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Wedderburn decomposition of LCM-rings

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Abstract. In this paper we extend in this paper a result of Zelinsky to the class of linearly compact, monocompact rings of prime characteristic.

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1 Introduction

A subtle fact of the theory of algebras over a field is the Wedderburn–Mal'cev Theorem (see, e.g., [3, 4]). This Theorem was extended also to classes of topological rings (see, e.g., [1, 10, 13]). The aim of this paper is an extension of the Wedderburn Theorem to the class of bounded, linearly compact, monocompact rings.

2 Notation and conventions

All topological ring are assumed to be Hausdorff and associative (and not necessarily with identity). If R is a topological ring and $S \subseteq R$, then by $\langle S \rangle$ the closed subring of R generated by S is denoted.

A *monocompact* ring [11] is a topological ring R which is the reunion of its compact subrings (equivalently, for each element $x \in R$ the subring $\langle x \rangle$ is compact).

A topological ring R is *called linearly compact* [8] if it has a fundamental system of neighborhoods of zero consisting of left ideals and every filter base consisting of cosets relative to closed left ideals has a non-empty intersection.

A topological ring R is called *hereditarily linearly compact* [1] if every closed subring is a linearly compact ring.

The class of hereditarily linearly compact rings is intermediate between compact totally disconnected rings and linearly compact rings.

A topological ring R is called *topologically locally finite* [11] provided for every finite subset F the subring $\langle F \rangle$ is compact.

Recall that an element of a topological ring is called *topologically nilpotent* provided $x^n \to 0$.

The connected component of zero of a topological Abelian group R is denoted by R_0.

As usual, a *local* ring is a ring with identity having a unique maximal left ideal.

A *semiprimitive* ring is a ring with identity whose Jacobson radical is zero.

If n is a natural number and R is a ring, then $M(n, R)$ denotes the ring of $n \times n$ matrices over R.

The symbol $A \cong_{\text{top}} B$ means that topological rings A and B are isomorphic.

3 Semiprimitive LCM-rings

Definition 3.1. *A* LCM*-ring is a linearly compact, monocompact ring, having a fundamental system of neighborhoods of zero consisting of ideals.*

Example 3.2. *Let $\mathbb{F}$ be any field which is an infinite algebraic extension of a finite field. Then the ring* $\begin{bmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{bmatrix}$ *is a discrete* LCM*-ring.*

Lemma 3.3. *For any topologically nilpotent element a of a compact ring R and any integers $c_n \in \mathbb{Z}$, the series $\sum_{n=1}^{\infty} c_n a^n$ converges.*

Proof. We may consider without loss of generality that R is quasi-regular. We may consider that $R = \langle a \rangle$.

The ring R/R_0 has a a fundamental system of neighborhood of zero consisting of ideals. Evidently, R/R_0 is topologically nilpotent. By theorem of Kaplansky [11, Theorem 2.5.7], $R_0 R = R R_0 = 0$, hence R is quasi-regular and so is topologically nilpotent. Since R is complete, by the Cauchy criterion, the series $\sum_{n=1}^{\infty} c_n a^n$ is convergent. □

Recall that a ring R is called SBI-*ring* [6] if for any $a \in J(R)$ there exists an $x \in J(R)$ such that:

(i) $x^2 + x = a$;

(ii) for all $z \in J(R)$, $az = za$ implies $xz = zx$.

Theorem 3.4. *Any topological ring R whose Jacobson radical $J(R)$ is monocompact is a* SBI*-ring.*

Proof. Let $a \in J(R)$. Then $\langle a \rangle \subseteq J(R)$ and $\langle a \rangle$ is compact. Consider the sequence

$$c_1 = 1, \quad c_k = -\sum_{i=1}^{k-1} c_i c_{k-i}, \quad k = 2, 3, ...,$$

of integers. Then, by Lemma 3.3, $x = \sum_{n=1}^{\infty} c_n a^n$ exists. Evidently, $x^2 + x = a$ and for all $z \in J(R)$, $az = za$ implies $xz = zx$. □

Corollary 3.5. *(see [6, p. 125]). Any compact ring is a* SBI*-ring.*

Corollary 3.6. *Any countably compact ring with identity is a* SBI*-ring.*

Lemma 3.7. *If R is a* LCM*-ring, R' is a topological ring having a fundamental system of neighborhoods of zero consisting of left ideals and $f : R \to R'$ is a surjective continuous homomorphism, then R' is a* LCM*-ring, too.*

Proof. Indeed, R' is linearly compact. Since R is the union of its compact subrings, R is the union of its compact subrings, too. Therefore R' is monocompact, hence R' is LCM-ring. □

Corollary 3.8. *If R is a* LCM*-ring and V is an open ideal, then the quotient ring R/V is a discrete* LCM*-ring.*

Lemma 3.9. *Any discrete* LCM*-ring R is locally finite.*

Proof. For every $x \in R$, the subring $\langle x \rangle$ is finite, hence $J(R)$ is a nilring. By [11, Theorem 2.9.30], $J(R)$ is a locally nilpotent ideal and so $J(R)$ is locally finite.

The ring $R/J(R)$ is isomorphic to a finite product $M(n_1, \Delta_1) \times \cdots \times M(n_k, \Delta_k)$, where $\Delta_1, \cdots, \Delta_k$ are algebraic extensions of finite fields. It follows that $R/J(R)$ is locally finite. Since the class of locally finite rings is closed under extensions, R is a locally finite ring. □

Problem 3.10. *Let R be a linearly compact ring and I be a closed left topological nilideal. Is then I locally topologically nilpotent?*

In the case when R has a fundamental system of neighborhoods of zero consisting of ideals, the Problem 3.10 has a positive answer, according to [11, Theorem 2.9.30].

Theorem 3.11. *If R is a* LCM*-ring then R is topologically locally finite.*

Proof. By Corollary 3.8 and Lemma 3.9, for every open ideal V, the quotient ring R/V is a locally finite ring. Then $R \cong_{\text{top}} \varprojlim R/V$ is a topologically finite ring. □

Corollary 3.12. *Any* LCM*-ring is a* SBI*-ring.*

Lemma 3.13. *The Jacobson radical $J(R)$ of a* LCM*-ring R is monocompact.*

Proof. By Leptin's Theorem [7], $J(R)$ is a closed ideal of R. □

Theorem 3.14. *If R is a* LCM*-ring with identity, $R/J(R)$ is topologically isomorphic to $M(n, \Delta)$ where Δ is a division ring, then $R \cong_{\text{top}} M(n, P)$ where P is a* LCM*-ring and $P/J(P)$ is topologically isomorphic to Δ.*

Proof. By Lemma 3.13, $J(R)$ is monocompact and by Theorem 3.4, R is a SBI-ring.

By [5, Theorem 3.8.1], $R \cong M(n, P)$, where P is a ring with identity and $P/J(P) \cong S$. We identify R with $M(n, P)$. By [11, Theorem 2.6.65], there exists a topology $\mathfrak{T}_0$ on P such that the ring $M(n, P)$ is equipped with the canonical topology of a matrix ring. Since $(P, \mathfrak{T}_0)$ is topologically isomorphic to $e\, M(n, P)\, e$ for some idempotent e, $(P, \mathfrak{T}_0)$ is a LCM-ring. □

Lemma 3.15. *If R is a left linearly compact discrete ring, then any family $\{e_\alpha : \alpha \in \Omega\}$ of orthogonal idempotents of R is finite.*

Proof. Indeed, if there exists a sequence $(e_{\alpha_n})_{n\geq 1}$ of pairwise different, non-zero elements of $\{e_\alpha : \alpha \in \Omega\}$, then $\sum\limits_{n=1}^{\infty} Re_{\alpha_n}$ is an infinite direct sum of left ideals, a contradiction. □

Lemma 3.16. *If R is a* LCM-*ring then each family $\{e_\alpha : \alpha \in \Omega\}$ of orthogonal idempotents is summable.*

Proof. Since R has a fundamental system of neighborhoods $\mathcal{B}$ of zero consisting of ideals,

$$R \cong_{\text{top}} \varprojlim \{R/V : V \in \mathcal{B}\} \subseteq \prod_{V\in\mathcal{B}} R/V.$$

For each $W \in \mathcal{B}$ denote by pr_W the canonical projection of $\prod\limits_{V\in\mathcal{B}} R/V$ on R/W. Since, $\{\text{pr}_V(e_\alpha) : \alpha \in \Omega\}$ is a family of orthogonal idempotents, by Lemma 3.15, this family is finite, therefore it is summable. By [2, Proposition 3.5.4], the family $\{e_\alpha : \alpha \in \Omega\}$ is summable in $\prod\limits_{V\in\mathcal{B}} R/V$. Since $\varprojlim \{R/V : V \in \mathcal{B}\}$ is a closed subring in $\prod\limits_{V\in\mathcal{B}} R/V$, this family is summable in $\varprojlim \{R/V : V \in \mathcal{B}\}$,too. □

Theorem 3.17. *An* LCM-*ring R is semiprimitive if and only if*

$$R \cong_{\text{top}} \prod_{\alpha\in\Omega} M(n_\alpha, R_\alpha),$$

where each R_α is an algebraic extension of a finite field.

Proof. Suppose that R is semiprimitive. By Leptin's Theorem [7],

$$R \cong_{\text{top}} \prod_{\alpha\in\Omega} M(n_\alpha, R_\alpha),$$

where each R_α is a division ring. Since each R_α is monocompact, every subring of R_α generated by one element is finite. Therefore, each R_α has a finite characteristic and can be regarded as an algebra over a finite field F_α. Since R_α is monocompact, R_α is an algebraic algebra over F_α and by Theorem of Jacobson [5, Theorem 7.12.2], R_α is commutative.

Conversely, let $R \cong_{\text{top}} \prod\limits_{\alpha\in\Omega} M(n_\alpha, R_\alpha)$ where each R_α is an algebraic extension of a finite field. Let $x = (x_\alpha)_{\alpha\in\Omega} \in \prod\limits_{\alpha\in\Omega} M(n_\alpha, R_\alpha)$. Then for every $\beta \in \Omega$,

$$\text{pr}_\beta \langle x\rangle \subseteq \left\langle \text{pr}_\beta (x)\right\rangle = \langle x_\beta\rangle,$$

hence $\langle x\rangle \subseteq \prod\limits_{\alpha\in\Omega} \langle \text{pr}_\alpha (x)\rangle$. Since every subring $\langle \text{pr}_\alpha (x)\rangle$ is finite, by Theorem of Tihonov, $\prod\limits_{\alpha\in\Omega} \langle \text{pr}_\alpha (x)\rangle$ is compact and so $\langle x\rangle$ is compact. □

4 Wedderburn decomposition of LCM-rings

We say that a topological ring R *admits a Wedderburn decomposition in the category of topological rings* [1] if the Jacobson radical $J(R)$ is closed and there exists a closed subring S such that:

(i) $R = S + J(R)$;

(ii) $S \cap J(R) = 0$;

(iii) the restriction of the canonical homomorphism $\varphi : R \to R/J(R)$ to S is a topological isomorphism.

We will use below the following Theorem of Zelinsky:

Theorem 4.1 (13)**.** . *If R is a compact ring of prime characteristic p, then there exists a compact subring S of R such that $R = S + J(R)$.*

Lemma 4.2. *If R is a local* LCM*-ring, $R/J(R)$ is finite and* $\operatorname{char} R = p$ *is a prime number, then there exists a finite subring F of R which is a field, such that $R = F + J(R)$.*

Proof. The group of units $U(R/J(R))$ of the field $R/J(R)$ is cyclic. Denote by ϕ the canonical homomorphism of R onto $R/J(R)$. Let $\theta \in R$ such that $\phi(\theta)$ is a generator of $U(R/J(R))$. The subring $\langle\theta\rangle$ is compact; evidently, $R = \langle\theta\rangle + J(R)$. By Theorem **??**, there exists a subfield F of $\langle\theta\rangle$ such that $\langle\theta\rangle = F + J(\langle\theta\rangle)$. Since $J(\langle\theta\rangle)$ is topologically nil, $J(\langle\theta\rangle) \subseteq J(R)$, hence $R = F + J(R)$. □

Remark 4.3. *If K is a compact subring of R, then $J(K) = J(R) \cap K$.*

Indeed, since $J(R) \cap K$ is a topologically nil ideal of K, we obtain that $J(R) \cap K \subseteq J(K)$. Conversely, since $K/(J(R) \cap K) \cong_{\text{top}} (K + J(R))/J(R)$ and $(K + J(R))/J(R)$ is a subfield of $R/J(R)$ or 0, we obtain that $J(K) \subseteq J(R) \cap K$.

Lemma 4.4. *Let R be a local* LCM*-ring of prime characteristic p. Then there exists a finite subring F of R which is a field, such that $R = F + J(R)$.*

Proof. Denote by φ the canonical homomorphism of R onto $R/J(R)$.

Consider that $R/J(R) = \bigcup_{i=1}^{\infty} K_i$, where each K_i is a finite subfield of $R/J(R)$ and $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$.

Let $K_1 = \langle\varphi(x_1)\rangle$ and consider the subring $\langle x_1\rangle$ of R. By Theorem of Zelinsky, there exists a finite subring S_1 of $\langle x_1\rangle$ such that

$$\langle x_1\rangle = S_1 + J(\langle x_1\rangle).$$

Assume that we have constructed for a positive integer n, a set $\{S_1, \cdots, S_n\}$ of subrings of R which are finite fields, such that:

(i) $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n$;

(ii) $\varphi(S_i) = K_i$, $i = 1, ..., n$.

Since K_{n+1} is a finite field, there exists $x_{n+1} \in R$ such that $K_{n+1} = \langle \varphi(x_{n+1}) \rangle$. Again, by Theorem of Zelinsky there exists a finite subring P of $\langle x_1 \rangle$ which is a field such that

$$\langle x_{n+1} \rangle = P + J(\langle x_{n+1} \rangle).$$

Evidently, $\varphi(P) = \varphi(\langle x_{n+1} \rangle) = K_{n+1}$. Therefore P is isomorphic to K_{n+1}. We note that P contains a subfield Q isomorphic to K_n. Consider the subring $\langle Q, S_n \rangle$ of R. By Theorem 3.11, $\langle Q, S_n \rangle$ is compact. Since $R/J(R)$ contains a unique subfield isomorphic to K_n, we obtain that $\varphi(Q) = \varphi(S_n) = K_n$. It follows that $\varphi(\langle Q, S_n \rangle) \subseteq \langle \varphi(Q), \varphi(S_n) \rangle = K_n$. Since $\varphi(\langle Q, S_n \rangle) \supseteq \varphi(S_n) = K_n$, we obtain that $\varphi(\langle Q, S_n \rangle) = K_n$. Since $\varphi(Q) = \varphi(S_n) = K_n$,

$$\langle Q, S_n \rangle = Q + J(\langle Q, S_n \rangle) = S_n + J(\langle Q, S_n \rangle).$$

By Theorem of Mal'cev (see, for example [11, Theorem 2.10.3]), there exists $a \in J(\langle Q, S_n \rangle) \subseteq J(R)$, such that

$$S_n = (1+a)^{-1} Q \ (1+a).$$

Then

$$(1+a)^{-1} P \ (1+a) \supseteq (1+a)^{-1} Q \ (1+a) = S_n.$$

Put

$$S_{n+1} = (1+a)^{-1} P \ (1+a).$$

Then S_{n+1} is a field, $S_n \subseteq S_{n+1}$ and

$$\varphi(S_{n+1}) = (\varphi(1+a))^{-1} \varphi(P) \ \varphi(1+a) = \varphi(P) = K_{n+1}.$$

We constructed a sequence

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq \cdots$$

of subrings of R which are finite fields and $\varphi(S_n) = K_n$, for each $n \in \mathbb{N}$. Then $S = \bigcup_{i=1}^{\infty} S_i$ is a subring of R which is a field and

$$\varphi\left(\bigcup_{i=1}^{\infty} S_i\right) = \bigcup_{i=1}^{\infty} \varphi(S_i) = \bigcup_{i=1}^{\infty} K_i = R/J(R).$$

Therefore $R = S + J(R)$. Since $S \cap J(R) = 0$ and $J(R)$ is open in R, we obtain that R is a topological direct sum of R and $J(R)$. $\square$

Lemma 4.5. *Let R be a* LCM*-ring with identity such that $R/J(R) \cong M(n, \Delta)$ where Δ is a division ring. Then there exists a subring S of R isomorphic to $M(n, \Delta)$ such that $R = S + J(R)$.*

Proof. By Theorem 3.14, there exists a local LCM-ring P such that $R \cong_{\text{top}} M(n, P)$. By Lemma 4.4, there exists a subring S of P such that $P = S + J(P)$ a topological direct sum of S and $J(P)$. We identify R with $M(n, P)$. Then

$$M(n, P) = M(n, S) + M(n, J(P))$$

and

$$M(n, S) \cap M(n, J(P)) = 0.$$

The subring $M(n, S)$ is discrete and $J(M(n, P)) = M(n, J(P))$. □

Theorem 4.6. *Let $f : R \to R'$ be a continuous homomorphism of a* LCM-*ring R with identity e on a* LCM-*ring R' with identity e' and $Ker f \subseteq J(R)$. If $\{e'_\alpha : \alpha \in \Omega\}$ is a family of orthogonal idempotents, $e' = \sum_{\alpha \in \Omega} e'_\alpha$, then there exists a family $\{e_\alpha : \alpha \in \Omega\}$ of orthogonal idempotents such that $e = \sum_{\alpha \in \Omega} e_\alpha$, $f(e_\alpha) = e'_\alpha$, $\alpha \in \Omega$.*

The proof of this Theorem is analogous to the proof of Theorem 2.6.57 from [11].

The following Theorem was proved for compact rings by Z.S. Lipkina [9].

Theorem 4.7. *Let R be an arbitrary* LCM-*ring. Then there exists a closed subring A, topologically isomorphic to a product of primary* LCM-*rings such that $R = A + J(R)$.*

The proof of this Theorem is analogous to the proof of Theorem 2.6.58 from [11].

Theorem 4.8. *Let R be a* LCM-*ring of prime characteristic. Then there exists a closed subring S such that $R = S \oplus J(R)$ (a topological direct group sum).*

Proof. By Theorem 4.7, there exists a closed subring A, such that $A \cong_{\text{top}} \prod_{\alpha \in \Omega} R_\alpha$, where each R_α is a primary ring and $R = A + J(R)$. By Lemma 4.5, for each $\alpha \in \Omega$, there exists a subring S_α, such that $R_\alpha = S_\alpha + J(R_\alpha)$. Since $J\left(\prod_{\alpha \in \Omega} R_\alpha\right) = \prod_{\alpha \in \Omega} J(R_\alpha)$, there exists a subring S of the ring A, topologically isomorphic to $\prod_{\alpha \in \Omega} S_\alpha$, such that $A = S + J(A)$.

We note that $J(A) \subseteq J(R)$. Indeed, since

$$R/J(R) = (A + J(R))/J(R) \cong A/(A \cap J(R)),$$

$A/(A \cap J(R))$ is semiprimitive, hence $J(A) \subseteq J(R)$.

Therefore

$$R = A + J(R) = S + J(A) + J(R) = S + J(R)$$

and, evidently, $S \cap J(R) = 0$.

We affirm that this sum is a topological direct sum. Indeed, let $\varphi : R \to R/J(R)$ the canonical homomorphism. Since $\varphi|_S : S \to R/J(R)$ is a continuous isomorphism of semiprimitive linearly compact rings, $\varphi|_S$ is a topological isomorphism. By [1, Lemma 13], this sum is a topological. □

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