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Totally bounded rings and their groups of units

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Abstract. We will present here some recent results concerning totally bounded topological rings. Most results will be presented but not proved.

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All topological spaces are assumed to be Tychonoff. Topological groups are assumed to be Hausdorff. Topological rings are assumed to be associative and Hausdorff. The Jacobson radical of a ring R will be denoted $J(R)$. The symbol $R = A \oplus B$ means that the group R is a topological direct sum of its subgroups A and B.

A topological space X is called:
pseudo-compact provided each real-valued function on it is bounded;
countably compact provided each countable open cover has a finite subcover.

The closure of a subset A of a topological space X will be denoted by $\overline{A}$. If R is a ring and A its subset, then $\langle A\rangle$ stands for the subring of R generated by A.

We will examine endomorphisms of linear spaces over finite fields by using of pointwise topology.

Let k be a finite field and V be a linear k-space. Recall that *the pointwise topology* on End$V$ is given by a fundamental system of neighbourhoods of zero consisting of subsets of the form $T(K) = \{\alpha : \alpha \in \mathrm{End}V, \alpha(K) = 0\}$, where K runs all finite subsets of V. We will consider End$V$ as a topological ring with the pointwise topology. Below $GL(V)$ stands for the topological group of all invertible elements of End$V$ with respect to the pointwise topology.

The pointwise topology allows to study some endomorphisms of V.

Definition 1. *An element α of EndV is called: topologically nilpotent provided it is a topologically nilpotent element of EndV; compact provided the subring $\overline{\langle\alpha\rangle}$ is compact; topologically unipotent provided the element $1-\alpha$ is topologically nilpotent; semisimple provided the subring $\overline{\langle\alpha\rangle}$ is a compact semiprimitive ring.*

Recall that an element $\alpha \in \mathrm{End}V$ is called *locally finite* provided V is decomposed in a direct sum of α-invariant finite-dimensional subspaces.

Remark 1. It follows from ([18], Theorem 19.4) that if k is a finite field, V a left vector k-space, then $\alpha \in \mathrm{End}V$ is compact $\Leftrightarrow$ for every $v \in V$, the subset $\overline{\langle\alpha\rangle}\, v$ is finite.

We note that every locally finite element of End$V$ is compact. The following example shows that the reverse affirmation is not true:

Let k be any finite field and V a linear k-space of countable infinite dimension. Fix a countable base $\{v_i : i \in \omega\}$. Let $\alpha \in \mathrm{End}V, \alpha(v_0) = 0$, and $\alpha(v_i) = v_{i-1}$ for $i > 0$. Each nonzero α-invariant linear subspace of V contains kv_0, hence V cannot be decomposed in a direct sum of finite-dimensional α-invariant subspaces, i.e., α is not locally finite. It follows from Remark 1 that α is compact.

Proposition 1. *Let V be a linear space over a finite field k and $\alpha \in \mathrm{End}V$ a compact element. Then:*

i) there exist unique elements α_s, α_n such that $\alpha = \alpha_s + \alpha_n$, where α_s is semisimple, α_n is topologically nilpotent, and $\alpha_s\alpha_n = \alpha_n\alpha_s$;

ii) if $\alpha \in GL(V)$, then there exist unique elements $\alpha_s, \alpha_u \in GL(V)$ satisfying the conditions: α_s is semisimple, α_u topologically unipotent, $\alpha = \alpha_s\alpha_u$, and $\alpha_s\alpha_u = \alpha_u\alpha_s$.

In analogy with the theory of linear algebraic groups we shall call the decomposition $\alpha = \alpha_s + \alpha_n$ *the additive Jordan decomposition* of α and the decomposition $\alpha = \alpha_s\alpha_u$ for an invertible α *the multiplicative Jordan decomposition* for α.

Theorem 2. *Let $R = S \oplus J(R)$ be Wedderburn-Mal'cev decomposition of a compact ring with identity of prime characteristic. Then $U(R) = U(S) \cdot (1 + J(R))$, $U(S) \cap (1 + J(R)) = 1$, i.e., $U(R)$ is a semidirect topological product of $U(S)$ and $1 + J(R)$.*

Theorem 3. *Let V be a linear space over a finite field k. If H is a closed subgroup of $GL(V), x \in H, x$ is compact, $x = x_s x_u$ its multiplicative Jordan decomposition then $x_s \in H$ and $x_u \in H$.*

Theorem 4 [15]. *Let R be a countably compact ring with identity. The following conditions are equivalent:*

1) $U(R)$ is a torsion group;

2) R has a finite characteristic and there exist two different positive integers n and k such that the ring R satisfies the identity $x^n = x^k$;

3) R is a locally finite ring;

4) for every $x \in R$ the subring $\langle x \rangle$ is finite.

Theorem 5 [15]. *Let R be a compact ring with identity. Then the following conditions are equivalent:*

1) $U(R)$ is a torsion group;

2) $R(+)$ is a torsion group and $R/J(R) \cong_{top} L_1^{m_1} \times \cdots \times L_n^{m_n}$, where $L_1, \cdots, L_n$ are finite simple rings and $m_1, \ldots, m_n$ are arbitrary cardinal numbers.

In 1988–1997 there appeared a number of interesting papers of Jo-Ann Cohen, Kwangil Koh and I. W. Lorimer concerning groups of units of compact rings with identity [1–11].

A topological ring R is a *semidirect product* of a subring S and an ideal I provided R is a topological group sum of S and I.

The following Theorem of A.Tripe generalizes a result obtained by Jo-Ann Cohen and K.Koh [10]:

Theorem 6 [14]. *Let R be a countably compact ring with identity. Then the group $U(R)$ is simple iff R is a boolean ring or it is topologically isomorphic to one of the following rings:*

1) $A_0 \times I$ where A_0 is a finite field of cardinality 3 or 2^m where $2^m - 1$ is a prime number (=a prime number of Mersenne);

2) $A_0 \times I$ where A_0 is the ring of $n \times n$ matrices over $\mathbb{Z}/(2), n \geq 3$;

3) a semidirect product of I and $\mathbb{Z}/(4)$;

4) a semidirect product of I and $\mathbb{Z}/(2)[x]/(x^2)$;

5) a semidirect product of I and $M(2, \mathbb{Z}/(2))$,

where in all cases I is a countably compact boolean ring.

There are examples showing that in 3), 4), 5) the semidirect product cannot be replaced by direct products.

There is a gap between pseudo-compactness and countable compactness:

Theorem 7 [16]. *Let k be a finite field and X a set of cardinality $2^{\omega\cdot}$ Then the ring $k[X]$ of polynomials over X with coefficients from k admits a pseudo-compact ring topology.*

As follows from Chevalley's Theorem ([19], Chapter II, Theorem13) if $(R, \mathcal{T})$ is a commutative compact Noetherian ring and $\mathcal{T}_1$ is a ring topology on R such that $(R, \mathcal{T}_1)$ has a fundamental system of neighbourhoods of zero then $\mathcal{T} \leq \mathcal{T}_1$. We extend this assertion to the noncommutative case:

Theorem 8. *Let $(R, \mathcal{T})$ be a compact left Noetherian ring with identity. If $\mathcal{T}_1$ is a ring topology on R and $(R, \mathcal{T}_1)$ has a fundamental system of neighbourhoods of zero consisting of left ideals then $\mathcal{T} \leq \mathcal{T}_1$.*

Proof. Any left ideal of R is closed in $(R, \mathcal{T})$. Let $\{V_\alpha\}_{\alpha\in\Omega}$ be a fundamental system of neighbourhoods of zero of $(R, \mathcal{T}_1)$ consisting of left ideals. If V is an open ideal of $(R, \mathcal{T})$, then $\{0\} \subseteq \cap_{\alpha\in\Omega} V_\alpha \subseteq V$. By compactness of $(R, \mathcal{T})$ there exist $\alpha_1 \dots, \alpha_n \in \Omega$ such that $V_{\alpha_1} \cap \dots \cap V_{\alpha_n} \subseteq V$. Therefore V is an open ideal of $(R, \mathcal{T}_1)$. Since V was arbitrarily, $\mathcal{T} \leq \mathcal{T}_1$. □

Recall that if a, b are two elements of a boolean ring R, then put $a \leq b$ if $ab = a$. An element a of a boolean ring R is called an *atom* provided $a \neq 0$ and for each $x \in R$, $x \leq a$, $x \neq a$, $x = 0$. A boolean ring R is called *atomic* provided for each $x \in R$, $x \neq 0$, there exists at least one atom a such that $a \leq x$.

Theorem 9. *Let R be an atomic boolean ring. Then there exists a totally bounded ring topology $\mathcal{T}_0$ on R such that $\mathcal{T}_0 \leq \mathcal{T}_1$ for each Hausdorff ring topology $\mathcal{T}_1$ on R.*

Proof. Consider the family $\mathfrak{B}$ consisting of ideals of the form $\mathrm{Ann}(a)$, $a \in R$. Each element of B is a maximal ideal of R, hence it is cofinite. We note that B is a filter base. Indeed, let $a_1, a_2, ..., a_n \in R$. Then there exists $a \in R$ such that $Ra_1 + Ra_2 + ... + Ra_n = Ra$. Evidently, $\mathrm{Ann}(a) = \mathrm{Ann}(a_1) \cap \mathrm{Ann}(a_2) \cap ... \cap \mathrm{Ann}(a_n)$. We affirm that $\cap\mathfrak{B} = 0$: Let $0 \neq x \in R$. If a is an atom of R, $a \leq x$, then $xa = a$, hence $x \notin \mathrm{Ann}(a) = R(1-a) \in B$. It follows that $\mathfrak{B}$ gives a totally bounded ring topology $\mathcal{T}_0$ on R. If $\mathcal{T}_1$ is another $\mathcal{T}_1$-ring topology on R, then each $\mathrm{Ann}(a)$ is closed in $(R,\mathcal{T}_1)$ and cofinite, hence $\mathrm{Ann}(a)$ is open in $(R,\mathcal{T}_1)$ and so $\mathcal{T}_0 \leq \mathcal{T}_1$. □

Corollary 10. *The quasicomponent of any atomic topological $\mathcal{T}_1$-ring is equal to zero.*

The notion of the Bohr compactification of a topological ring was introduced by Holm [12, 13].

Definition 2. *Let $(R, \mathcal{T})$ be a topological ring. A pair $((bR, b\mathcal{T}), b_R)$ with the following properties is called a Bohr compactification of $(R, \mathcal{T})$:*

1) $(bR, b\mathcal{T})$ is a compact ring;

2) b_R is a continuous homomorphism from $(R, \mathcal{T})$ onto a dense subring $(bR, b\mathcal{T})$;

3) for every continuous homomorphism α of $(R, \mathcal{T})$ into a compact ring C there exists a continuous homomorphism $\hat{\alpha} : (bR, b\mathcal{T}) \to C$ such that $\hat{\alpha} \circ b_R = \alpha$.

Theorem 11. *Every topological ring $(R, \mathcal{T})$ has a Bohr compactification unique up to an isomorphism.*

It is interesting to calculate the Bohr compactification of concrete topological rings.

Theorem 12 [12]. *The Bohr compactification $b(\mathbb{Z}, \mathcal{T}_d)$ of the ring of integers is isomorphic to $\prod_{p\in P} \mathbb{Z}_p$ as a topological ring.*

Theorem 13 [12]. *Let R be a ring furnished with the discrete topology, bR its Bohr compactification, $P(R)$ the lattice of all precompact ring topologies on R, and $C(bR)$ the lattice of closed ideals of bR. Then there is a lattice antiisomorphism $\Phi : P(R) \to C(bR)$ such that $bR/\Phi(\mathcal{T})$ is isomorphic to the completion of $(R, \mathcal{T})$.*

In [12] was introduced the concept of a van der Waerden ring: A compact ring $(R, \mathcal{T})$ is called a vdW-*ring* provided each ring homomorphism $h : (R, \mathcal{T}) \to (K, \mathcal{U})$ with $(K, \mathcal{U})$ is continuous.

Fix a faithful indexing $\{R_n : n \in \omega\}$ of all matrix rings over finite fields. By Theorem of Kaplansky a semiprimitive ring R is of the form $R = \prod_{n\in I} R_n^{\alpha_n}$ for suitable $I = I(R) \subseteq \omega$ and cardinal numbers $\alpha_n = \alpha_n(R_n)$.

Theorem 14 [12]. *Let R be a compact semiprimitive ring with Kaplansky representation $R = \prod_{n\in I} R_n^{\alpha_n}$. In order that R be a vdW-ring it is necessary and sufficient that each α_n be finite.*

A compact ring with identity is a vdW-ring iff every cofinite ideal is open.

Theorem 15 [12, 17]. *A compact semisimple ring admits a unique pseudo-compact topology iff it is metrizable.*

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