}(V)$ satisfying the conditions: $\alpha_s$ is semisimple, $\alpha_u$ topologically unipotent, $\alpha = \alpha_s \alpha_u$, and $\alpha_s \alpha_u = \alpha_u \alpha_s$.

In analogy with the theory of linear algebraic groups we shall call the decomposition $\alpha = \alpha_s + \alpha_n$ the additive Jordan decomposition of $\alpha$ and the decomposition $\alpha = \alpha_s \alpha_u$ for an invertible $\alpha$ the multiplicative Jordan decomposition for $\alpha$.

**Theorem 2.** Let $R = S \oplus J(R)$ be Wedderburn-Mal’cev decomposition of a compact ring with identity of prime characteristic. Then $U(R) = U(S) \cdot (1 + J(R)), U(S) \cap (1 + J(R)) = 1$, i.e., $U(R)$ is a semidirect topological product of $U(S)$ and $1 + J(R)$.

**Theorem 3.** Let $V$ be a linear space over a finite field $k$. If $H$ is a closed subgroup of $\text{GL}(V), x \in H, x$ is compact, $x = x_s x_u$ its multiplicative Jordan decomposition then $x_s \in H$ and $x_u \in H$.

**Theorem 4** [15]. Let $R$ be a countably compact ring with identity. The following conditions are equivalent:

1) $U(R)$ is a torsion group;

2) $R$ has a finite characteristic and there exist two different positive integers $n$ and $k$ such that the ring $R$ satisfies the identity $x^n = x^k$;

3) $R$ is a locally finite ring;

4) for every $x \in R$ the subring $(x)$ is finite.

**Theorem 5** [15]. Let $R$ be a compact ring with identity. Then the following conditions are equivalent:

1) $U(R)$ is a torsion group;

2) $R(+) \text{ is a torsion group and } R/J(R) \cong_{\text{top}} L_1^{m_1} \times \cdots \times L_n^{m_n}$, where $L_1, \ldots, L_n$ are finite simple rings and $m_1, \ldots, m_n$ are arbitrary cardinal numbers.

A topological ring $R$ is a *semidirect product* of a subring $S$ and an ideal $I$ provided $R$ is a topological group sum of $S$ and $I$.

The following Theorem of A.Tripe generalizes a result obtained by Jo-Ann Cohen and K.Koh [10]:

**Theorem 6 [14].** Let $R$ be a countably compact ring with identity. Then the group $U(R)$ is simple iff $R$ is a boolean ring or it is topologically isomorphic to one of the following rings:

1) $A_0 \times I$ where $A_0$ is a finite field of cardinality 3 or $2^m$ where $2^m - 1$ is a prime number (a prime number of Mersenne);
2) $A_0 \times I$ where $A_0$ is the ring of $n \times n$ matrices over $\mathbb{Z}/(2)$, $n \geq 3$;
3) a semidirect product of $I$ and $\mathbb{Z}/(4)$;
4) a semidirect product of $I$ and $\mathbb{Z}/(2)[x]/(x^2)$;
5) a semidirect product of $I$ and $M(2, \mathbb{Z}/(2))$,

where in all cases $I$ is a countably compact boolean ring.

There are examples showing that in 3), 4), 5) the semidirect product cannot be replaced by direct products.

There is a gap between pseudo-compactness and countable compactness:

**Theorem 7 [16].** Let $k$ be a finite field and $X$ a set of cardinality $2^{\omega}$. Then the ring $k[X]$ of polynomials over $X$ with coefficients from $k$ admits a pseudo-compact ring topology.

As follows from Chevalley’s Theorem ([19], Chapter II, Theorem 13) if $(R, T)$ is a commutative compact Noetherian ring and $T_1$ is a ring topology on $R$ such that $(R, T_1)$ has a fundamental system of neighbourhoods of zero then $T \leq T_1$. We extend this assertion to the noncommutative case:

**Theorem 8.** Let $(R, T)$ be a compact left Noetherian ring with identity. If $T_1$ is a ring topology on $R$ and $(R, T_1)$ has a fundamental system of neighbourhoods of zero consisting of left ideals then $T \leq T_1$.

**Proof.** Any left ideal of $R$ is closed in $(R, T)$. Let $\{V_\alpha\}_{\alpha \in \Omega}$ be a fundamental system of neighbourhoods of zero of $(R, T_1)$ consisting of left ideals. If $V$ is an open ideal of $(R, T)$, then $\{0\} \subseteq \cap_{\alpha \in \Omega} V_\alpha \subseteq V$. By compactness of $(R, T)$ there exist $\alpha_1, \ldots, \alpha_n \in \Omega$ such that $V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} \subseteq V$. Therefore $V$ is an open ideal of $(R, T_1)$. Since $V$ was arbitrarily, $T \leq T_1$.

Recall that if $a, b$ are two elements of a boolean ring $R$, then put $a \leq b$ if $ab = a$.

An element $a$ of a boolean ring $R$ is called an *atom* provided $a \neq 0$ and for each $x \in R$, $x \leq a$, $x \neq a$, $x = 0$. A boolean ring $R$ is called *atomic* provided for each $x \in R$, $x \neq 0$, there exists at least one atom $a$ such that $a \leq x$.

**Theorem 9.** Let $R$ be an atomic boolean ring. Then there exists a totally bounded ring topology $T_0$ on $R$ such that $T_0 \leq T_1$ for each Hausdorff ring topology $T_1$ on $R$. 

Proof. Consider the family \( \mathcal{B} \) consisting of ideals of the form \( \text{Ann}(a), a \in R \). Each element of \( B \) is a maximal ideal of \( R \), hence it is cofinite. We note that \( B \) is a filter base. Indeed, let \( a_1, a_2, \ldots, a_n \in R \). Then there exists \( a \in R \) such that \( Ra_1 + Ra_2 + \ldots + Ra_n = Ra \). Evidently, \( \text{Ann}(a) = \text{Ann}(a_1) \cap \text{Ann}(a_2) \cap \ldots \cap \text{Ann}(a_n) \).

We affirm that \( \cap \mathcal{B} = 0 \). Let \( 0 \neq x \in R \). If \( a \) is an atom of \( R \), \( a \leq x \), then \( xa = a \), hence \( x \notin \text{Ann}(a) = R(1 - a) \in B \). It follows that \( \mathcal{B} \) gives a totally bounded ring topology \( T_0 \) on \( R \). If \( T_1 \) is another \( T_1 \)-ring topology on \( R \), then each \( \text{Ann}(a) \) is closed in \( (R,T_1) \) and cofinite, hence \( \text{Ann}(a) \) is open in \( (R,T_1) \) and so \( T_0 \leq T_1 \).

Corollary 10. The quasicomponent of any atomic topological \( T_1 \)-ring is equal to zero.

The notion of the Bohr compactification of a topological ring was introduced by Holm [12, 13].

Definition 2. Let \( (R,T) \) be a topological ring. A pair \( ((bR,bT),bR) \) with the following properties is called a Bohr compactification of \( (R,T) \):

1. \( (bR,bT) \) is a compact ring;
2. \( bT \) is a continuous homomorphism from \( (R,T) \) onto a dense subring \( (bR,bT) \);
3. for every continuous homomorphism \( \alpha \) of \( (R,T) \) into a compact ring \( C \) there exists a continuous homomorphism \( \hat{\alpha} : (bR,bT) \to C \) such that \( \hat{\alpha} \circ bR = \alpha \).

Theorem 11. Every topological ring \( (R,T) \) has a Bohr compactification unique up to an isomorphism.

It is interesting to calculate the Bohr compactification of concrete topological rings.

Theorem 12 [12]. The Bohr compactification \( b(\mathbb{Z},T_d) \) of the ring of integers is isomorphic to \( \prod_{p \in P} \mathbb{Z}_p \) as a topological ring.

Theorem 13 [12]. Let \( R \) be a ring furnished with the discrete topology, \( bR \) its Bohr compactification, \( P(R) \) the lattice of all precompact ring topologies on \( R \), and \( C(bR) \) the lattice of closed ideals of \( bR \). Then there is a lattice antiisomorphism \( \Phi : P(R) \to C(bR) \) such that \( bR/\Phi(T) \) is isomorphic to the completion of \( (R,T) \).

In [12] was introduced the concept of a van der Waerden ring: A compact ring \( (R,T) \) is called a vdW-ring provided each ring homomorphism \( h : (R,T) \to (K,U) \) with \( (K,U) \) continuous.

Fix a faithful indexing \( \{ R_n : n \in \omega \} \) of all matrix rings over finite fields. By Theorem of Kaplansky a semiprimitive ring \( R \) is of the form \( R = \prod_{n \in I} R_n^{\alpha_n} \) for suitable \( I = I(R) \subseteq \omega \) and cardinal numbers \( \alpha_n = \alpha_n(R_n) \).

Theorem 14 [12]. Let \( R \) be a compact semiprimitive ring with Kaplansky representation \( R = \prod_{n \in I} R_n^{\alpha_n} \). In order that \( R \) be a vdW-ring it is necessary and sufficient that each \( \alpha_n \) be finite.

A compact ring with identity is a vdW-ring iff every cofinite ideal is open.

Theorem 15 [12, 17]. A compact semisimple ring admits a unique pseudo-compact topology iff it is metrizable.
References


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Received November 25, 2004